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An Exercise in Automorphic Cohomology — the Case GL_2 over a Quaternion Algebra

To Stephen S. Kudla on the occasion of his 60th birthday

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Abstract

The cohomology of an arithmetic subgroup of a connected reductive algebraic group defined over a number field may be interpreted in terms of the space of automorphic forms. Taking the adelic point of view, we consider the automorphic cohomology of the general linear group GL_4 over a totally real number field and its inner form GL_2 over a quaternion division algebra. In particular, we give the structural description of the automorphic cohomology, study the contribution of the Eisenstein series, and obtain a non-vanishing result for the cuspidal cohomology of the inner form. The main tool for passing to the inner form is the Jacquet-Langlands correspondence, which is made explicit in our case of interest.

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0 Introduction

0.1 Automorphic cohomology

Let G be a connected reductive algebraic $\mathbb{Q}$-group. The case of interest for us will be the group $Res_{k/\mathbb{Q}}H$ obtained from a connected reductive algebraic group H defined over an algebraic number field k by restriction of scalars. The automorphic cohomology $H^*(G, E)$ of G, is usually defined as the relative Lie algebra

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cohomology group

$$H^*(G,E) = H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_E \otimes_{\mathbb{C}} E),$$

of G where $\mathcal{A}_E$ denotes the space of automorphic forms on $G(k)\backslash G(\mathbb{A})$ with respect to coefficient system originating with a finite-dimensional algebraic representation of G. We refer to Section 1 in Chapter I for unexplained notation $\mathfrak{m}_G$ and $K_{\mathbb{R}}$. This cohomology reflects a deep relation between the cohomology of arithmetic subgroups of G and the corresponding theory of automorphic forms.

By use of the notion of cuspidal support for an automorphic form one obtains a finer decomposition of the automorphic cohomology. Let $\mathcal{C}$ be the set of classes of associate parabolic $\mathbb{Q}$-subgroups of G. Given $\{P\}$ a class of associate parabolic $\mathbb{Q}$-subgroups of G, let $\phi = \{\phi_R\}_{R\in\{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$ as defined in [10, Section 1.2.]. The set of all such collections $\phi = \{\phi_R\}_{R\in\{P\}}$ is denoted by $\Phi_{E,\{P\}}$. Given a class $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups of G, we denote by $V_G(\{P\})$ the space of smooth functions on $G(\mathbb{A})$ of uniform moderate growth which are negligible along every parabolic $\mathbb{Q}$-subgroup $Q \notin \{P\}$. Then, given any $\phi \in \Phi_{E,\{P\}}$, we let

$$\mathcal{A}_{E,\{P\},\phi} = \left\{ f \in V_G(\{P\}) \mid f_R \in \bigoplus_{\pi\in\phi_R} L^2_{\mathrm{cusp},\pi}(L_R, \omega_\pi) \otimes S(\check{\mathfrak{a}}_R^G) \quad \forall R \in \{P\} \right\}$$

be the space of functions in $V_G(\{P\})$ whose constant term along each $R \in \{P\}$ belongs to the isotypic components attached to the elements $\pi \in \phi_R$ of the space of cuspidal automorphic forms on the Levi components L_R. Then, by [10, Theorem 1.4 resp. 2.3] or [24, Theorem in III, 2.6], the automorphic cohomology $H^*(G,E)$ has a direct sum decomposition

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_{\mathbb{R}}, \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E)$$

where, given $\{P\} \in \mathcal{C}$, the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$.

The summand in the direct sum decomposition of the cohomology $H^*(G,E)$ that is indexed by the full group $\{G\}$ is called the cuspidal cohomology of G with coefficients in E, and denoted by $H^*_{\mathrm{cusp}}(G,E)$. Due to the results in [8], the cohomology classes in the remaining summands can be described by suitable derivatives of Eisenstein series or residues of these. These classes span the so called Eisenstein cohomology, to be denoted $H^*_{\mathrm{Eis}}(G,E)$.

0.2 The case $GL(2)$ and inner forms

We now suppose that the semi-simple rank $\mathrm{rk}_{\mathbb{Q}} G = 1$. This is the case for $G = Res_{k/\mathbb{Q}} H$ if $\mathrm{rk}_k H = 1$. Then there is exactly one $G(\mathbb{Q})$-conjugacy class $\mathcal{P}$ of proper parabolic $\mathbb{Q}$-subgroups of G, and the associate class $\{P\}$ of such a parabolic $\mathbb{Q}$-subgroup P coincides with $\mathcal{P}$. Thus the decomposition above reduces to direct sum decomposition

$$H^*(G,E) = H^*_{\text{cusp}}(G,E) \oplus \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E).$$

In the case of the general linear group $H = GL(2)/k$, Harder describes in [17] in detail which types (in the sense of [26]) of Eisenstein cohomology classes occur and how their actual construction is related to the analytic properties of certain Euler products (or automorphic L-functions) attached to π. In the general case of a k-rank one group H, the results in [16] present a premature form of the decomposition above. However, the internal structure of the Eisenstein cohomology, in particular, the very existence of residual Eisenstein cohomology classes are still open questions in this generality.

By use of the twisted trace formula one can detect cuspidal automorphic representations for the group $GL(2)/k$ (or variants thereof) which give rise to non-vanishing cuspidal cohomology classes. Then one can use the Jacquet-Langlands correspondence [18] between cuspidal automorphic representations for $GL(2)$ and automorphic representations for its inner forms to construct non-vanishing cohomology classes for these inner forms. This had, for example, interesting applications in the study of arithmetically defined compact hyperbolic 3-manifolds in [20], resp. [28].

0.3 The case $GL(2)$ over a central division algebra

In this paper our object of concern is the following case: Let D be a central division k-algebra of degree d defined over an algebraic number field k. Then the connected reductive k-group $GL(2,D)$ is of semi-simple k-rank 1; it is a k-form of the general linear k-group $GL(2d)$. Up to conjugacy, a minimal parabolic k-subgroup Q of $GL(2,D)$ has the form $Q = LN$ with Levi component $L \cong GL(1,D) \times GL(1,D)$, and N the unipotent radical of Q. In recent work, Badulescu [1] and Badulescu-Renard [2] established a generalization of the global Jacquet-Langlands correspondence to the case $GL(n)/k$ and its inner forms. It is based on the local Jacquet-Langlands correspondence for unitary representations which they define. Unlike the local Jacquet-Langlands correspondence of Deligne-Kazhdan-Vignéras [7] for square-integrable representations which is a bijection, this generalization is neither injective nor surjective, and there are unitary representations on both sides not involved in the correspondence. Nevertheless, it suffices for defining the global Jacquet-Langlands correspondence. For more details see Section 2 and Section 3 in Chapter II.

However, it is our objective to understand what the implications of the general Jacquet-Langlands correspondence are for the investigation of the automorphic cohomology of the group $Res_{k/\mathbb{Q}}GL(2,D)$ obtained from $GL(2,D)$ by restriction of scalars. Of course, a detailed understanding of the automorphic cohomology of the group $Res_{k/\mathbb{Q}}GL(2d)/k$ obtained from the split k-group $GL(2d)$ by restriction of scalars is fundamental for such an analysis.

Being modest in our aim we focus on the case of a quaternion division algebra D over k. Then the group $H' = GL(2, D)$ is an inner form of the general linear group $H = GL(4)/k$ of semi-simple k-rank 3. We are interested in the automorphic cohomology of the $\mathbb{Q}$-groups $G = Res_{k/\mathbb{Q}}H$ and $G' = Res_{k/\mathbb{Q}}H'$ obtained from H and H', respectively, by restriction of scalars. Starting from the decomposition of the automorphic cohomology along the cuspidal support as described in Section 0.1 in both cases, namely G and G', we make a comparison of the internal structure of the individual summands involved in this description. The general Jacquet-Langlands correspondence, made explicit in the case of the group H and its inner form H', provides a relation between the automorphic representations on both sides. However, due to some subtle issues, this relation is not at all carried over to the cohomological frame work. This investigation has to be seen as a first attempt to understand where the obstacles for a direct "cohomological comparison" are. Some of them, for example, originate in the still not well understood cohomological contribution of automorphic representations which occur in the discrete spectrum of the underlying algebraic k-group but are non-cuspidal.

Our work includes

— a structural description of the automorphic cohomology of the group $G = Res_{k/\mathbb{Q}}GL(4)/k$, in particular, of the Eisenstein cohomology,
— an explicit description (up to infinitesimal equivalence) of the irreducible unitary representations of $GL_4(\mathbb{R})$ with non-zero cohomology,
— making explicit the local and the global Jacquet-Langlands correspondence in the case $GL(4)/k$,
— a structural description of the automorphic cohomology of the group $G' = Res_{k/\mathbb{Q}}GL(2, D)$, in particular, of the Eisenstein cohomology,
— a non-vanishing result for the cuspidal cohomology $H^*_{\mathrm{cusp}}(G', \mathbb{C})$ of $G' = Res_{k/\mathbb{Q}}GL(2, D)$
— understanding in which way residues of Eisenstein series may give rise to non-trivial classes in the automorphic cohomology of $G' = Res_{k/\mathbb{Q}}GL(2, D)$ and $G = Res_{k/\mathbb{Q}}GL(4)/k$ respectively, and how their very existence may be understood in terms of the global Jacquet-Langlands correspondence.

These results are contained in Part III of this paper. In Part I, we define the automorphic cohomology, and recall its decomposition along the cuspidal support. We also discuss some background material in the theory of Eisenstein series. Part II deals with the general Jacquet-Langlands correspondence. Although the local Jacquet-Langlands correspondence for unitary representations seems quite complicated, it gives the crucial local ingredients for the definition of the global Jacquet-Langlands correspondence. This global correspondence between discrete spectra of a general linear group and its inner form is defined and proved in Badulescu [1] and Badulescu-Renard [2]. It seems much more natural than the local correspondences required for its definition. We precisely describe the local and global correspondence in the case $GL(4)/k$. This amounts to an explicit enumeration of the unitary representations involved in this correspondence.

Notation

1. Let k be an arbitrary finite extension of the field $\mathbb{Q}$ of rational numbers. The set of places of k will be denoted by V, while V_∞ (resp. V_f) will refer to the set of archimedean (resp. non-archimedean) places of k. The completion of k at a place $v \in V$ is denoted by k_v, and its ring of integers by $\mathcal{O}_v$, $v \in V_f$. In the case $k = \mathbb{Q}$, the ring of integers of $\mathbb{Q}_v$ will be denoted by $\mathbb{Z}_v$, $v \in V_f$. Let $\mathbb{A}_k$ (resp. $\mathbb{I}_k$) be the ring of adèles (resp. the group of idèles) of k. We denote by $\mathbb{A}_{k,f}$ the finite adèles of k. For $k = \mathbb{Q}$ we suppress the subscript from the notation.

   Let D be a division algebra central over k. Let d be the degree of D over k. We denote by V_D a finite set of places $v \in V$ of k such that D does not split at $v \in V_D$, and splits at $v \notin V_D$. In other words, $D \otimes_k k_v$ is isomorphic to the matrix algebra $M_d(k_v)$ of $d \times d$ matrices over k_v at places $v \notin V_D$, while it is isomorphic to a matrix algebra of smaller square matrices with entries in a division algebra D_v over k_v at places $v \in V_D$.

2. The algebraic groups we consider will be linear groups, i.e., such a group H defined over a field k is affine viewed as an algebraic variety. We fix an embedding $\rho : H \longrightarrow GL_n$ (defined over k) of G into some general linear group.

   If H is an algebraic group defined over a field k, and k' is a commutative k-algebra containing k, we denote by $H(k')$ the group of k'-valued points of H. When k' is a field containing k we denote by H/k' the k' algebraic group $H \times_k k'$ obtained from H by extending the ground field from k to k'.

   On the other hand, if H is an algebraic group defined over a field k, and k' a field contained in k, we denote by $Res_{k/k'}H$ the algebraic group defined over k' obtained from H by restriction of scalars from k to k'.

3. Let G be a connected reductive algebraic group defined over $\mathbb{Q}$. Suppose that a minimal parabolic $\mathbb{Q}$-subgroup P_0 of G and a Levi decomposition $P_0 = L_0 N_0$ of P_0 over $\mathbb{Q}$ have been fixed. By definition, a standard parabolic $\mathbb{Q}$-subgroup of G is a parabolic $\mathbb{Q}$-subgroup P of G with $P_0 \subset P$. Then P has a unique Levi decomposition $P = L_P N_P$ over $\mathbb{Q}$ such that $L_P \supset L_0$. When the dependency on the parabolic subgroup is clear from the context, we suppress the subscript P from the notation.

   Let A_P be the maximal $\mathbb{Q}$-split torus in the center of L_P. In the case of the minimal parabolic $\mathbb{Q}$-subgroup P_0 we write $A_0 = A_{P_0}$. Then there is a unique Langlands decomposition $P = M_P A_P N_P$ with $M_P \supset M_0$ and $A_P \subset A_0$.

   Let $\mathfrak{g}, \mathfrak{p}, ...$ denote the Lie algebras of $G(\mathbb{R}), P(\mathbb{R}), \ldots$, respectively. The Lie algebras of the factors in the Langlands decomposition of P will be denoted by $\mathfrak{m}_P, \mathfrak{a}_P, \mathfrak{n}_P$, and $\mathfrak{l}_P = \mathfrak{m}_P + \mathfrak{a}_P$. We put $\check{\mathfrak{a}}_0 = X^*(P_0) \otimes \mathbb{R}$, where X^* denotes the group of $\mathbb{Q}$-rational characters, and similarly for a given standard parabolic $\mathbb{Q}$-subgroup $P \supset P_0, \check{\mathfrak{a}}_P = X^*(P) \otimes \mathbb{R}$. Then $\mathfrak{a}_P = X_*(A_P) \otimes \mathbb{R}$, where X_* denotes the group of $\mathbb{Q}$-rational cocharacters, and $\mathfrak{a}_0 = X_*(A_0) \otimes \mathbb{R}$

are in a natural way in duality with $\check{\mathfrak{a}}_P$ and $\check{\mathfrak{a}}_0$; the pairing is denoted by $\langle\ ,\ \rangle$. In particular, $\mathfrak{a}_P$ and $\mathfrak{a}_0$ are independent of the Langlands decomposition up to canonical isomorphism. The inclusion $A_P \subset A_0$ defines $\mathfrak{a}_P \to \mathfrak{a}_0$, and the restriction of characters of P to P_0 defines $\check{\mathfrak{a}}_P \to \check{\mathfrak{a}}_0$ which is inverse to the dual of the previous map. Thus, one has direct sum decompositions $\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P$ and $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P$ respectively. Let $\mathfrak{a}_P^Q$ be the intersection of $\mathfrak{a}_P$ and $\mathfrak{a}_0^Q$ in $\mathfrak{a}_0$. Similar notation is used for $\check{\mathfrak{a}}$. By $\mathfrak{m}_G$ we denote the intersection $\cap\ker(d\chi)$ of the kernels of the differentials of all rational characters $\chi \in X^*(G)$. Then we put $\mathfrak{a}_P^G := \mathfrak{a}_P \cap \mathfrak{m}_G$; its dimension is called the rank of P. We use $\check{\mathfrak{a}}_P^G$ for its dual. We denote by $\Phi \subset X^*(A_0) \subset \check{\mathfrak{a}}_0$ the set of roots of A_0 in $\mathfrak{g}$; it is a root system in the vector space $\check{\mathfrak{a}}_0$. The ordering on Φ is fixed so that Φ^+ coincides with the set of roots of A_0 in P_0. Let $\Delta \subset \Phi$ be the set of simple positive roots. Let $\mathfrak{a}_0^{G+} \subset \mathfrak{a}_0^G$ and $\check{\mathfrak{a}}_0^{G+} \subset \check{\mathfrak{a}}_0^G$ be the open positive Weyl chambers determined by the choice of P_0. Similarly, we denote the corresponding positive Weyl chambers by $\mathfrak{a}_P^{G+} \subset \mathfrak{a}_P^G$ and $\check{\mathfrak{a}}_P^{G+} \subset \check{\mathfrak{a}}_P^G$. If P is a standard parabolic $\mathbb{Q}$-subgroup of G the Weyl group of A_0 in L_P is denoted by W_P. In particular, we write $W = W_G$ for the Weyl group of the root system Φ. Note that W_P is a subgroup of W.

