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On Eisenstein series and the cohomology of arithmetic groups

Sur les séries d'Eisenstein et la cohomologie des groupes arithmétiques

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ABSTRACT

The automorphic cohomology of a reductive $\mathbb{Q}$-group G captures essential analytic aspects of the arithmetic subgroups of G. The subspace spanned by all possible residues and principal values of derivatives of Eisenstein series, attached to cuspidal automorphic forms π on the Levi factor of proper parabolic $\mathbb{Q}$-subgroups of G, forms the Eisenstein cohomology. We show that non-trivial classes can only arise if the point of evaluation features a "half-integral" property. Consequently, only the analytic behavior of the automorphic L-functions at half-integral arguments matters whether an Eisenstein series attached to a globally generic π gives rise to a residual class or not.

RÉSUMÉ

La cohomologie automorphe d'un $\mathbb{Q}$-groupe réductif G détecte des propriétés analytiques essentielles des sous-groupes arithmétiques de G. La cohomologie d'Eisenstein est le sous-espace engendré par tous les résidus ainsi que par les valeurs principales des dérivées des séries d'Eisenstein, attachées aux formes automorphes cuspidales π sur les facteurs de Levi des $\mathbb{Q}$-sous-groupes paraboliques propres de G. Nous montrons que les classes non triviales ne peuvent provenir que des évaluations aux points « demi-entiers ». Ainsi, savoir si une série d'Eisenstein attachée à une forme π générique donne lieu à une classe résiduelle ou non, ne dépend que du comportement analytique de fonctions L automorphes en des points demi-entiers.

1. Eisenstein cohomology of arithmetic groups

Let G be a connected reductive algebraic group defined over $\mathbb{Q}$. Let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ at a place v of $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$, and $\mathbb{A}_f$ the finite adèles. We fix a choice of a minimal parabolic $\mathbb{Q}$-subgroup P_0 of G with Levi decomposition $P_0 = L_0 N_0$, and a choice of a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ such that K is in good position with respect to P_0 (cf. [6, Sect. I.1.4]). Here K_v is a maximal compact subgroup of $G(\mathbb{Q}_v)$, and we write $K_{\mathbb{R}}$ for K_v at the archimedean place $v = \infty$ of $\mathbb{Q}$. Let M_G be the connected component of the intersection of the kernels of all $\mathbb{Q}$-rational characters of G, and $\mathfrak{m}_G$ its Lie algebra. Let A_G be a maximal $\mathbb{Q}$-split torus in the center of G.

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Let E be a finite-dimensional irreducible representation of $G(\mathbb{C})$ of highest weight Λ. Let J_E be the annihilator of the dual representation of E in the center of the universal enveloping algebra of $\mathfrak{m}_G$. Let $\mathcal{A}_E$ be the space of automorphic forms on $A_G(\mathbb{R})^\circ G(\mathbb{Q})\backslash G(\mathbb{A})$ (cf. [6,1]) annihilated by a power of J_E. It carries the structure of an $(\mathfrak{m}_G, K_\mathbb{R}, G(\mathbb{A}_f))$-module. The automorphic cohomology of G with coefficients in E is defined as the Lie algebra cohomology

$$H^*(G,E) = H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_E \otimes E).$$

As proved in [2, Thm. 1.4, resp. 2.3], this cohomology decomposes according to the decomposition of the space of automorphic forms with respect to their cuspidal support. More precisely, let $\mathcal{C}$ be the set of associate classes of parabolic $\mathbb{Q}$-subgroups of G, and, given a class $\{P\} \in \mathcal{C}$, represented by a parabolic $\mathbb{Q}$-subgroup P with Levi decomposition $P = L_P N_P$, let $\Phi_{E,\{P\}}$ be the set of associate classes $\phi = \{\phi_Q\}_{Q\in\{P\}}$ of cuspidal automorphic representations of the Levi factors of $Q \in \{P\}$ as in [2, Sect. 1.2]. Let $\mathcal{A}_{E,\{P\},\phi}$ be the subspace of $\mathcal{A}_E$ consisting of automorphic forms whose constant term along a parabolic $\mathbb{Q}$-subgroup Q of G is orthogonal to the space of cuspidal automorphic forms on $L_Q(\mathbb{A})$ if $Q \notin \{P\}$, and belongs to the ϕ_Q-isotypic component of that space if $Q \in \{P\}$. Then