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I. Automorphic Forms, Eisenstein Series, and Cohomology

In this chapter we recall the definitions and review important properties of automorphic forms, automorphic cohomology, and Eisenstein series on a connected reductive linear algebraic group defined over $\mathbb{Q}$. Although in the rest of the paper we also consider reductive groups defined over an arbitrary algebraic number field k, this will suffice because it can be considered as an algebraic group over $\mathbb{Q}$ via the restriction of scalars from k to $\mathbb{Q}$. Then the $\mathbb{A}_k$-points of a k-group coincide with $\mathbb{A}$-points of the restriction of scalars viewed as a $\mathbb{Q}$-group.

1 Spaces of automorphic forms

1.1 Parabolic subgroups

Let G be a connected reductive linear algebraic group defined over $\mathbb{Q}$. Fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}$ and a Levi subgroup L_0 of P_0 defined over $\mathbb{Q}$. One has the Levi decomposition $P_0 = L_0 N_0$ with unipotent radical N_0. By definition, a standard parabolic subgroup P of G is a parabolic subgroup P of G defined over $\mathbb{Q}$ that contains P_0. Analogously, a standard Levi subgroup L_P of G is a Levi subgroup of any standard parabolic subgroup P of G such that L_P contains L_0. A given standard parabolic subgroup P of G has a unique standard Levi subgroup L_P. We denote by $P = L_P N_P$ the corresponding Levi decomposition of P over $\mathbb{Q}$.

1.2 Iwasawa decomposition

By definition, the adèle group $G(\mathbb{A})$ of the group G is the direct product of the group $G(\mathbb{R})$ of real points of G and the restricted product $\prod'_{v\in V_f} G(\mathbb{Q}_v) =: G(\mathbb{A}_f)$ with respect to the maximal compact subgroups $G(\mathbb{Z}_v) \subset G(\mathbb{Q}_v)$, $v \in V_f$. We fix a maximal compact subgroup K of $G(\mathbb{A})$ subject to the following condition. Since it is of the form $K = \prod_{v\in V} K_v$ where K_v is a maximal compact subgroup of $G(\mathbb{Q}_v), v \in V$, we suppose (as we may) that $K_v = G(\mathbb{Z}_v)$ for almost all finite places $v \in V_f$. If v is archimedean we write $K_{\mathbb{R}}$ instead of K_v, and we write $K_f = \prod_{v\in V_f} K_v$. We may assume that the group K is in "good position" relative to P_0 (cf. [24, I, 1.6]).

For a given standard parabolic subgroup $P = L_P N_P$ of G one has the Iwasawa decomposition $G(\mathbb{A}) = L_P(\mathbb{A}) N_P(\mathbb{A}) K$. Then we can define the standard height function $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ on $G(\mathbb{A})$ by $\prod_{v\in V} |\chi(l)|_v = e^{<\chi, H_P(lnk)>}$ for any character $\chi \in X^*(L_P) \subset \check{\mathfrak{a}}_P$.

1.3 Lie algebras

We denote by M_G the connected component of the intersection of the kernels of all $\mathbb{Q}$-rational characters of G, and by $\mathfrak{m}_G = \mathrm{Lie}(M_G(\mathbb{R}))$ the corresponding Lie algebra. Note that the maximal $\mathbb{Q}$-split torus A_G in the center of G reduces to the identity if G is a semi-simple group. In such a case, $\mathfrak{m}_G = \mathrm{Lie}(G(\mathbb{R}))$. In general, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{a}_G \oplus \mathfrak{m}_G$ of Lie algebras where $\mathfrak{a}_G$ denotes the Lie algebra of $A_G(\mathbb{R})$. In particular, $\mathfrak{m}_G$ coincides with $\mathrm{Lie}(A_G(\mathbb{R})^\circ \setminus G(\mathbb{R}))$. The maximal compact subgroup $K_{\mathbb{R}}$ of $G(\mathbb{R})$ may be viewed as a subgroup of $A_G(\mathbb{R})^\circ \setminus G(\mathbb{R})$. A character $\chi \in X^*(G)$ defines a homomorphism $G(\mathbb{A}) \to \mathbb{I}$ of $G(\mathbb{A})$ into the group of idèles, also denoted by χ. We denote by $G(\mathbb{A})^1$ the subgroup

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) \mid |\chi(g)|_{\mathbb{A}} = 1, \chi \in X^*(G)\}$$

of $G(\mathbb{A})$. One has a decomposition $G(\mathbb{A}) = A_G(\mathbb{R})^\circ \times G(\mathbb{A})^1$ as a direct product, and the group $G(\mathbb{A})^1$ can be identified with $A_G(\mathbb{R})^\circ \backslash G(\mathbb{A})$. In an analogous way, $\mathfrak{m}_G$ can be identified with the Lie algebra $\mathrm{Lie}(A_G(\mathbb{R})^\circ \setminus (G(\mathbb{A}) \cap G(\mathbb{R})))$.

1.4 Functions of uniform moderate growth

We fix a height $\| \ \|$ on the adèle group $G(\mathbb{A})$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the complexification of the real Lie algebra $\mathfrak{g}$. By definition, a C^∞-function $f : G(\mathbb{A}) \to \mathbb{C}$ is of uniform moderate growth on $G(\mathbb{Q})\backslash G(\mathbb{A})$ if

— f is K-finite (i.e., the set $\{f_k, k \in K\}$, where $f_k(g) = f(gk)$, spans a finite-dimensional space),
— there exists a constant $r > 0$, $r \in \mathbb{R}$, such that for all elements $D \in \mathcal{U}(\mathfrak{g})$ there is $c_D \in \mathbb{R}$ with $|Df(g)| \le c_D \|g\|^r$ for all $g \in G(\mathbb{A})$,
— f is invariant under left translation by elements of $G(\mathbb{Q})$.

We denote the space of such functions of uniform moderate growth by $V_{\mathrm{umg}}(G)$. We write

$$V_G = C^\infty_{\mathrm{umg}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}))$$

for the space of smooth complex-valued functions of uniform moderate growth on $G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A})$. The space V_G carries in a natural way the structure of a $(\mathfrak{g}, K_{\mathbb{R}}; G(\mathbb{A}_f))$-module.

1.5 Automorphic forms

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We call a function $f \in V_{\mathrm{umg}}(G)$ an automorphic form on $G(\mathbb{A})$ if there exists an ideal $J \subset \mathcal{Z}(\mathfrak{g})$ of finite codimension that annihilates f. We denote the space of automorphic forms on $G(\mathbb{A})$ by $\mathcal{A}(G)$.

For a given character of $A_G(\mathbb{R})^\circ$, that is, a continuous homomorphism $\chi : A_G(\mathbb{R})^\circ \to \mathbb{C}^\times$, let $V(G,\chi)$ (resp. $\mathcal{A}(G,\chi)$) denote the subspace of functions f in $V_{\mathrm{umg}}(G)$ (resp. $\mathcal{A}(G)$) so that $f(ag) = \chi(a)f(g)$ for all $a \in A_G(\mathbb{R})^\circ$ and each $g \in G(\mathbb{A})$. In the case of the trivial character $\chi = 1$, we have $V_G = V(G,1)$, and we write $\mathcal{A}_G = \mathcal{A}(G,1)$.

1.6 Constant term

Let $P = L_P N_P$ be a standard parabolic $\mathbb{Q}$-subgroup of G. For a measurable locally integrable function f on $G(\mathbb{Q})\backslash G(\mathbb{A})$, the constant term of f along P is the function f_P on $N_P(\mathbb{A})L_P(\mathbb{Q})\backslash G(\mathbb{A})$ defined by

$$f_P : g \longmapsto \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} f(ng)dn\ ,\ g \in G(\mathbb{A}),$$

where the Haar measure dn on $N_P(\mathbb{A})$ is normalized in such a way that one has $\mathrm{vol}_{dn}\ (N_P(\mathbb{Q})\backslash N_P(\mathbb{A})) = 1$. The assignment $f \longmapsto f_P$ is compatible with the actions of $\mathfrak{g}, K_\mathbb{R}$ and $G(\mathbb{A}_f)$ on these functions (if they are defined). If f is smooth (or has moderate growth) then f_P is smooth (or has moderate growth).

1.7 Cuspidal automorphic forms

For an automorphic form $f \in \mathcal{A}(G)$ we say that f is cuspidal if $f_P \equiv 0$ for all proper standard parabolic $\mathbb{Q}$-subgroups of G. We denote the space of all cuspidal automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ by $\mathcal{A}_{\mathrm{cusp}}(G)$. The space $\mathcal{A}_{\mathrm{cusp}}(G)$ is equipped with a natural $(\mathfrak{g}, K_\mathbb{R}, G(\mathbb{A}_f))$-module structure.

1.8 L^2 automorphic forms

Let Z be the center of G, and ω a unitary character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$. We consider the space of L^2 automorphic forms with central character ω. It is the Hilbert space consisting of classes of measurable functions $f : G(\mathbb{A}) \to \mathbb{C}$ such that

— $f(\gamma g) = f(g)$ for $\gamma \in G(\mathbb{Q})$ and $g \in G(\mathbb{A})$,
— $f(zg) = \omega(z)f(g)$ for $z \in Z(\mathbb{A})$ and $g \in G(\mathbb{A})$,
— f is square-integrable modulo center, i.e., $\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |f(g)|^2\, dg < \infty$.

We denote this space by $L^2(G, \omega)$. The group $G(\mathbb{A})$ acts on $L^2(G, \omega)$ by right translations. Let $L^2_{\mathrm{disc}}(G, \omega)$ be the subspace of $L^2(G, \omega)$ which is the sum of all irreducible subrepresentations. It is called the discrete spectrum of G. Its orthogonal complement is the continuous spectrum of G denoted by $L^2_{\mathrm{cont}}(G, \omega)$.

Let $L^2_{\mathrm{cusp}}(G, \omega)$ be the subspace of the space $L^2(G, \omega)$ consisting of cuspidal square-integrable automorphic forms, i.e., those classes in $L^2(G, \omega)$ represented by a measurable function f on $G(\mathbb{A})$ whose constant term $f_P(g) = 0$ for almost all $g \in G(\mathbb{A})$ along all proper parabolic $\mathbb{Q}$-subgroups P. It is a closed $G(\mathbb{A})$-invariant subspace of $L^2(G, \omega)$ called the cuspidal spectrum of G. Gelfand, Graev and Piatetski-Shapiro proved in [11] that it is semi-simple, and each irreducible subrepresentation appears with finite multiplicity. Hence, $L^2_{\mathrm{cusp}}(G, \omega)$ is a subspace of $L^2_{\mathrm{disc}}(G, \omega)$. The orthogonal complement of $L^2_{\mathrm{cusp}}(G, \omega)$ in $L^2_{\mathrm{disc}}(G, \omega)$ is the residual spectrum of G, denoted by $L^2_{\mathrm{res}}(G, \omega)$.

Any cuspidal automorphic form in $\mathcal{A}_{\mathrm{cusp}}(G)$, with a given central character ω, is square-integrable modulo center, since it is of rapid decay. On the other hand, the smooth K-finite functions in an irreducible subrepresentation of $L^2_{\mathrm{cusp}}(G, \omega)$ belong to $\mathcal{A}_{\mathrm{cusp}}(G)$. Such space of smooth K-finite functions in an irreducible subrepresentation of $L^2_{\mathrm{cusp}}(G, \omega)$ is not a representation of $G(\mathbb{A})$, because the $K_\mathbb{R}$-finiteness is not preserved. However, it is a $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$-submodule of $\mathcal{A}_{\mathrm{cusp}}(G)$, and, by abuse of language, we call it a cuspidal automorphic representation of $G(\mathbb{A})$. Similarly, the space of smooth $K_\mathbb{R}$-finite functions in an irreducible subrepresentation of $L^2_{\mathrm{disc}}(G, \omega)$ is only a $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$-submodule of the space $\mathcal{A}(G)$. Nevertheless, we call it an automorphic representation of $G(\mathbb{A})$ belonging to the discrete spectrum. See [5] for more details.

1.9 Decomposition along associate classes of parabolic subgroups

Two parabolic $\mathbb{Q}$-subgroups P and Q of G are said to be associate if their reductive components are conjugate by an element in $G(\mathbb{Q})$. This is equivalent to the condition that their split components are $G(\mathbb{Q})$-conjugate. This notion induces an equivalence relation on the set $\mathcal{P}(G)$ of parabolic $\mathbb{Q}$-subgroups of G. Given $P \in \mathcal{P}(G)$, we denote its equivalence class by $\{P\}$, to be called the associate class of P. Let $\mathcal{C}$ be the set of classes of associate parabolic $\mathbb{Q}$-subgroups of G. For $\{P\} \in \mathcal{C}$ denote by $V_G(\{P\})$ the space of elements in V_G that are negligible along Q for every parabolic $\mathbb{Q}$-subgroup Q in G, $Q \notin \{P\}$, that is, given $Q = L_Q N_Q$, for all $g \in G(\mathbb{A})$ the function $l \mapsto f_Q(lg)$ is orthogonal to the space of cuspidal functions on $A_G(\mathbb{R})^\circ L_Q(\mathbb{Q}) \setminus L_Q(\mathbb{A})$.

The space $V_G(\{P\})$, $\{P\} \in \mathcal{C}$, is a submodule in V_G with respect to its natural structure as $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$-module. It is known that the $\sum V_G(\{P\})$, $\{P\} \in \mathcal{C}$, forms a direct sum. Finally, one has a decomposition as a direct sum of $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$-modules

$$V_G = \bigoplus_{\{P\} \in \mathcal{C}} V_G(\{P\}).$$

This was first proved in [22], see [6, Theorem 2.4], for a variant of the original proof. This decomposition gives rise to a decomposition on the subspace $\mathcal{A}_G$ of V_G.

2 Automorphic cohomology

2.1 Definition of automorphic cohomology

Let (ν, E) be an irreducible finite-dimensional algebraic representation of $G(\mathbb{C})$ in a complex vector space. We suppose that $A_G(\mathbb{R})^\circ$ acts by a character on E, to be denoted by χ^{-1}. Let $J_E \subset \mathcal{Z}(\mathfrak{g})$ be the annihilator of the dual representation of E in $\mathcal{Z}(\mathfrak{g})$. Let $\mathcal{A}_E \subset V_G$ be the subspace of functions $f \in V_G$ which are annihilated by a power of J_E. Then the spaces $\mathcal{A}_E \otimes_\mathbb{C} E$ and $V_G \otimes_\mathbb{C} E$ both are naturally equipped with an $(\mathfrak{m}_G, K_\mathbb{R})$-module structure. By [8, Theorem 18], the inclusion

$$\mathcal{A}_E \otimes_\mathbb{C} E \longrightarrow V_G \otimes_\mathbb{C} E$$

of the space of automorphic forms on G (with respect to (ν, E)) in the space of functions of uniform moderate growth induces an isomorphism on the level of $(\mathfrak{m}_G, K_\mathbb{R})$-cohomology, that is,

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_E \otimes_\mathbb{C} E) \xrightarrow{\sim} H^*(\mathfrak{m}_G, K_\mathbb{R}; V_G \otimes_\mathbb{C} E). \tag{2.1}$$

Both cohomology spaces carry a $G(\mathbb{A}_f)$-module structure induced by the one on $\mathcal{A}_E$ and V_G respectively, and the isomorphism is compatible with this $G(\mathbb{A}_f)$-module structure.

By Borel's regularization theorem, the latter group can be identified with the group

$$H^*(\mathfrak{m}_G, K_{\mathbb{R}}; C^\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \backslash G(\mathbb{A})) \otimes_{\mathbb{C}} E)$$

as a $G(\mathbb{A}_f)$-module up to a twist. In fact, as explained in [10] we keep in mind that these cohomology groups have an interpretation as the inductive limit of the deRham cohomology groups $H^*(X_C, E)$ of the orbit space

$$X_C := G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})/K_{\mathbb{R}}C$$

with coefficients in the local system given by the representation (ν, E), where C ranges over the open compact subgroups of $G(\mathbb{A}_f)$. Thus, it is a natural framework to study the cohomology of congruence subgroups of $G(\mathbb{Q})$.

2.2 Decomposition along the cuspidal support

Let $\{P\}$ be a class of associate parabolic $\mathbb{Q}$-subgroups of G, and let $\phi = \{\phi_R\}_{R\in\{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$ as defined in [10, Section 1.2].

The set of all such collections $\phi = \{\phi_R\}_{R\in\{P\}}$ is denoted by $\Phi_{E,\{P\}}$. Given a class $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups of G, and any $\phi \in \Phi_{E,\{P\}}$, we let

$$\mathcal{A}_{E,\{P\},\phi} = \left\{ f \in V_G(\{P\}) \mid f_R \in \bigoplus_{\pi\in\phi_R} L^2_{\mathrm{cusp},\pi}(L_R, \omega_\pi) \otimes S(\check{\mathfrak{a}}_R^G) \quad \forall R \in \{P\} \right\}$$

be the space of functions of uniform moderate growth whose constant term along each $R \in \{P\}$ belongs to the isotypic components attached to the elements $\pi \in \phi_R$, where ω_π is the central character of π. Then we have the following result ([10, Theorem 1.4 resp. 2.3], or [24, Theorem in III, 2.6])

Theorem 2.1. *The automorphic cohomology $H^*(G, E)$ has a direct sum decomposition*

$$H^*(G, E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E)$$

where, given $\{P\} \in \mathcal{C}$, the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$.

The summand in the direct sum decomposition of the cohomology $H^*(G, E)$ that is indexed by the full group $\{G\}$ will be called the *cuspidal cohomology of G with coefficients in E*, to be denoted $H^*_{\mathrm{cusp}}(G, E)$. The decompostion of $H^*(G, E)$ according to the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$ - subgroups of G exhibits a natural complement to the cuspidal cohomology, namely the summands indexed by $\{P\} \in \mathcal{C}, \{P\} \neq \{G\}$. Due to the results in [8] that these cohomology classes can be described by suitable derivatives of Eisenstein series or residues of these, one calls this complement

$$H^*_{\mathrm{Eis}}(G, E) := \bigoplus_{\{P\}\in\mathcal{C}, P\neq G} H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\}} \otimes_{\mathbb{C}} E)$$

the Eisenstein cohomology of G with coefficients in E.

3 Eisenstein series

In this section, following [21] and [24], we recall some facts regarding the analytic properties of Eisenstein series attached to cuspidal automorphic representations on the Levi components of proper parabolic $\mathbb{Q}$-subgroups of a connected reductive algebraic $\mathbb{Q}$-group G. Special attention is given to the case of maximal proper parabolic $\mathbb{Q}$-subgroup. We use these results to study the spaces $\mathcal{A}_{E,\{P\},\phi}$, $\phi \in \Phi_{E,\{P\}}$, which are of importance in the study of automorphic cohomology. We retain the notation of previous sections.

3.1 Definition of Eisenstein series

Let P be a standard parabolic $\mathbb{Q}$-subgroup of G. We write $P = L_P N_P$ for its Levi decomposition. Let π be a cuspidal automorphic representation of $L_P(\mathbb{A})$. More precisely, this is an irreducible $(\mathfrak{l}, K_\mathbb{R}; L_P(\mathbb{A}_f))$-module realized on the space of K-finite smooth functions in an irreducible subrepresentation of $L^2_{\mathrm{cusp}}(L_P,\omega)$ for some central character ω. We denote by V_π the π-isotypic subspace of the space $L^2_{\mathrm{cusp}}(L_P,\omega)$.

We suppose that π is normalized in such a way that the differential of the restriction of the central character of π to $A_P(\mathbb{R})^+$ is trivial. This assumption is just a convenient choice of coordinates, which makes the poles of the Eisenstein series attached to π real. As explained in [10, Section 1.3], it can be achieved by replacing π by an appropriate unitary twist. The twist just moves the poles of the Eisenstein series along the imaginary axis.

As in [10, Section 1.3], consider the space W_π of right K-finite smooth functions

$$f : N_P(\mathbb{A})L_P(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that for every $g \in G(\mathbb{A})$ the function

$$f_g(l) = f(lg)$$

on $L_P(\mathbb{Q})\backslash L_P(\mathbb{A})$ belongs to the π-isotypic subspace V_π of the space of cuspidal automorphic forms on $L_P(\mathbb{A})$. Then, for $f \in W_\pi$, and $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, and for each $g \in G(\mathbb{A})$, one defines (at least formally) the Eisenstein series as

$$E_P^G(f,\lambda)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle H_P(\gamma g), \lambda + \rho_P \rangle} f(\gamma g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_\lambda(\gamma g),$$

where $f_\lambda(g) = f(g) e^{\langle H_P(g), \lambda + \rho_P\rangle}$, and ρ_P is the half-sum of positive roots with respect to P_0 in the root system Φ of G which are not positive roots of L_P. The pairing $\langle \cdot, \cdot \rangle$ is the natural pairing of $\check{\mathfrak{a}}_{P_0,\mathbb{C}}$ and $\mathfrak{a}_{P_0,\mathbb{C}}$.

This Eisenstein series converges absolutely and uniformly in g if the real part $Re(\lambda)$ is sufficiently regular, i.e., lies deep enough inside the positive Weyl chamber $\check{\mathfrak{a}}_P^{G+}$ defined by P. The assignment $\lambda \longmapsto E_P^G(f,\lambda)(g)$ defines a map that is holomorphic in the region of absolute convergence of the defining series and has a meromorphic continuation to all of $\check{\mathfrak{a}}_{P,\mathbb{C}}$. We refer to [24, Section IV.1] for proofs of these facts.

3.2 Filtration of $\mathcal{A}_{E,\{P\},\phi}$

In the case of the given parabolic $\mathbb{Q}$-subgroup P, and the given associate class $\phi \in \Phi_{E,\{P\}}$ containing π, the space $\mathcal{A}_{E,\{P\},\phi}$ introduced in Section 2 can be described using Eisenstein series attached to π as in [10, Section 1.3]. More precisely, there is a unique $\lambda_0 \in \check{\mathfrak{a}}_P$ such that the Eisenstein series $E(f,\lambda_0)$ attached to π (or its residue if there is a pole at $\lambda = \lambda_0$) is annihilated by J_E. There exists a polynomial $q(\lambda)$ on $\check{\mathfrak{a}}_P$ such that $q(\lambda)E(f,\lambda)$ is holomorphic at $\lambda = \lambda_0$. Then, the space $\mathcal{A}_{E,\{P\},\phi}$ is spanned by all coefficients in the Taylor expansion of $q(\lambda)E(f,\lambda)$ around $\lambda = \lambda_0$. This definition is independent on the choice of q, as well as the choice of a representative P for the associate class $\{P\}$ and a representative $\pi \in \phi_P$ for ϕ. These coefficients are in fact all possible residues and main values of derivatives of the Eisenstein series $E(f,\lambda)$ attached to π.

The $(\mathfrak{m}_G, K_{\mathbb{R}}; G(\mathbb{A}_f))$-module $\mathcal{A}_{E,\{P\}}$ has a filtration defined in [8, Section 6]. However, we use a slight modification as in [10, Section 5.2], where the filtration is given along the cuspidal support. According to the decomposition of $\mathcal{A}_{E,\{P\}}$ along the cuspidal support as in Section 1.8, it suffices to give the filtration of the $(\mathfrak{m}_G, K_{\mathbb{R}}; G(\mathbb{A}_f))$-modules $\mathcal{A}_{E,\{P\},\phi}$, where $\phi \in \Phi_{E,\{P\}}$ is the associate class of π. In this paper we consider only the (possibly trivial) lowest filtration step

$$\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi},$$

given as the $(\mathfrak{m}_G, K_{\mathbb{R}}; G(\mathbb{A}_f))$-module $\mathcal{L}_{E,\{P\},\phi}$ consisting of square-integrable automorphic forms on G supported in $\{P\}$ and the associate class of π. It is non-trivial if and only if the Eisenstein series $E(f,\lambda)$ has a square-integrable residue at $\lambda = \lambda_0$ for some choice of $f \in W_\pi$. In that case, it is isomorphic to the space of these residues when f ranges over W_π.

We remark that the fact that the space $\mathcal{L}_{E,\{P\},\phi}$ is a filtration step in the filtration defined in [8, Section 6] follows from the fact that J_E is the annihilator of a finite-dimensional representation. Indeed, the infinitesimal character of any finite-dimensional representation of $G(\mathbb{C})$ is represented by an element inside the open positive Weyl chamber $\check{\mathfrak{a}}_0^{G+}$, and thus, the restrictions of the elements in its Weyl group orbit to any $\check{\mathfrak{a}}_P$ with $P \neq G$ are non-zero. This shows that the function T used for defining the filtration in [8, Section 6] obtains its minimal value if and only if we form the residual Eisenstein series from a residual representation of $G(\mathbb{A})$ itself, since only then the evaluation point is zero. This means that we obtain the residual representations of $G(\mathbb{A})$ supported in π alone in the lowest filtration step. These form $\mathcal{L}_{E,\{P\},\phi}$. However, this argument does not hold in general for any ideal of finite-codimension in $\mathcal{Z}(\mathfrak{g})$ because the infinitesimal characters which are annihilated by such ideal might be on the boundaries of the Weyl chambers. In that case there really exist non-square-integrable automorphic forms in the lowest filtration step.

3.3 Eisenstein series of relative rank one

Let $P = P_\alpha$, denote the standard maximal parabolic $\mathbb{Q}$-subgroup of G which corresponds to the subset $\Delta \setminus \{\alpha\} \subset \Delta$, where α is a simple root in Φ. In this special case there is a convenient choice of an isomorphism $\check{\mathfrak{a}}^G_{P,\mathbb{C}} \cong \mathbb{C}$. Let ρ_P be the half-sum of positive roots in Φ which are not positive roots of L_P. We choose

$$\widetilde{\rho}_P = \langle \rho_P, \alpha^\vee \rangle^{-1} \rho_P$$

as a basis for $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$ following Shahidi's convention in [31]. Here $\alpha^\vee$ is the coroot dual to α, and $\langle \cdot, \cdot \rangle$ the natural pairing. We always identify accordingly $s \in \mathbb{C}$ with $\lambda_s = \widetilde{\rho}_P \otimes s \in \check{\mathfrak{a}}^G_{P,\mathbb{C}}$.

As in the general case, for a cuspidal automorphic representation π of $L_P(\mathbb{A})$, which is normalized as above, one defines the space W_π, and then, for $f \in W_\pi$, and $\lambda_s \in \check{\mathfrak{a}}^G_{P,\mathbb{C}}$, and for each $g \in G(\mathbb{A})$, one defines (at least formally) the Eisenstein series by the same series as above. This Eisenstein series converges absolutely and uniformly in g if the real part $Re(s)$ is sufficiently regular, i.e., lies deep enough inside the positive Weyl chamber defined by P. The assignment $s \longmapsto E^G_P(f, \lambda_s)(g)$ defines a map that is holomorphic in the region of absolute convergence of the defining series and has a meromorphic continuation to all of $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$. It has a finite number of simple poles in the real interval $0 < \lambda_s \leq \rho_P$, i.e., all the remaining poles lie in the region $Re(s) < 0$. We refer to [24, Section IV.1] for proofs of these facts.

In this case the filtration of the space $\mathcal{A}_{E,\{P\},\phi}$ defined in [8, Section 6] is a two-step filtration

$$\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi},$$

as in [10, Section 5.2]. The space $\mathcal{L}_{E,\{P\},\phi}$ is spanned by the residues at $s > 0$ of the Eisenstein series attached to a function f such that for every $g \in G(\mathbb{A})$ the functions f_g defined above belong to the π-isotypic subspace of the space of cuspidal automorphic forms on $L_P(\mathbb{A})$. Those residues are square-integrable automorphic forms by [24, Section I.4.11]. The quotient $\mathcal{A}_{E,\{P\},\phi}/\mathcal{L}_{E,\{P\},\phi}$ is spanned by the principal values of the derivatives of such Eisenstein series at $Re(s) \geq 0$.