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes E).$$

For $\{P\} \neq \{G\}$, the cohomology classes in a summand $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes E)$ are constructed from the residues or principal values of the derivatives of Eisenstein series attached to a cuspidal automorphic representation π of $L_P(\mathbb{A})$ belonging to an associate class $\phi \in \Phi_{E,\{P\}}$. Thus, the family of these summands is called the Eisenstein cohomology. We assume, as we may, that π is normalized in such a way that the poles of the Eisenstein series attached to π are real.

As proved in [5, Sect. 3], from the representation theoretic point of view, the study of a summand in the above decomposition of the automorphic cohomology, reduces to the study of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^*\big(\mathfrak{p}, K_\mathbb{R} \cap P(\mathbb{R}); V_\pi \otimes H^*(\mathfrak{n}_P, E) \otimes S\big(\check{\mathfrak{a}}_P^G\big)\big),$$

where $\mathfrak{p}$, $\mathfrak{n}_P$ are the Lie algebras of P and N_P, V_π is the π-isotypic subspace of the space of cuspidal automorphic forms on $L_P(\mathbb{A})$, and $S(\check{\mathfrak{a}}_P^G)$ is the symmetric algebra of $\check{\mathfrak{a}}_P^G$ endowed with the $(\mathfrak{m}_G, K_\mathbb{R})$-module structure as in [1, p. 218]. Here $\check{\mathfrak{a}}_P^G$ is the dual of $\mathfrak{a}_P \cap \mathfrak{m}_G$, where $\mathfrak{a}_P$ is the Lie algebra of the maximal split torus A_P in the center of L_P.

2. Necessary conditions for non-vanishing

The necessary conditions for non-vanishing of cohomology classes are given in terms of the absolute root system of G. Hence, for simplicity of exposition, we assume from this point on that G is $\mathbb{Q}$-split. Let Ψ be the absolute root system of G with respect to L_0, Ψ^+ and Δ the positive and simple roots determined by P_0. Let ρ_{P_0} be the half-sum of positive roots. Let W be the absolute Weyl group of G. Let P be a standard (i.e. containing P_0) proper parabolic $\mathbb{Q}$-subgroup of G, with Levi decomposition $P = L_P N_P$. Let W^P be the set of minimal coset representatives for $W_{L_P}\backslash W$ (cf. [3]), where W_{L_P} is the absolute Weyl group of L_P. For $w \in W^P$, let F_{μ_w} be a representation of the Levi factor $L_P(\mathbb{C})$ of highest weight $\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0}$. Let $\check{\mathfrak{a}}_P = X^*(P) \otimes \mathbb{R}$, where $X^*(P)$ denotes the group of $\mathbb{Q}$-rational characters of P. Representation theoretical arguments show

Proposition 2.1. *The space $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes_\mathbb{C} E)$ is trivial except possibly if there exists a representative $w \in W^P$ such that F_{μ_w} is isomorphic to its complex conjugate contragredient representation $F^*_{\mu_w}$, and so that for any $\pi \in \phi$ the infinitesimal characters of its infinite component π_∞ and $F^*_{\mu_w}$ coincide.*

Proposition 2.2. *(See Thm. 4.11 in [7].) If the two necessary conditions in Proposition* 2.1 *are satisfied for certain $w \in W^P$, then the only possibly non-trivial cohomology classes are those obtained from the residues or the principal values of the derivatives of the Eisenstein series attached to π as in* [4] *or* [6, Sect. II.1.5] *at the value $s_w = (-w(\Lambda + \rho_{P_0}))|_{\check{\mathfrak{a}}_P}$ of its complex parameter.*