3.4 Square-integrable cohomology

Having defined the space of square-integrable automorphic forms $\mathcal{L}_{E,\{P\},\phi}$, it is natural to consider its contribution to automorphic cohomology. Let

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P\},\phi} \otimes_\mathbb{C} E)$$

be the relative Lie algebra cohomology of $\mathcal{L}_{E,\{P\},\phi}$. The inclusion $\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi}$ induces a map in the cohomology

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P\},\phi} \otimes_\mathbb{C} E) \to H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes_\mathbb{C} E).$$

We are interested in the image of that map which we denote by

$$H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes_\mathbb{C} E).$$

Let

$$H_{\mathrm{Eis},(\mathrm{sq})}(G,E)=\bigoplus_{\{P\}\neq\{G\}}\bigoplus_{\phi\in\Phi_{E,\{P\}}}H^*_{(\mathrm{sq})}(\mathfrak{m}_G,K_{\mathbb{R}};\mathcal{A}_{E,\{P\},\phi}\otimes_{\mathbb{C}}E).$$

Since every cuspidal automorphic form in $\mathcal{A}_{\mathrm{cusp}}(G)$ is square-integrable, we have $\mathcal{L}_{E,\{G\},\phi}=\mathcal{A}_{E,\{G\},\phi}$, and the cuspidal cohomology coincides with

$$H^*_{\mathrm{cusp}}(G,E)=\bigoplus_{\phi\in\Phi_{E,\{G\}}}H^*_{(\mathrm{sq})}(\mathfrak{m}_G,K_{\mathbb{R}};\mathcal{L}_{E,\{G\},\phi}\otimes_{\mathbb{C}}E).$$

Thus we define square-integrable automorphic cohomology of G with respect to E as

$$H^*_{(\mathrm{sq})}(G,E)=H^*_{\mathrm{cusp}}(G,E)\oplus H_{\mathrm{Eis},(\mathrm{sq})}(G,E).$$

II. Jacquet-Langlands Correspondence

In this chapter, k denotes an algebraic number field, and D a division algebra central over k of degree d. We would like to compare the automorphic cohomology of the k-split general linear group defined over a number field k with the automorphic cohomology of its inner form — the general linear group over a division algebra D. In particular, we are interested in the case $GL(2,D)$, where D is a quaternion division algebra central over k. The correspondence between automorphic representations belonging to the discrete spectrum of $GL_4(\mathbb{A}_k)$ and its inner form is given by the global Jacquet-Langlands correspondence between general linear groups and their inner forms defined in Badulescu [1] and Badulescu and Renard [2]. The former paper deals with the case where D splits at all $v\in V_\infty$. The latter paper removes this assumption. We begin with a precise definition of considered groups.

1 A k-rank one form of the general linear group: $GL(2,D)$

1.1 The general linear group

Let $GL(n)$ be the general linear group defined over k. It is a connected reductive algebraic k-group of semi-simple k-rank $n-1$, where $n\geq 2$. Let Q_0 be the minimal parabolic k-subgroup consisting of upper triangular non-singular matrices, let S be the maximal torus of diagonal matrices, and let $Q_0=L_{Q_0}N_{Q_0}$ be its Levi decomposition where $L_{Q_0}:=S$ and N_{Q_0} denotes the unipotent radical of Q_0. Let Φ, Φ^+, Δ denote the corresponding sets of roots, positive roots, simple roots, respectively. The set Δ is given as $\Delta=\{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}\}$ where α_i denotes the usual projection $S\to k^\times$ given by the assignment $\mathrm{diag}(t_1,\ldots,t_n)\mapsto t_i/t_{i+1}$. Let W be the Weyl group of $GL(n)$ with respect to S.

The conjugacy classes with respect to $GL(n,k)$ in the set $\mathcal{P}(GL(n))$ of parabolic k-subgroups are in one-to-one correspondence with the subsets of Δ.

Corresponding to $J \subset \Delta$ there is the class represented by the standard parabolic subgroup Q_J. We let $S_J = (\cap_{\alpha \in J} \ker\alpha)^\circ$, and we denote the centralizer of S_J by $L_{Q_J} := Z_{GL(n)}(S_J)$. Then Q_J is the semidirect product of its unipotent radical N_{Q_J} by L_{Q_J}, a so called Levi decomposition of Q_J. The group L_{Q_J} is reductive, a Levi subgroup of Q_J. Notice that the characters of S in N_{Q_J} are exactly the positive roots which contain at least one simple root not in J. Since any Q in $\mathcal{P}(GL(n))$ is $GL(n,k)$-conjugate to a unique Q_J the corresponding subset $J \subset \Delta$ is called the type of Q, to be denoted $J(Q)$. The groups Q_J are the standard parabolic $\mathbb{Q}$-subgroups of $GL(n)$ determined by the choice of S and the set Δ of simple roots.

One has the following description: Let $\rho = (r_1, \dots, r_l)$ be an ordered partition of n into positive integers, i.e., an ordered sequence of positive integers so that $r_1 + \dots + r_l = n$. The corresponding standard parabolic subgroup Q_ρ consists of all matrices in $GL(n,k)$ admitting a block decomposition in the form $(p_{i,j})$ with $p_{i,j}$ an $(r_i \times r_j)$-matrix, and $p_{i,j} = 0$ for $i > j$. Every parabolic subgroup of $GL(n)$ is conjugate to a subgroup of this type. More precisely, Q_ρ is of type $J_\rho = \Delta \setminus \{\alpha_{r_1+\dots+r_i} : i = 1, \cdots, l-1\}$, and the assignment $\rho \mapsto J_\rho$ defines a bijection between partitions of n and subsets of Δ. The standard Levi subgroup L_{Q_ρ} of Q_ρ is the subgroup of matrices in Q_ρ where each block above the block diagonal is zero, i.e., $p_{i,j} = 0$ for $i < j$. Thus, there is an isomorphism $L_{Q_\rho} \cong GL(r_1) \times \dots \times GL(r_l)$. A so called cuspidal parabolic subgroup corresponds up to conjugacy to the case where $r_i = 1$ or 2 for $i = 1, \dots, l$.

In particular, if R is a maximal parabolic k-subgroup of $GL(n)$ of type $\Delta \setminus \{\alpha_j\}$ then it is conjugate to the standard maximal parabolic k-subgroup

$$Q_j := Q_{\Delta \setminus \{\alpha_j\}} = \{(a_{ik}) \in GL(n) \,|\, a_{ik} = 0 \text{ for } k \leq j < i\}, \quad j = 1, \dots, n-1.$$

We say that R is of type j. Note that in this case the Levi subgroup is isomorphic to $GL(j) \times GL(n-j)$. The associate class $\{Q_j\}$ of maximal parabolic k-subgroups R associated to Q_j consists of the groups R of type j and $n-j$. If $n = 2m$ is even the elements R in $\{Q_m\}$ are conjugate to its opposite parabolic k-subgroup R^{opp}, that is, the conjugacy class of Q_m is self-opposite. Otherwise, for general n, the group Q_j is not conjugate to Q_j^{opp}. We note that there are $[\frac{n}{2}]$ associate classes of maximal parabolic k-subgroups in $GL(n)$.

In terms of ordered partitions of n, the parabolic k-subgroups Q_ρ and $Q_{\rho'}$ of $GL(n)$, corresponding to $\rho = (r_1, \dots, r_l)$ and $\rho' = (r'_1, \dots, r'_m)$, are associate if and only if $l = m$ and there is a permutation p of l letters such that $r'_i = r_{p(i)}$ for all $i = 1, \dots, l$. Thus, associate classes of parabolic k-subgroups of $GL(n)$ are parameterized by unordered partitions of n into positive integers.

Example – the case $H = GL(4)$. In the case of the general linear group $H = GL(4)$ there are three conjugacy classes of maximal parabolic k-subgroups; they are represented by the standard parabolic groups Q_1, Q_2, and Q_3. Since Q_1 and Q_3 are associate, one has the two associate classes $\{Q_1\}$ and $\{Q_2\}$. The minimal parabolic k-subgroups of H form one associate class $\{Q_0\}$ represented by the standard parabolic subgroup $Q_0 := Q_\rho$ with $\rho = (1,1,1,1)$. Finally, the three conjugacy classes of the standard parabolic k-subgroups of H corresponding to the ordered sequences $(2,1,1),(1,2,1)$ and $(1,1,2)$ respectively form one associate class, to be denoted $\{Q_{\{\alpha_1\}}\}$, where $Q_{\{\alpha_1\}}$ is a parabolic k-subgroup of H of parabolic rank 2 and of type $J = \{\alpha_1\}$.

1.2 GL_2 over a central division algebra

Let A be a central simple algebra of degree d over an algebraic number field k. Given a positive integer q, let $GL(q,A)$ be the connected reductive algebraic k-group whose group $GL(q,A)(l)$ of rational points over a commutative k-algebra l containing k equals the group

$$GL_q(A_l) = \left\{x \in M_q(A_l) \mid \mathrm{nrd}_{M_q(A_l)}(x) \neq 0\right\},$$

where $A_l = A \otimes_k l$, and $\mathrm{nrd}_{M_q(A_l)}$ is the reduced norm on the $q \times q$ matrix algebra with entries in A_l. If $q = 1$ then $GL_1(A_l)$ is the group of invertible elements in the l-algebra A_l. The reduced norm defines a surjective k-morphism $GL(q,A) \longrightarrow \mathbb{G}_m$ of k-groups, whose kernel is a connected semi-simple algebraic k-group, to be denoted $SL(q,A)$. We have

$$SL(q,A)(l) = SL_q(A_l) = \left\{x \in M_q(A_l) \mid \mathrm{nrd}_{M_q(A_l)}(x) = 1\right\}.$$

Let D be a central division k-algebra of degree d. Then the connected reductive k-group $GL(2,D)$ is of semi-simple k-rank 1. The group $Z'(k)$ of k-rational points of its center Z' is given by

$$Z'(k) = \left\{g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in k^\times 1_D\right\}.$$

We fix a maximal k-split torus $S' \subset GL(2,D)$ subject to

$$S'(k) = \left\{g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \mid \lambda, \mu \in k^\times 1_D\right\}.$$

For the centralizer $L' := Z_{GL(2,D)}(S')$ of S' we have

$$L'(k) = \left\{g = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \mid x, y \in D^\times\right\}.$$

We may (and will) identify L' with the k-group $GL(1,D) \times GL(1,D)$.

Let $\Phi_k = \Phi(GL(2,D),S') \subset X^*(S')$ be the set of roots of $GL(2,D)$ with respect to S'. The set $\{\alpha\}$ is a basis for Φ_k where α denotes the non-trivial character $S'/k \to \mathbb{G}_m/k$ defined by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \mapsto \lambda\mu^{-1}$. The corresponding minimal parabolic k-subgroup determined by $\{\alpha\}$ is denoted by Q'. Its Levi factor is $L_{Q'} = L'$, and we have a Levi decomposition of Q' into the semidirect product $L_{Q'}N_{Q'}$ of its unipotent radical $N_{Q'}$ by $L_{Q'}$.

Let l be a splitting field of D, thus, there is an isomorphism $\psi : D \otimes_k l \to M_d(l)$ of l-algebras, where $M_d(l)$ is the $d \times d$ matrix algebra with entries in l. We fix this isomorphism ψ once and for all. We denote by the same letter the isomorphism

$$\psi : GL(2,D) \times_k l \longrightarrow GL(n)/l, \text{ with } n = 2d$$

of algebraic l-groups induced by ψ. The group $GL(2,D)$ is a k-form of the general linear k-group $GL(2d)$. The image of the l-group $Q' \times_k l$ under ψ is the standard parabolic l-subgroup $Q_d = Q_{\Delta\setminus\{\alpha_d\}}$ in the notation introduced in the previous subsection for the general linear group. Its Levi subgroup L_{Q_d}/l is isomorphic to $GL(d)/l \times GL(d)/l$.

Example – the case $H' = GL(2, D)$, D is a quaternion division algebra. We include this example to fix the notation for the rest of the paper. Let D be a quaternion division algebra central over k. Let V_D be the finite set of places of k at which D does not split. Thus, V_D is the set of places v of k at which $D \otimes_k k_v \cong D_v$, where D_v is the unique (up to isomorphism) quaternion division algebra over k_v. For $v \notin V_D$ we have $D \otimes_k k_v \cong M_2(k_v)$, where $M_2(k_v)$ is the 2×2 matrix algebra over k_v. Let H' denote the algebraic k-group $GL(2, D)$. It is an inner form of the algebraic k-group $H = GL(4)$. The only conjugacy class of parabolic subgroups of H' is represented by $Q' = L_{Q'} N_{Q'}$, where $L_{Q'} \cong GL(1, D) \times GL(1, D)$ is the Levi factor. It is an inner form of the parabolic subgroup Q_2 of H, with the Levi factor $L_{Q_2} \cong GL(2) \times GL(2)$.

We write $H'(\mathbb{A}_k)$ for the adèlic points of H'. Also we have $H'(k_v) \cong GL_2(D_v)$ for $v \in V_D$, and $H'(k_v) \cong GL_4(k_v)$ for $v \notin V_D$. The adèlic points of $GL(1, D)$ are denoted by $D^\times_{\mathbb{A}_k}$. Also $GL(1, D)(k_v) \cong D_v^\times$ for $v \in V_D$, and $GL(1, D)(k_v) \cong GL_2(k_v)$ for $v \notin V_D$.

2 Local Jacquet-Langlands correspondence

We retain the notation of the previous section. Classical local Jacquet-Langlands correspondence is a bijection between the set of isomorphism classes of square-integrable representations of $GL_n(k_v)$ and $GL_{n/d_v}(D_v)$, where k_v is the completion of k at a non-archimedean place v, and D_v is a central simple division algebra of degree d_v over k_v, where d_v divides n. This generalization of original local Jacquet-Langlands correspondence between $GL_2(k_v)$ and the multiplicative group of the quaternion division algebra over k_v (cf. [18]) is obtained by Deligne, Kazhdan and Vignéras in [7]. It is defined by a certain character relation.

For finite places $v \in V_f$ this correspondence is generalized to unitary representations by Badulescu in [1], and for infinite places $v \in V_\infty$, which means real places since there is no division algebra over $\mathbb{C}$, by Badulescu and Renard in [2]. Although we use the term unitary representation at all places, one should have in mind that at infinite places these are in fact Harish-Chandra modules. The correspondence for unitary representations is no longer injective nor surjective, and there exist on both sides unitary representations which are not involved in the correspondence. However, it is again defined using the same character relation, and the reason for missing representations is that, in some cases, this relation does not respect unitarity.

Since we are mainly interested in the infinite places, where the representations should be cohomological, we recall here in detail the local correspondence between $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$ of Badulescu and Renard [2], where $\mathbb{H}$ denotes the Hamilton quaternions. The latter group is isomorphic to $H'(k_v)$ for places $v \in V_\infty$ at which D does not split, i.e., $v \in V_\infty \cap V_D$. For finite places we also recall the correspondence at the level of detail needed later on when dealing with the global correspondence.

2.1 Unitary dual of $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$

The unitary dual of $GL(n)$ over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ is classified by Vogan [37]. However, as in [2], we use the description and notation of [35], which is in analogy with the one introduced by Tadić when describing the unitary dual of $GL(n)$ over a p-adic field in [33] and discussing the unitary dual of $GL(n)$ over a p-adic division algebra in [34].

For a unitary character χ of $\mathbb{R}^\times$, let $\chi \circ \det_n$ denote the corresponding character of $GL_n(\mathbb{R})$ where $\det_n$ is the determinant on $GL_n(\mathbb{R})$, and let $\chi \circ \mathrm{nrd}_n$ denote the corresponding character of $GL_n(\mathbb{H})$, where nrd_n is the reduced norm on $GL_n(\mathbb{H})$.

For a unitary character χ of $\mathbb{R}^\times$, let $\pi(\chi, \alpha)$, where $0 < \alpha < 1/2$, denote the complementary series representation of $GL_2(\mathbb{R})$, i.e., the induced representation of $GL_2(\mathbb{R})$ from the character $\chi|\cdot|^\alpha \otimes \chi|\cdot|^{-\alpha}$ of $\mathbb{R}^\times \times \mathbb{R}^\times$. It is irreducible and unitary.

Similarly, if ρ is either a unitary square-integrable representation δ, or a unitary character $\chi \circ \det_2$, of $GL_2(\mathbb{R})$, and $0 < \alpha < 1/2$, let $\pi(\rho, \alpha)$ be the induced representation of $GL_4(\mathbb{R})$ from the representation $\rho|\det_2|^\alpha \otimes \rho|\det_2|^{-\alpha}$ of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$. It is also irreducible and unitary.

Finally, for a unitary square-integrable representation δ of $GL_2(\mathbb{R})$ we denote by $u(\delta, 2)$ the unique irreducible quotient of the induced representation of $GL_4(\mathbb{R})$ from the representation $\delta|\det|^{1/2} \otimes \delta|\det|^{-1/2}$ of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$. Then $u(\delta, 2)$ is a unitary representation.

We introduce the following sets of (isomorphism classes of) unitary representations (cf. [35] and [2])

$$\begin{aligned}
\mathcal{U}_1 &= \{\chi\}, \\
\mathcal{U}_2 &= \{\chi \circ \det_2,\ \pi(\chi, \alpha),\ \delta : \quad 0 < \alpha < 1/2\}, \\
\mathcal{U}_3 &= \{\chi \circ \det_3\}, \\
\mathcal{U}_4 &= \{\chi \circ \det_4,\ \pi(\chi \circ \det_2, \alpha),\ \pi(\delta, \alpha),\ u(\delta, 2) : \quad 0 < \alpha < 1/2\},
\end{aligned}$$

where χ ranges over all unitary characters of $\mathbb{R}^\times$, and δ over all unitary square-integrable representations of $GL_2(\mathbb{R})$. The representations in $\mathcal{U}_i$ are representations of $GL_i(\mathbb{R})$.

We introduce the same notation for $GL_2(\mathbb{H})$. If $\chi \circ \mathrm{nrd}_1$ is a character of $\mathbb{H}^\times$, let $\pi(\chi \circ \mathrm{nrd}_1, \beta)$ with $0 < \beta < 1$ denote the corresponding complementary series representation of $GL_2(\mathbb{H})$. It is irreducible and unitary. Note that in this case the complementary series is "longer" and goes all the way to 1.

If δ' is an irreducible unitary representation of $\mathbb{H}^\times$ which is not one-dimensional (it is certainly finite-dimensional), then $\pi(\delta', \alpha)$, where $0 < \alpha < 1/2$, denotes the corresponding complementary series representation of $GL_2(\mathbb{H})$. It is irreducible and unitary. For such δ', we let $u(\delta', 2)$ denote the unique irreducible quotient of the induced representation of $GL_2(\mathbb{H})$ from the representation $\delta' \mathrm{nrd}_1^{1/2} \otimes \delta' \mathrm{nrd}_1^{-1/2}$ of $\mathbb{H}^\times \times \mathbb{H}^\times$. It is a unitary representation.

Then we define the following sets of (isomorphism classes of) unitary representations (cf. [2])

$$\mathcal{U}_1' = \{\chi \circ \mathrm{nrd}_1,\, \delta'\}\,,$$
$$\mathcal{U}_2' = \{\chi \circ \mathrm{nrd}_2,\, \pi(\chi \circ \mathrm{nrd}_1, \beta),\, \pi(\delta', \alpha),\, u(\delta', 2) \;:\; 0 < \alpha < 1/2,\, 0 < \beta < 1\}\,,$$

where χ ranges over all unitary characters of $\mathbb{R}^\times$, and δ' over all irreducible unitary representations of $\mathbb{H}^\times$ which are not one-dimensional. The representations in $\mathcal{U}_i'$ are representations of $GL_i(\mathbb{H})$.

Now the unitary duals of $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$ are given in the following Theorem.

Theorem 2.1. *The sets $\mathcal{U}_i$ for $i = 1, 2, 3, 4$, and $\mathcal{U}_j'$ for $j = 1, 2$, consist of irreducible unitary representations. Every representation induced from a tensor product of representations either in sets $\mathcal{U}_i$, or sets $\mathcal{U}_j'$, to the appropriate general linear group is irreducible and unitary. Every irreducible unitary representation of $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$ is obtained in a unique way, up to the order of factors, as such an induced representation.*

2.2 Jacquet-Langlands correspondence for $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$

Now we follow [2] to define the archimedean local Jacquet-Langlands correspondence for unitary representations in our case. It is more convenient to define a map from representations of $GL_4(\mathbb{R})$ to those of $GL_2(\mathbb{H})$. However, this map is not defined on all irreducible unitary representations of $GL_4(\mathbb{R})$. Those (irreducible) unitary representations for which the map is defined are called locally compatible.

By Theorem 2.1, every irreducible unitary representation of $GL_4(\mathbb{R})$ is induced from a tensor product of unique (up to permutation) elements of $\mathcal{U}_i$, $i = 1, 2, 3, 4$. The Jacquet-Langlands correspondence respects this induction process in a sense that the Jacquet-Langlands of such an induced representation is the induced representation of the Jacquet-Langlands of the elements of $\mathcal{U}_i$ from which we induce. Therefore, it suffices to define the correspondence for representations in $\mathcal{U}_i$. We call these the basic unitary representations. Also, if just one of the basic unitary representations from which we induce is not compatible, then the whole induced representation is not compatible.

In the description of the local Jacquet-Langlands correspondence at a real place we use the notation D_m for square-integrable representations of $GL_2(\mathbb{R})$, where $m \geq 2$ is an integer. The square-integrable representation D_m is characterized by the fact that its restriction to the maximal compact subgroup $O(2)$ of $GL_2(\mathbb{R})$ is of the form

$$D_m\big|_{O(2)} \cong \bigoplus_{\substack{j \equiv m \mod 2 \\ j \geq m}} W_j,$$

where W_j, for $j \geq 2$, is the irreducible representation of $O(2)$ obtained as the representation fully induced from the character $z \mapsto z^j$ of the index two subgroup $U(1)$ in $O(2)$. Hence, m is called the lowest $O(2)$-type of D_m, even though $O(2)$

is not commutative. Another characterization of D_m is that it is the unique irreducible subrepresentation of the representation of $GL_2(\mathbb{R})$ induced from the character $|\cdot|^{\frac{m-1}{2}}\,\mathrm{sgn}^m \otimes |\cdot|^{-\frac{m-1}{2}}$ of the maximal split torus.

The elements of $\mathcal{U}_1$ and $\mathcal{U}_3$ are not compatible. Also the complementary series representation $\pi(\chi,\alpha)\in\mathcal{U}_2$ is not compatible. The characters $\chi\circ\det_2\in\mathcal{U}_2$ and $\chi\circ\det_4\in\mathcal{U}_4$ correspond to the characters $\chi\circ\mathrm{nrd}_1\in\mathcal{U}_1'$ and $\chi\circ\mathrm{nrd}_2\in\mathcal{U}_2'$, respectively. The complementary series representation $\pi(\chi\circ\det_2,\alpha)\in\mathcal{U}_4$ corresponds to $\pi(\chi\circ\mathrm{nrd}_1,\alpha)$.

It remains to define the correspondence for representations involving a unitary square-integrable representation δ. In that case the correspondence depends on δ in the following way. If $\delta = D_2(\chi\circ\det_2)\in\mathcal{U}_2$ is of lowest $O(2)$-type 2, then by the classical Jacquet-Langlands correspondence it corresponds to the character $\chi\circ\mathrm{nrd}_1\in\mathcal{U}_1'$ of $\mathbb{H}^\times$, where χ is a unitary character of $\mathbb{R}^\times$. Observe that D_2 corresponds to the trivial character of $\mathbb{H}^\times$. In this case we also have $\pi(\delta,\alpha)\in\mathcal{U}_4$ corresponds to $\pi(\chi\circ\mathrm{nrd}_1,\alpha)\in\mathcal{U}_2$, while $u(\delta,2)\in\mathcal{U}_4$ corresponds to $\pi(\chi\circ\mathrm{nrd}_1,1/2)\in\mathcal{U}_2$.

On the other hand, if $\delta = D_m(\chi\circ\det_2)\in\mathcal{U}_2$ is of lowest $O(2)$-type $m>2$, then it corresponds to $\delta' = D_m'(\chi\circ\mathrm{nrd}_1)\in\mathcal{U}_1'$, which is not a one-dimensional representation of $\mathbb{H}^\times$. In this case $\pi(\delta,\alpha)\in\mathcal{U}_4$ and $u(\delta,2)\in\mathcal{U}_4$ correspond to $\pi(\delta',\alpha)\in\mathcal{U}_2'$ and $u(\delta',2)\in\mathcal{U}_2'$.

Table 1 Local Jacquet-Langlands correspondence at an infinite (real) place

Set	Repn	J.-L. corr.
$\mathcal{U}_1$	χ	$-$
$\mathcal{U}_2$	$\chi\circ\det_2$	$\chi\circ\mathrm{nrd}_1$
	$\pi(\chi,\alpha)$	$-$
	$\delta\cong D_2(\chi\circ\det_2)$	$\chi\circ\mathrm{nrd}_1$
	$\delta\cong D_m(\chi\circ\det_2), m>2$	$D_m'(\chi\circ\mathrm{nrd}_1)$
$\mathcal{U}_3$	$\chi\circ\det_3$	$-$
$\mathcal{U}_4$	$\chi\circ\det_4$	$\chi\circ\mathrm{nrd}_2$
	$\pi(\chi\circ\det_2,\alpha)$	$\pi(\chi\circ\mathrm{nrd}_1,\alpha)$
	$\pi(\delta,\alpha),\delta\cong D_2(\chi\circ\det_2)$	$\pi(\chi\circ\mathrm{nrd}_1,\alpha)$
	$\pi(\delta,\alpha),\delta\cong D_m(\chi\circ\det_2), m>2$	$\pi(\delta',\alpha),\delta'\cong D_m'(\chi\circ\mathrm{nrd}_1)$
	$u(\delta,2),\delta\cong D_2(\chi\circ\det_2)$	$\pi(\chi\circ\mathrm{nrd}_1,1/2)$
	$u(\delta,2),\delta\cong D_m(\chi\circ\det_2), m>2$	$u(\delta',2),\delta'\cong D_m'(\chi\circ\mathrm{nrd}_1)$

For the convenience of the reader, we summarize the correspondence for representations in $\mathcal{U}_i$, $i=1,\ 2,\ 3,\ 4$, in Table 1. The definition shows that, even restricted to the set of compatible unitary representations of $GL_4(\mathbb{R})$, the Jacquet-Langlands correspondence is neither injective, nor surjective. Injectivity fails, for example, for the trivial and the sign character of $GL_4(\mathbb{R})$ which are both mapped to the trivial character of $GL_2(\mathbb{H})$. Surjectivity fails since the complementary series representation of $GL_2(\mathbb{H})$ attached to a character of $\mathbb{H}^\times$ and $1/2<\beta<1$ are not in the image.

2.3 Unitary dual of $GL_4(k_v)$ and $GL_2(D_v)$ at $v \in V_f$

The unitary dual of the general linear group over a p-adic field is obtained by Tadić in [33], while over a p-adic division algebra he described in [34] the unitary dual depending on certain conjectures, which were finally proved by Sécherre in [30] and Badulescu and Renard in [3]. However, in our small rank case, the description follows directly from [7]. We consider the case of D_v the unique (up to isomorphism) quaternion division algebra over k_v.

As in the archimedean case, the unitary dual of $GL_4(k_v)$ consists of representations fully induced from certain basic unitary representations of appropriate general linear groups. These basic unitary representations are again divided into four sets $\mathcal{V}_i$, $i = 1, 2, 3, 4$, where $\mathcal{V}_i$ contains the basic unitary representations of $GL_i(k_v)$, as follows.

$$\begin{aligned}
\mathcal{V}_1 &= \{\chi\}\,, \\
\mathcal{V}_2 &= \{\chi \circ \det_2,\, \pi(\chi, \alpha),\, \delta_2 : \quad 0 < \alpha < 1/2\}\,, \\
\mathcal{V}_3 &= \{\chi \circ \det_3,\, \delta_3\}\,, \\
\mathcal{V}_4 &= \{\chi \circ \det_4,\, \pi(\chi \circ \det_2, \alpha),\, \pi(\delta_2, \alpha),\, u(\delta_2, 2),\, \delta_4 : \quad 0 < \alpha < 1/2\}\,,
\end{aligned}$$

where χ ranges over all unitary characters of $k_v^\times$, while δ_i ranges over all unitary square-integrable representations of $GL_i(k_v)$. Note that at a finite place there are square-integrable representation of $GL_i(k_v)$ for any i, and supercuspidal representations are also square-integrable. Since the notation here is in obvious analogy to the case $k_v = \mathbb{R}$ in Section 2.1, we do not repeat the explanation.