3. Evaluation points and automorphic L-functions at half-integral arguments

We retain the assumption that G is $\mathbb{Q}$-split, and restrict our attention to classical groups. More precisely, G is the $\mathbb{Q}$-split general linear group GL_n $(n > 1)$, the symplectic group Sp_n, the odd special orthogonal group SO_{2n+1}, or the even special orthogonal group SO_{2n} $(n > 1)$. Let $e_k \in \check{\mathfrak{a}}_{P_0}$, for $k = 1, \ldots, n$, be the projection of L_0 to its kth component. The standard parabolic $\mathbb{Q}$-subgroups of G are in bijection with the subsets of the set Δ of simple roots. Let $1 \leqslant R_1 < \cdots < R_d \leqslant n$ be integers, and $R_d = n$ if $G = GL_n$. Let P be a standard parabolic $\mathbb{Q}$-subgroup of G corresponding to $\Theta_P = \Delta \setminus \{\alpha_{R_1}, \ldots, \alpha_{R_d}\}$, where α_R is the Rth root in the standard ordering of simple roots, except in the case $G = GL_n$ where $\Theta_P = \Delta \setminus \{\alpha_{R_1}, \ldots, \alpha_{R_{d-1}}\}$. For simplicity of exposition, if $G = SO_{2n}$ we exclude the case $R_d = n - 1$. Let π be a cuspidal automorphic representation of $L_P(\mathbb{A})$ belonging to an associate class $\phi \in \Phi_{E,\{P\}}$.

Theorem 3.1. *Let $s_w = -w(\Lambda + \rho_{P_0})|_{\check{\mathfrak{a}}_P}$ be the evaluation point written in the basis $\{e_1, \dots, e_n\}$ for $\check{\mathfrak{a}}_{P_0}$ as*

$$s_w = t_1 \sum_{l_1=1}^{R_1} e_{l_1} + t_2 \sum_{l_2=R_1+1}^{R_2} e_{l_2} + \dots + t_d \sum_{l_d=R_{d-1}+1}^{R_d} e_{l_d},$$

where $t_1, \dots, t_d \in \mathbb{R}$. Then the residue or a derivative of the Eisenstein series attached to π, evaluated at s_w, can possibly give rise to a non-trivial cohomology class in the space $H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\},\phi} \otimes E)$ only if s_w has the property that $t_l \in \frac{1}{2}\mathbb{Z}$ for $l = 1, \dots, d$, except in the case $G = GL_n$ where we have $t_k - t_l \in \frac{1}{2}\mathbb{Z}$ for $1 \leqslant k < l \leqslant d$.

The main technical tool in the proof is a combinatorial description of the sets W^P for classical groups which enables us to give explicit formulas for the action of $w \in W^P$ on $\check{\mathfrak{a}}_{P_0}$. The divisibility properties of the coefficients of the evaluation point $s_w = -w(\Lambda + \rho_{P_0})|_{\check{\mathfrak{a}}_P}$ can be controlled using the explicit formula for the action of $w \in W^P$ and the necessary condition $F^*_{\mu_w} \cong F_{\mu_w}$ in Proposition 2.1.

This theorem shows that for computing Eisenstein cohomology one only needs to consider the Eisenstein series at evaluation points of a very special form. In particular, if π is globally generic, the Langlands–Shahidi method relates the poles of the Eisenstein series attached to π to the analytic properties of certain automorphic L-functions. The point of evaluation s_w occurs in the arguments of those L-functions as $k_\beta \langle s_w, \beta^\vee \rangle$, where $\beta \in \Psi^+_{\mathrm{red}}(G, A_P)$ ranges over the positive roots in the reduced root system of G with respect to A_P, and either $k_\beta = 1$ or $k_\beta \in \{1, 2\}$ depending on β. Therefore, in all cases $\langle s_w, \beta^\vee \rangle \in \frac{1}{2}\mathbb{Z}$. Note that for different β, different L-functions appear. Moreover, if the symmetric or exterior square L-function appears with $k_\beta = 1$, then $R_d = n$, and either $G = SO_{2n+1}$ with β of the form $\beta = e_{R_k}$, or $G = SO_{2n}$ with β of the form $\beta = e_{R_{k-1}} + e_{R_k}$ and $R_k - R_{k-1} \geqslant 2$. Thus, in these two cases, in fact, $\langle s_w, \beta^\vee \rangle = 2t_k \in \mathbb{Z}$. Although the analytic properties of all the L-functions in the Langlands–Shahidi normalizing factors are not completely understood (e.g. the poles inside $0 < s < 1$ for the symmetric and exterior square L-functions), it turns out, due to Theorem 3.1, that they are known at the evaluation points which are relevant for cohomology. We discuss an example in the next section.