Similarly, we define sets $\mathcal{V}'_i$, for $i = 1, 2$, of basic unitary representation involved in the unitary dual of $GL_2(D_v)$. We have

$$\begin{aligned}
\mathcal{V}'_1 &= \{\chi \circ \mathrm{nrd}_1,\, \rho'\} \\
\mathcal{V}'_2 &= \{\chi \circ \mathrm{nrd}_2,\, \pi(\chi \circ \mathrm{nrd}_1, \beta),\, \pi(\rho', \alpha),\, u(\rho', 2),\, \delta' \,: 0 < \alpha < 1/2, 0 < \beta < 1\}\,,
\end{aligned}$$

where χ ranges over all unitary characters of $k_v^\times$, ρ' over all irreducible unitary representations of $D_v^\times$ which are not one-dimensional, and δ' over all unitary square-integrable representations of $GL_2(D_v)$. The representations in $\mathcal{V}'_i$ are representations of $GL_i(D_v)$.

2.4 Jacquet-Langlands correspondence for $GL_4(k_v)$ and $GL_2(D_v)$ at $v \in V_f$

In the description of the global Jacquet-Langlands correspondence in Section 3 below, we also need some information at non-split finite places $v \in V_f \cap V_D$. For such places $H'(k_v) \cong GL_2(D_v)$, where D_v is the unique (up to isomorphism) quaternion division algebra over k_v, and the correspondence is a special case of the local result in [1].

Just as in the archimedean case, the correspondence respects induction from unitary representations of smaller general linear groups. Hence, it suffices to define it on the sets $\mathcal{V}_i$, for $i = 1, 2, 3, 4$, of basic unitary representations given in Section 2.3. The correspondence is summarized in Table 2.

Table 2 Local Jacquet-Langlands correspondence at a finite place

Set	Repn	J.–L. corr.
$\mathcal{V}_1$	χ	–
$\mathcal{V}_2$	$\chi \circ \det_2$	$\chi \circ \mathrm{nrd}_1$
	$\pi(\chi, \alpha)$	–
	$\delta_2 \cong St_2(\chi \circ \det_2)$	$\chi \circ \mathrm{nrd}_1$
	δ_2 supercuspidal	ρ'
$\mathcal{V}_3$	$\chi \circ \det_3$	–
	δ_3	–
$\mathcal{V}_4$	$\chi \circ \det_4$	$\chi \circ \mathrm{nrd}_2$
	$\pi(\chi \circ \det_2, \alpha)$	$\pi(\chi \circ \mathrm{nrd}_1, \alpha)$
	$\pi(\delta_2, \alpha), \delta_2 \cong St_2(\chi \circ \det_2)$	$\pi(\chi \circ \mathrm{nrd}_1, \alpha)$
	$\pi(\delta_2, \alpha), \delta_2$ supercuspidal	$\pi(\rho', \alpha)$
	$u(\delta_2, 2), \delta_2 \cong St_2(\chi \circ \det_2)$	$\pi(\chi \circ \mathrm{nrd}_1, 1/2)$
	$u(\delta_2, 2), \delta_2$ supercuspidal	$u(\rho', 2)$
	δ_4	δ'

Let us explain the unexplained notation used in Table 2. The square-integrable representation St_2 of $GL_2(k_v)$ is the Steinberg representation, which is the unique irreducible subrepresentation of the induced representation from the character $|\cdot|^{1/2} \otimes |\cdot|^{-1/2}$ of $k_v^\times \times k_v^\times$. Every unitary square-integrable representation which is not supercuspidal is a twist of St_2 by a unitary character χ of $k_v^\times$, i.e., it is of the form $St_2(\chi \circ \det_2)$. For a supercuspidal representation δ_2 of $GL_2(k_v)$ we denote by ρ' the corresponding representation of $D_v^\times$. It is not one-dimensional by the original Jacquet-Langlands correspondence as in [18]. Finally, the correspondence between δ_4 and δ' is just the bijection on square-integrable representations as defined in [7]. In [7] the correspondence is made more precise using the Bernstein-Zelevinsky classification (cf. [40], [4]).

3 Global Jacquet-Langlands correspondence

Although the local Jacquet-Langlands correspondence for unitary representations seems quite complicated, it gives the crucial local ingredients for the definition of the global Jacquet-Langlands correspondence. This global correspondence between discrete spectra of a general linear group and its inner form is defined and proved in Badulescu [1] and Badulescu-Renard [2]. It seems much more natural than the local correspondences required for its definition.

In what follows we define the global Jacquet-Langlands correspondence for the irreducible constituents of the discrete spectrum of $H(\mathbb{A}_k)$ and $H'(\mathbb{A}_k)$. The definition uses the local correspondence of Section 2, which is defined for Harish-Chandra modules at infinite places $v \in V_\infty$, and smooth representations at finite places $v \in V_f$. Hence, one should have in mind, when dealing with irreducible constituents of the discrete spectrum, that we actually pass to the underlying $(\mathfrak{g}, K_{\mathbb{R}}; G(\mathbb{A}_f))$-module without mentioning that explicitly.

3.1 Jacquet-Langlands correspondence between $GL_2(\mathbb{A}_k)$ and $D^\times_{\mathbb{A}_k}$

We first recall the original global correspondence of Jacquet and Langlands [18]. Note that all irreducible automorphic representations of $D^\times_{\mathbb{A}_k}$ are cuspidal. Originally the correspondence is a bijection between (cuspidal) automorphic representations of $D^\times_{\mathbb{A}_k}$ which are not one-dimensional and so-called compatible cuspidal automorphic representations of $GL_2(\mathbb{A}_k)$. This is extended in [1] and [2] to one-dimensional automorphic representations of $D^\times_{\mathbb{A}_k}$. These correspond to the residual representations of $GL_2(\mathbb{A}_k)$, which are all one-dimensional as well.

Theorem 3.1. (1) *There is a unique bijection between (cuspidal) automorphic representations of $D^\times_{\mathbb{A}_k}$ which are not one-dimensional, and cuspidal automorphic representations of $GL_2(\mathbb{A}_k)$ with square-integrable local component at each place where D does not split, such that if $\pi' \cong \otimes_v \pi'_v$ corresponds to $\pi \cong \otimes_v \pi_v$, then $\pi'_v \cong \pi_v$ at $v \notin V_D$, and π'_v corresponds to π_v by the local Jacquet-Langlands correspondence at $v \in V_D$.*

(2) *There is a unique extension of the bijection in (1) to an injection of all (cuspidal) automorphic representations of $D^\times_{\mathbb{A}_k}$ into automorphic representations of $GL_2(\mathbb{A}_k)$ belonging to the discrete spectrum such that if π' corresponds to π, then π is compatible, and the two local conditions of (1) are satisfied. More precisely, this extension maps one-dimensional representation $\chi \circ \mathrm{nrd}$ to $\chi \circ \det$, where χ is a unitary character of $k^\times \backslash \mathbb{I}_k$.*

3.2 Global correspondence

Again we restrict our attention to the case of the global Jacquet-Langlands correspondence between $H'(\mathbb{A}_k)$ and $H(\mathbb{A}_k)$. The center of both groups is isomorphic to the group of idèles $\mathbb{I}_k$ via the isomorphism that assigns to an element $x \in \mathbb{I}_k$ the scalar matrix of the appropriate size with x on the diagonal. Hence, we may view the central characters of discrete spectrum automorphic representations of both groups as unitary characters of $k^\times \backslash \mathbb{I}_k$. We fix such a central character ω. It is preserved under global Jacquet-Langlands correspondence.

We say that an irreducible constituent of $L^2_{\mathrm{disc}}(H,\omega)$ is (globally) compatible with respect to D if every local component π_v of π at a place $v \in V_D$ is locally compatible as a unitary representation of $H(k_v) \cong GL_4(k_v)$, i.e., there is a unitary representation π'_v of $H'(k_v) \cong GL_2(D_v)$ corresponding to π_v by the local Jacquet-Langlands correspondence. In our case at hand, the main result of [1] regarding Jacquet-Langlands correspondence is as follows.

Theorem 3.2. *There is a unique map Ξ from the set of irreducible constituents of $L^2_{\mathrm{disc}}(H',\omega)$ to the set of irreducible constituents of $L^2_{\mathrm{disc}}(H,\omega)$, such that if $\pi = \Xi(\pi')$ then*

- *π is compatible (with respect to D),*
- *$\pi_v \cong \pi'_v$ for $v \notin V_D$,*
- *π_v corresponds to π'_v by the local Jacquet-Langlands correspondence at $v \in V_D$.*

The map Ξ is injective, and the image consists of all compatible constituents of $L^2_{\text{disc}}(H,\omega)$ with respect to D.

The following more precise description of the global correspondence is also proved in [1]. For a positive integer l let R_l be the standard parabolic k-subgroup of $GL(ln)/k$ corresponding to the partition $\rho = (n, n, \ldots, n)$ of ln. Its Levi factor L_{R_l} is isomorphic to the direct product of l copies of $GL(n)/k$. Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_k)$. Let

$$\underline{t}_l = \left(\frac{l-1}{2}, \frac{l-3}{2}, \ldots, -\frac{l-1}{2}\right) \in \check{\mathfrak{a}}_{R_l}.$$

Then, we denote by $I(l,\pi)$ the representation of $GL_{ln}(\mathbb{A}_k)$ induced from the representation

$$\pi|\det|^{\frac{l-1}{2}} \otimes \pi|\det|^{\frac{l-3}{2}} \otimes \cdots \otimes \pi|\det|^{-\frac{l-1}{2}}$$

of the Levi factor $L_{R_l}(\mathbb{A}_k)$. This representation has a unique irreducible quotient which we denote by $J(l,\pi)$. It is a residual representation of $GL_{ln}(\mathbb{A}_k)$ if $l > 1$. For $l = 1$, we have by definition $I(1,\pi) = J(1,\pi) = \pi$. All residual representations of $GL_N(\mathbb{A}_k)$, for $N > 1$, are obtained in this way for some divisor $l > 1$ of N.

Theorem 3.3. *Let π be an irreducible cuspidal automorphic representation of the group $GL_n(\mathbb{A}_k)$. There is a unique positive integer $s_{\pi,D}$, depending only on π and the division algebra D, which is defined by the condition that $J(l,\pi)$ is globally compatible (with respect to D) if and only if $s_{\pi,D}$ divides l. Moreover, $s_{\pi,D}$ divides the degree d of the division algebra.*

A representation of the form $J(s_{\pi,D},\pi)$ of $GL_{ns_{\pi,D}}(\mathbb{A}_k)$ corresponds to a cuspidal automorphic representation π' of the inner form. A representation of the form $J(ms_{\pi,D},\pi)$, with $m > 1$, corresponds to a residual representation $J'(m,\pi')$ of the inner form, where m stands in this case for the point

$$\underline{t}'_m = \left(s_{\pi,D}\frac{m-1}{2}, s_{\pi,D}\frac{m-3}{2}, \ldots, -s_{\pi,D}\frac{m-1}{2}\right),$$

and the notation $J'(m,\pi')$ for inner forms is in an obvious analogy with the split case.

3.3 Discrete spectrum of $GL_4(\mathbb{A}_k)$

In order to describe the global Jacquet-Langlands correspondence more precisely, we require the description of the discrete spectrum $L^2_{\text{disc}}(H,\omega)$ of $H(\mathbb{A}_k)$. In [23], Mœglin and Waldspurger describe the residual part of the discrete spectrum for $GL_n(\mathbb{A}_k)$. The decomposition into irreducibles of the cuspidal part was first proved by Gelfand, Graev and Piatetski-Shapiro in [11]. In [32] Shalika proved that each representation appears with multiplicity one.

Theorem 3.4. *The discrete spectrum $L^2_{\text{disc}}(H,\omega)$ of $H(\mathbb{A}_k)$ decomposes into*

$$L^2_{\text{disc}}(H,\omega) \cong L^2_{\text{cusp}}(H,\omega) \oplus L^2_{\text{res}}(H,\omega),$$

where $L^2_{\mathrm{cusp}}(H,\omega)$ is the cuspidal spectrum consisting of cuspidal elements, and $L^2_{\mathrm{res}}(H,\omega)$ is its orthogonal complement called the residual spectrum. The cuspidal part $L^2_{\mathrm{cusp}}(H,\omega)$ decomposes into a Hilbert space direct sum of all irreducible cuspidal automorphic representations of $H(\mathbb{A}_k)$ with central character ω, each appearing with multiplicity one. The residual part $L^2_{\mathrm{res}}(H,\omega)$ decomposes along the cuspidal support into

$$L^2_{\mathrm{res}}(H,\omega) \cong L^2_{\mathrm{res},\{Q_0\}}(H,\omega) \oplus L^2_{\mathrm{res},\{Q_2\}}(H,\omega),$$

where

$$L^2_{\mathrm{res},\{Q_0\}}(H,\omega) \cong \bigoplus_{\mu} \mu \circ \det,$$

and the sum ranges over all unitary characters μ of $k^\times \backslash \mathbb{I}_k$ such that $\mu^4 = \omega$, while

$$L^2_{\mathrm{res},\{Q_2\}}(H,\omega) \cong \bigoplus_{\sigma} J(2,\sigma),$$

and the sum ranges over all irreducible cuspidal automorphic representations σ of $GL_2(\mathbb{A}_k)$ with central character ω_σ such that $\omega_\sigma^2 = \omega$.

3.4 Jacquet-Langlands correspondence for $H'(\mathbb{A}_k)$ and $H(\mathbb{A}_k)$

In our case at hand, we can make the correspondence Ξ described in the two theorems of Section 3.2 more explicit. Namely, we have the following explicit description of the global Jacquet-Langlands correspondence.

Proposition 3.5. *Let π be an irreducible constituent of $L^2_{\mathrm{disc}}(H,\omega)$. In view of Theorem 3.4 we have the following possibilities.*

(1) *If π is cuspidal, then π is compatible with respect to D if and only if at all non-split places $v \in V_D$ the local component π_v is one of the following:*
   - *$v \in V_f$ and π_v is a unitary square integrable representation of $H(k_v) \cong GL_4(k_v)$,*
   - *π_v is a tempered representation of $H(k_v) \cong GL_4(k_v)$ fully induced from two unitary square-integrable representations of $GL_2(k_v)$,*
   - *π_v is a complementary series representation of $H(k_v) \cong GL_4(k_v)$ attached to a unitary square-integrable representation of $GL_2(k_v)$ and a real number $0 < \alpha < 1/2$.*

   Assume that π is compatible, and $\pi = \Xi(\pi')$. The local component π'_v of π' at $v \in V_D$ is according to the form of π_v one of the following
   - *a unitary square integrable representation of $H'(k_v) \cong GL_2(D_v)$ (this can be explicitly described in terms of Zelevinsky segments),*
   - *a tempered representation of $H'(k_v) \cong GL_2(D_v)$ fully induced from a tensor product of two unitary representations of $D_v^\times$,*
   - *a complementary series representation of $H'(k_v) \cong GL_2(D_v)$ attached to a unitary representation of $D_v^\times$ and a real number $0 < \alpha < 1/2$.*

(2) *If $\pi \cong J(2,\sigma)$, where σ is a cuspidal automorphic representation of $GL_2(\mathbb{A}_k)$, then π is always compatible with respect to D.*
   (a) *If σ_v is square-integrable at all non-split places $v \in V_D$, let σ' be the cuspidal automorphic representation of $D^\times_{\mathbb{A}_k}$ corresponding to σ by the classical Jacquet-Langlands correspondence. Note that σ' is not one-dimensional. Then π corresponds to the residual representation $J'(2,\sigma')$ of $H'(\mathbb{A}_k)$.*
   (b) *If there is a non-split place $v \in V_D$ such that σ_v is not square-integrable, then π corresponds to a cuspidal automorphic representation π' of the group $H'(\mathbb{A}_k)$. The local component π'_v of π' at $v \in V_D$ such that σ_v is not square-integrable is*
   - *either a tempered representation of $H'(k_v) \cong GL_2(D_v)$, which is fully induced from a tensor product of two unitary characters of $D_v^\times$,*
   - *or a complementary series representation of $H'(k_v) \cong GL_2(D_v)$, attached to a unitary character of $D_v^\times$ and a real number $0 < \alpha < 1/2$.*

(3) *If $\pi \cong \mu \circ \det$, where μ is a unitary character of $k^\times \backslash \mathbb{I}_k$, then π is always compatible with respect to D. It corresponds by the global Jacquet-Langlands correspondence to the one-dimensional residual representation $\mu \circ \mathrm{nrd}$ of $H'(\mathbb{A}_k)$.*

Proof. We prove each part of the proposition separately.

(1) Any cuspidal automorphic representation π of $GL_4(\mathbb{A}_k)$ is generic (cf. [32]). Hence, its local components are generic as well. The generic unitary dual, obtained in [33] over a p-adic field and [37] over an archimedean field, is of the same form. More precisely, for any $v \in V$ the local component π_v is a fully induced representation from certain basic unitary representations of the form χ, δ_i, $\pi(\chi,\alpha)$, $\pi(\delta_2,\alpha)$, where $i = 2,3,4$ if $v \in V_f$, and $i = 2$ if $v \in V_\infty$ (since at the real place only δ_2 exist).
   Compatibility of π is determined by the local compatibility at $v \in V_D$. A local component π_v at $v \in V_D$ is compatible if and only if the basic unitary representation involved are compatible. This rules out the possibility of χ and $\pi(\chi,\alpha)$. Hence, in order to obtain a representation of $GL_4(k_v)$, there are only the three possibilities given in the theorem (the first one can occur only for $v \in V_D \cap V_f$ since there is no δ_4 at a real place).
   If π is compatible, then Theorem 3.3 shows that π' is cuspidal, because $\pi = J(1,\pi)$. The description of local components π'_v at $v \in V_D$ follows directly from Section 2.2 and Section 2.4.

(2) The fact that $J(2,\sigma)$ is always compatible follows from the second Theorem in Section 3.2. Indeed, it shows that $s_{\sigma,D}$ divides the degree of D. Thus, in the case of a quaternion division algebra, either $s_{\sigma,D} = 1$ or $s_{\sigma,D} = 2$, and in both cases $J(2,\sigma)$ is compatible.
   (a) If $s_{\sigma,D} = 1$, then σ itself is compatible. This is the situation of the original Jacquet-Langlands correspondence (cf. [18]). The condition of compatibility is precisely the condition on σ_v given in (2a). The rest of the claim directly follows from Theorem 3.3.

(b) If $s_{\sigma,D} = 2$, then σ is not compatible, which is exactly the opposite of the previous case, as claimed. Again Theorem 3.3 shows that in this case π' is cuspidal.
For describing the local components π'_v we consider only places $v \in V_D$ at which σ_v is not square-integrable, because the other possibility is covered by the original Jacquet-Langlands correspondence as in part (2a). Then σ_v is either a tempered representation fully induced from two unitary characters χ_1 and χ_2 of $k_v^\times$, or a complementary series representation $\pi(\chi, \alpha)$, where χ is a unitary character of $k_v^\times$ and $0 < \alpha < 1/2$.
Now π_v, for $v \in V_D$, can be written either as a fully induced representation from $\chi_1 \circ \det_2$ and $\chi_2 \circ \det_2$, or as $\pi(\chi \circ \det_2, \alpha)$. The claim follows from Section 2.2 and Section 2.4.

(3) Finally, the unitary characters $\mu \circ \det$ of $H(\mathbb{A}_k)$ are compatible, because their local components are also characters of $H(k_v) \cong GL_4(k_v)$, and they correspond to characters of $H'(k_v) \cong GL_2(D_v)$ at all $v \in V_D$. Note that $s_{\mu,D} = 2$ for every unitary character μ of $k^\times \backslash \mathbb{I}_k$. Therefore, characters of $H'(\mathbb{A}_k)$ are not cuspidal. □

3.5 Discrete spectrum of $H'(\mathbb{A}_k)$

As a consequence of the global Jacquet-Langlands correspondence explicitly described in Proposition (3.5) we obtain the decomposition of the discrete spectrum of $H'(\mathbb{A}_k)$. In [1] it is also proved that it is multiplicity one.

Theorem 3.6. *The discrete spectrum $L^2_{\mathrm{disc}}(H', \omega)$ of $H'(\mathbb{A}_k)$ decomposes into*

$$L^2_{\mathrm{disc}}(H', \omega) \cong L^2_{\mathrm{cusp}}(H', \omega) \oplus L^2_{\mathrm{res}}(H', \omega),$$

where $L^2_{\mathrm{cusp}}(H', \omega)$ is the cuspidal spectrum consisting of cuspidal elements, and $L^2_{\mathrm{res}}(H', \omega)$ is its orthogonal complement called the residual spectrum. The cuspidal part $L^2_{\mathrm{cusp}}(H', \omega)$ decomposes into a Hilbert space direct sum of irreducible cuspidal automorphic representations with central character ω, each appearing with multiplicity one, and obtained by the global Jacquet-Langlands correspondence either from a cuspidal automorphic representation of $H(\mathbb{A}_k)$ as in part (1) *of Proposition* 3.5, *or from a residual automorphic representation $J(2, \sigma)$ of $H(\mathbb{A}_k)$ with σ as in part* (2b) *of Proposition* 3.5. *The residual part $L^2_{\mathrm{res}}(H', \omega)$ decomposes into a Hilbert space direct sum*

$$L^2_{\mathrm{res}}(H', \omega) \cong \left(\bigoplus_{\mu} \mu \circ \mathrm{nrd} \right) \oplus \left(\bigoplus_{\sigma'} J'(2, \sigma') \right),$$

where the first sum ranges over all unitary characters μ of $k^\times \backslash \mathbb{I}_k$ such that $\mu^4 = \omega$, and $\mu \circ \mathrm{nrd}$ is obtained by the Jacquet-Langlands correspondence from $\mu \circ \det$, while the second sum ranges over all cuspidal automorphic representations σ' of $D^\times_{\mathbb{A}_k}$ which are not one-dimensional, and $J'(2, \sigma')$ is obtained by the Jacquet-Langlands correspondence from $J(2, \sigma)$, where σ is as in part (2a) *of Proposition* 3.5, *with central character $\omega_\sigma = \omega_{\sigma'}$ such that $\omega_\sigma^2 = \omega$.*

3.6 Remark

We describe here the discrete spectrum of algebraic k-groups, although the definition in Chapter I, Section 1.8 refers only to $\mathbb{Q}$-groups. However, this generalization is straightforward, and can be obtained via the restriction of scalars from k to $\mathbb{Q}$.

III. Automorphic Cohomology of General Linear Groups — a Comparison

Since the automorphic cohomology was defined in Chapter I for connected reductive linear algebraic groups defined over $\mathbb{Q}$, we need to consider the restriction of scalars from k to $\mathbb{Q}$ when dealing with k-groups. Thus, except in Section 1, in this chapter we retain the notation of the previous one, namely $H = GL(4)/k$ and $H' = GL(2, D)/k$, where D is a quaternion division algebra central over a number field k. However, we let $G = Res_{k/\mathbb{Q}}H$ and $G' = Res_{k/\mathbb{Q}}H'$ be the $\mathbb{Q}$-groups obtained from H and H' by restriction of scalars, respectively. The goal of this chapter is to compare the automorphic cohomology $H^*(G, E)$ and $H^*(G', E)$, with respect to the trivial representation $E = \mathbb{C}$, and in particular, relate the possible non-trivial cohomology classes via the Jacquet-Langlands correspondence.

1 Cohomological representations at Archimedean places

We briefly discuss the constructive approach to the classification [39] of irreducible unitary representations of a connected real reductive Lie group with non-vanishing relative Lie algebra cohomology. This general result allows us to enumerate (up to infinitesimal equivalence) the irreducible unitary $(\mathfrak{m}_{H(\mathbb{R})}, O(4))$-modules with non-vanishing Lie algebra cohomology in an explicit way, where $O(4)$ is the maximal compact subgroup of $H(\mathbb{R}) \cong GL_4(\mathbb{R})$. We have to start off by determining the irreducible unitary $(\mathfrak{m}_{SL_4(\mathbb{R})}, SO(4))$-modules with non-vanishing Lie algebra cohomology using the results in [39].

In this section we use a different notation then in the rest of the paper. Namely, G denotes a connected real reductive Lie group, $K \subset G$ a maximal compact subgroup. Write $\mathfrak{g}$ for the Lie algebra of G, and write $\mathfrak{g}_{\mathbb{C}}$ for its complexification. Given an irreducible unitary representation (π, H_π) of G we denote the Harish-Chandra module of H_π (i.e., the set of K-finite vectors in the space of C^∞-vectors of H_π) by the same letter or by $H_{\pi,K}$. We denote by W_G (resp. W_K) the Weyl group of G (resp. K).

1.1 The classification up to infinitesimal equivalence

Let G be a connected real reductive Lie group (of Harish-Chandra's class), $K \subset G$ a maximal compact subgroup. Let θ_K be the Cartan involution corresponding to the maximal compact subgroup $K \subset G$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding

Cartan decomposition. By definition a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that $\theta_K \mathfrak{q} = \mathfrak{q}$, and $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{q}$ where $\overline{\mathfrak{q}}$ refers to the image of $\mathfrak{q}$ under complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. Write $\mathfrak{u}$ for the nilradical of $\mathfrak{q}$. Then $\mathfrak{l}_{\mathbb{C}}$ is the complexification of a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The normalizer of $\mathfrak{q}$ in G is connected since G is, and it coincides with the connected Lie subgroup L of G with Lie algebra $\mathfrak{l}$. The Levi subgroup L has the same rank as G, is preserved by the Cartan involution θ_K, and the restriction of θ_K to L is a Cartan involution. Moreover, the group L contains a maximal torus $T \subset K$. We will indicate below a construction of all possible θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ up to conjugation by K. There are only finitely many K-conjugacy classes of θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$.

A θ_K-stable parabolic subalgebra $\mathfrak{q}$ gives rise to an irreducible unitary representation $A_{\mathfrak{q}}$ of G. It is constructed via cohomological induction as $\mathcal{R}^S_{\mathfrak{q}}$ (see [39, Thm. 2.5]) and it is uniquely determined up to infinitesimal equivalence by the K-conjugacy class of $\mathfrak{q}$. In the case that the θ_K-stable parabolic subalgebra coincides with the full algebra, that is, $\mathfrak{q}_0 := \mathfrak{g}_{\mathbb{C}}$, we take $A_{\mathfrak{q}} = \mathbb{C}$. We denote the Harish-Chandra module of $A_{\mathfrak{q}}$ by the same letter or by $A_{\mathfrak{q},K}$. One has

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q},K}) = \mathrm{Hom}_{L\cap K}(\wedge^{j-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}) \tag{1.1}$$

where $R = R(\mathfrak{q}) := \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. Consequently, the Lie algebra cohomology with respect to the representation $A_{\mathfrak{q}}$ vanishes in degrees below $\dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ and above $\dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) + \dim(\mathfrak{l} \cap \mathfrak{p}_{\mathbb{C}})$. Suppose (π, H_π) is an irreducible unitary representation (π, H_π) of G with

$$H^*(\mathfrak{g}, K, H_{\pi,K}) \neq 0.$$

Then there is a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ so that $\pi \cong A_{\mathfrak{q}}$. One finds the construction of the representations $A_{\mathfrak{q}}$ in [25], a proof of their unitarity in [36] and the classification of the irreducible unitary representations of G with non-vanishing cohomology in [39].

Following [39] and [38, Section 4] we outline a construction of all θ_K-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ up to conjugation by K. Fix a maximal torus T in K. The centralizer H of T in G is a Cartan subgroup. According to the Cartan decomposition of $\mathfrak{g}$ we may write $H = TA$ with $A = H \cap (\exp \mathfrak{p})$. We denote the Lie algebra of T by $\mathfrak{t}_c$. Let $\Phi(\mathfrak{k}, \mathfrak{t}_c) =: \Phi_c$ be the system of roots for $\mathfrak{t}_c$ in $\mathfrak{k}_{\mathbb{C}}$, and fix a system $\Phi_c^+ := \Phi^+(\mathfrak{k}, \mathfrak{t}_c) \subset \Phi(\mathfrak{k}, \mathfrak{t}_c)$ of positive roots. Similarly, we write Φ_n for the set of non-zero weights of $\mathfrak{t}_c$ on $\mathfrak{p}_{\mathbb{C}}$.

Fix an element $x \in i(\mathfrak{t}_c)_{\mathbb{R}}$ that is dominant for K, that is, $\gamma(x) \geq 0$ for all $\gamma \in \Phi^+(\mathfrak{k}, \mathfrak{t}_c)$. Then the θ_K-stable parabolic subalgebra associated to x is defined by

$$\mathfrak{q}_x = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma \in \Phi, \gamma(x) \geq 0} \mathfrak{g}_{\mathbb{C},\gamma}$$

with $\Phi := \Phi_n \cup \Phi_c$. The corresponding Levi subalgebra $\overline{\mathfrak{q}_x} \cap \mathfrak{q}_x = (\mathfrak{l}_x)_{\mathbb{C}}$ is

$$(\mathfrak{l}_x)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma \in \Phi, \gamma(x) = 0} \mathfrak{g}_{\mathbb{C},\gamma}.$$

The Levi subgroup is then described by $L_x = \{g \in G \mid \mathrm{Ad}(g)(x) = x\}$.