4. An example: maximal parabolic subgroups of the symplectic group

We consider the $\mathbb{Q}$-split symplectic group Sp_n of $\mathbb{Q}$-rank $n \geqslant 2$. The highest weight Λ of the representation E of $Sp_n(\mathbb{C})$ is of the form $\Lambda = \sum_{k=1}^n \lambda_k e_k$, where all $\lambda_k \in \mathbb{Z}$ and $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n \geqslant 0$. Let $P_n = L_n N_n$ be the standard maximal proper parabolic $\mathbb{Q}$-subgroup of Sp_n with the Levi factor $L_n \cong GL_n$. Let π be a cuspidal automorphic representation of $L_n(\mathbb{A})$ in an associate class $\phi \in \Phi_{E,\{P_n\}}$.

Theorem 4.1. *Let $\mathcal{L}_{E,\{P_n\},\phi}$ be the subspace of $\mathcal{A}_{E,\{P_n\},\phi}$ consisting of square-integrable automorphic forms. The cohomology space $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes E)$ is trivial except possibly in the case where the following conditions are satisfied*:

(i) *the representation π is selfdual, $L(s, \pi, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \pi) \neq 0$,*
(ii) *the $\mathbb{Q}$-rank n of the algebraic group $Sp_n/\mathbb{Q}$ is even,*
(iii) *the highest weight Λ of E satisfies $\lambda_{2l-1} = \lambda_{2l}$ for all $l = 1, 2, \dots, n/2$,*
(iv) *the infinite component π_∞ of π is a tempered representation of $GL_n(\mathbb{R})$ fully induced from $n/2$ unitary discrete series representations of $GL_2(\mathbb{R})$ having the lowest $O(2)$-types $2\mu_l + 2n - 4l + 4$ for $l = 1, \dots, n/2$, where $\mu_l = \lambda_{2l-1} = \lambda_{2l}$.*

Square-integrable automorphic forms in $\mathcal{L}_{E,\{P_n\},\phi}$ are obtained as residues of Eisenstein series attached to π at the poles inside the open positive Weyl chamber in $\check{\mathfrak{a}}_{P_n}$. Since all cuspidal automorphic representations of $GL_n(\mathbb{A})$ are globally generic, the Langlands–Shahidi method implies that those poles coincide with the poles of the normalizing factor

$$\frac{L(s, \pi)}{L(1+s, \pi)\varepsilon(s, \pi)} \frac{L(2s, \pi, \wedge^2)}{L(1+2s, \pi, \wedge^2)\varepsilon(2s, \pi, \wedge^2)},$$

where $s > 0$ is identified with the character $\det^s \in \check{\mathfrak{a}}_{P_n}$. The poles of that ratio at $s > 0$ are among the poles of $L(2s, \pi, \wedge^2)$. However, this L-function has no poles for $2s > 1$, it has a simple pole at $2s = 1$ for π as in Theorem 4.1(i), but its analytic behavior inside the critical strip $0 < 2s < 1$ is not known. At this point the strength of Theorem 3.1 reveals, because it shows that possible poles inside $0 < 2s < 1$ play no role in understanding the cohomology space $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{A}_{E,\{P_n\},\phi} \otimes E)$. The rest of the theorem follows from the explicit formulas for the action of $w \in W^{P_n}$, Propositions 2.1 and 2.2, and $s_w = 1/2$.

A treatment of the other maximal proper parabolic subgroups P_r, with the Levi factor $L_r \cong GL_r \times Sp_{n-r}$ where $r < n$, is also carried through. In that case, given a globally generic $\pi \cong \tau \otimes \sigma$, the analytic behavior of the exterior square L-function $L(2s, \tau, \wedge^2)$ at $s = 1/2$, and the Rankin–Selberg L-function $L(s, \tau \times \sigma)$ at $s = 1$, plays a decisive role.

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