1.2 The case $GL_4(\mathbb{R})$

This general construction allows us to enumerate (up to infinitesimal equivalence) the irreducible unitary $(\mathfrak{m}_{GL_4(\mathbb{R})}, O(4))$-modules with non-vanishing Lie algebra cohomology in an explicit way, where $O(4)$ is a maximal compact subgroup of $GL_4(\mathbb{R})$. The final result also appears in the thesis [19, Section 2].

First we have to deal with the pair consisting of the semi-simple real Lie group $G = SL(4, \mathbb{R})$ and the maximal compact subgroup $K = SO(4)$. The Lie algebra $\mathfrak{so}_4$ is semi-simple of rank 2. Fix the maximal torus $T = SO(2) \times SO(2)$ in K. The centralizer H of T in G is a Cartan subgroup. Let $\{\gamma_1, \gamma_2\}$ be a basis for the system $\Phi(\mathfrak{k}, \mathfrak{t}_c)$ of positive roots for $\mathfrak{t}_c$ in $\mathfrak{k}_{\mathbb{C}}$. Given an element $x \in i(\mathfrak{t}_c)_{\mathbb{R}}$ that is dominant for K, that is, $\gamma(x) \geq 0$ for all $\gamma \in \Phi(\mathfrak{k}, \mathfrak{t}_c)$, the θ_K-stable parabolic subalgebra associated to x is defined by

$$\mathfrak{q}_x = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma \in \Phi, \gamma(x) \geq 0} \mathfrak{g}_{\mathbb{C},\gamma}$$

with $\Phi := \Phi(\mathfrak{g}, \mathfrak{t}_c)$. The corresponding Levi subalgebra $\overline{\mathfrak{q}_x} \cap \mathfrak{q}_x = (\mathfrak{l}_x)_{\mathbb{C}}$ is

$$(\mathfrak{l}_x)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma \in \Phi, \gamma(x) = 0} \mathfrak{g}_{\mathbb{C},\gamma}.$$

Following this construction one obtains as in [19, Section 2.1] the following list of θ_K-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ (up to conjugation by $K = SO(4)$), enumerated as $\mathfrak{q}_j$, $j = 1, \ldots, 6$, with

$$\begin{array}{ll}
(\mathfrak{l}_1)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} & \mathfrak{u}_1 = \{0\}, \\
(\mathfrak{l}_2)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},-\gamma_1} & \mathfrak{u}_2 = \mathfrak{g}_{\mathbb{C},\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2-\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2+\gamma_1}, \\
(\mathfrak{l}_3)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},-\gamma_2} & \mathfrak{u}_3 = \mathfrak{g}_{\mathbb{C},\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1-\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1+\gamma_2}, \\
(\mathfrak{l}_4)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2-\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1-\gamma_2} & \mathfrak{u}_4 = \mathfrak{g}_{\mathbb{C},\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1+\gamma_2}, \\
(\mathfrak{l}_5)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} & \mathfrak{u}_5 = \mathfrak{g}_{\mathbb{C},\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2-\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2+\gamma_1}, \\
(\mathfrak{l}_6)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} & \mathfrak{u}_6 = \mathfrak{g}_{\mathbb{C},\gamma_1} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_1-\gamma_2} \oplus \mathfrak{g}_{\mathbb{C},\gamma_2+\gamma_1}.
\end{array}$$

We denote by $A_{\mathfrak{q}_j}$ the corresponding $(\mathfrak{m}_G, SO(4))$-module. Second, by taking into account the induction functor Ind from the category of $(\mathfrak{m}_G, SO(4))$-modules to the category of $(\mathfrak{m}_G, O(4))$-modules, we now are in the position to determine (up to infinitesimal equivalence) the irreducible unitary $(\mathfrak{m}_G, O(4))$-modules with non-vanishing relative Lie algebra cohomology. Note that $\mathfrak{m}_G = \mathfrak{m}_{GL_4(\mathbb{R})}$

Proposition 1.1. (1) *The* $(\mathfrak{m}_G, O(4))$*-modules* $\mathrm{Ind}(A_{\mathfrak{q}_j})$, $j = 2, 3$ *and* $j = 5, 6$ *are irreducible, to be denoted by* $Y_{\mathfrak{q}_j}$. *One has* $Y_{\mathfrak{q}_2} \cong Y_{\mathfrak{q}_3}$ *and* $Y_{\mathfrak{q}_5} \cong Y_{\mathfrak{q}_6}$ *respectively.*

(2) *The* $(\mathfrak{m}_G, O(4))$*-module* $\mathrm{Ind}(A_{\mathfrak{q}_1})$ *splits into the two irreducible summands* $Y_{\mathfrak{q}_1} := \mathbb{C}$ *and* $Y^{det}_{\mathfrak{q}_1} := \mathbb{C}_{det}$.

(3) *The* $(\mathfrak{m}_G, O(4))$*-module* $\mathrm{Ind}(A_{\mathfrak{q}_4})$ *splits into two irreducible* $(\mathfrak{m}_G, O(4))$ *-modules, more precisely one has* $\mathrm{Ind}(A_{\mathfrak{q}_j}) = R^2_{\mathfrak{q}_4}(\mathbb{C}) \oplus R^2_{\mathfrak{q}_4}(\mathbb{C}_{det})$.

Proof. The first assertion follows from the fact that $L_j \cap O(4) = L_j \cap SO(4)$ for $j = 2, 3$ and $j = 5, 6$ and that $\mathfrak{q}_2$ is conjugate under $O(4)$ to $\mathfrak{q}_3$ respectively $\mathfrak{q}_5$ is conjugate under $O(4)$ to $\mathfrak{q}_6$. With regard to the other two assertions we observe that the $L_j \cap O(4)$-module Ind($\mathbb{C}$) obtained from the trivial $L_j \cap SO(4)$-module $\mathbb{C}$ splits into the two irreducibles $\mathbb{C}$ and $\mathbb{C}_{det}$ if $j = 1$ and 4. □

Having determined the unitary Harish-Chandra modules with non-zero cohomology for $GL_4(\mathbb{R})$, it is convenient to have their description in terms of the Langlands classification and in terms of the classification of unitary representations used in Chapter II. We summarize these results in Table 3. The first column is the notation used in the proposition above. The second column gives the data defining the Langlands quotient of the representation, i.e., the standard parabolic subgroup Q, a square-integrable representation δ of its Levi factor $L_Q(\mathbb{R})$, and an element $\nu \in \overline{\check{\mathfrak{a}}_Q^+}$ given in the basis consisting of the determinants on each general linear group appearing in the Levi factor L_Q. The third column lists the basic unitary representations, see Section 2.1 in Chapter II, appearing in the classification of the unitary representations. We also give in the last column the degrees in which the cohomology is non-zero. In all these degrees the cohomology space is isomorphic to $\mathbb{C}$.

Table 3 Unitary Harish-Chandra modules for $GL_4(\mathbb{R})$ with non-zero cohomology

Repn	Langlands data	Basic unit. repns	$H^q \neq 0$
$Y_{\mathfrak{q}_1}$	$(Q_0(\mathbb{R}), \mathbf{1} \otimes \ldots \otimes \mathbf{1}, (3/2, 1/2, -1/2, -3/2))$	$\mathbf{1} \circ \det_4$	$q = 0, 5$
$Y_{\mathfrak{q}_1}^{\det}$	$(Q_0(\mathbb{R}), \mathrm{sgn} \otimes \ldots \otimes \mathrm{sgn}, (3/2, 1/2, -1/2, -3/2))$	$\mathrm{sgn} \circ \det_4$	$q = 4, 9$
$Y_{\mathfrak{q}_2}$	$(Q_2(\mathbb{R}), D_3 \otimes D_3, (1/2, -1/2))$	$u(D_3, 2)$	$q = 3, 6$
$Y_{\mathfrak{q}_5}$	$(Q_2(\mathbb{R}), D_2 \otimes D_4, (0, 0))$	D_2, D_4	$q = 4, 5$
$R^2_{\mathfrak{q}_4}(\mathbb{C})$	$(Q_{\{\alpha_2\}}(\mathbb{R}), \mathbf{1} \otimes D_4 \otimes \mathbf{1}, (1/2, 0, -1/2))$	$\mathbf{1} \circ \det_2, D_4$	$q = 3, 4$
$R^2_{\mathfrak{q}_4}(\mathbb{C}_{\det})$	$(Q_{\{\alpha_2\}}(\mathbb{R}), \mathrm{sgn} \otimes D_4 \otimes \mathrm{sgn}, (1/2, 0, -1/2))$	$\mathrm{sgn} \circ \det_2, D_4$	$q = 5, 6$

1.3 The case $GL_2(\mathbb{H})$

The same general classification of unitary representations with non-zero cohomology can be applied to the real Lie group $G' = GL_2(\mathbb{H})$, and its maximal compact subgroup K'. It turns out that there are up to infinitesimal equivalence three such representations which we denote by $X_{\mathfrak{q}'_j}$, $j = 1, 2, 3$. We describe these representations in terms of the Langlands classification, and the classification of unitary representations given in Section 2.1 in Chapter II. Note that here the elements of $\check{\mathfrak{a}}_{Q'}$ are given in the basis consisting of the reduced norm on each copy of $\mathbb{H}^\times$. We also give the degrees in which the cohomology is non-vanishing. In those degrees the cohomology space is isomorphic to $\mathbb{C}$. As in the case $GL_4(\mathbb{R})$, we summarize the final result in Table 4, and leave the details to the reader.

Table 4 Unitary Harish-Chandra modules for $GL_2(\mathbb{H})$ with non-zero cohomology

Repn	Langlands data	Basic unit. repns	$H^q \neq 0$
$X_{\mathfrak{q}'_1}$	$(Q'(\mathbb{R}), \mathbf{1} \otimes \mathbf{1}, (1, -1))$	$\mathbf{1} \circ \mathrm{nrd}_2$	$q = 0, 5$
$X_{\mathfrak{q}'_2}$	$(Q'(\mathbb{R}), D'_3 \otimes D'_3, (1/2, -1/2))$	$u(D'_3, 2)$	$q = 1, 4$
$X_{\mathfrak{q}'_3}$	$(Q'(\mathbb{R}), \mathbf{1}_{\mathbb{H}^\times} \otimes D'_4, (0, 0))$	$\mathbf{1}_{\mathbb{H}^\times}, D'_4$	$q = 2, 3$

Comparing the description in terms of basic unitary representations for groups $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$, given in the third column of Table 3 and Table 4, it is clear that the representations with non-zero cohomology are related by the local Jacquet-Langlands correspondence described in Section 2.2 in Chapter II. More precisely, the first two representations in Table 3 correspond to $X_{\mathfrak{q}'_1}$, the third one correspond to $X_{\mathfrak{q}'_2}$, and the remaining three to $X_{\mathfrak{q}'_3}$. However, the degrees in which their cohomology is non-vanishing varies. Even if two representations of $GL_4(\mathbb{R})$ correspond to the same representation of $GL_2(\mathbb{H})$, the degrees in which the cohomology is non-vanishing are not the same.

2 The automorphic cohomology of the general linear group $G = Res_{k/\mathbb{Q}}GL_4/k$

Let $H = GL(4)/k$ be the general linear group defined over k. We consider in this section the automorphic cohomology of the $\mathbb{Q}$-group $G = Res_{k/\mathbb{Q}}H$ obtained by restriction of scalars. The standard parabolic $\mathbb{Q}$-subgroups of G are obtained from the standard parabolic k-subgroups of H by restriction of scalars. In particular, we denote by $P_i = Res_{k/\mathbb{Q}}Q_i$, for $i = 1, 2, 3$, the three maximal proper standard parabolic $\mathbb{Q}$-subgroups of G, by $P_0 = Res_{k/\mathbb{Q}}Q_0$ the minimal one, and by $P_{\{\alpha_i\}} = Res_{k/\mathbb{Q}}Q_{\{\alpha_i\}}$, for $i = 1, 2, 3$, the three intermediate ones. As for H, among maximal parabolic subgroups P_1 and P_3 are associate, while the intermediate ones $P_{\{\alpha_i\}}$ are all associate. Thus, there are four associate classes of proper parabolic $\mathbb{Q}$-subgroups of G, namely the class of minimal parabolic subgroups $\{P_0\}$, two classes of maximal ones $\{P_1\}$ and $\{P_2\}$, and one class of intermediate ones $\{P_{\{\alpha_1\}}\}$.

The automorphic cohomology $H^*(G, E)$ has a direct sum decomposition

$$H^*(G, E) = H^*_{\text{cusp}}(G, E) \oplus H^*_{\text{Eis}}(G, E)$$

where

$$H^*_{\text{Eis}}(G, E) := \bigoplus_{\{P\} \in \mathcal{C}, P \neq G} H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\}} \otimes_{\mathbb{C}} E)$$

is the Eisenstein cohomology of G with coefficients in E. The sum ranges over associate classes of proper parabolic $\mathbb{Q}$-subgroups of G. In this section, we discuss the internal structure of each of the corresponding summands in this decomposition of the automorphic cohomology.

2.1 Cuspidal cohomology

The cuspidal cohomology $H^*_{\text{cusp}}(G, E)$ decomposes as a direct sum as

$$H^*_{\text{cusp}}(G, E) = \bigoplus_{\phi \in \Phi_{E,\{G\}}} H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{G\},\phi} \otimes_{\mathbb{C}} E)$$

where the sum ranges over the set $\Phi_{E,\{G\}}$ of classes of associate irreducible cuspidal automorphic representations of G.

We define certain constants required to state a vanishing result for cuspidal cohomology of the general linear group. For a given real Lie group M with finitely many connected components and reductive Lie algebra $2q(M) = \dim M - \dim K_M$ where K_M is a maximal compact subgroup of M. Set $\ell_0(M) := \mathrm{rk}(M) - \mathrm{rk}(K_M)$, where rk denotes the absolute rank, and write $q_0(M) := \frac{1}{2}(2q(M) - \ell_0(M))$. In the case of the real Lie group $GL_n(\mathbb{R})$ these values can be made explicit; we refer to [27, Section 3.5]. The interval $[q_0(M), q_0(M) + \ell_0(M)]$ is centered around the middle dimension of the symmetric space associated to the Lie group M. For the group $H(\mathbb{R}) \cong GL_4(\mathbb{R})$ of interest to us, one obtains the interval $[4, 5]$.

Theorem 2.1. *Let $k/\mathbb{Q}$ be an algebraic number field, and let $G/\mathbb{Q}$ be the algebraic $\mathbb{Q}$-group $Res_{k/\mathbb{Q}}GL(n)$ obtained from the general linear group $GL(n)$ defined over k by restriction of scalars. Let (ν, E) be an irreducible finite dimensional algebraic representation of $G(\mathbb{C})$. Then*

$$H^j_{\mathrm{cusp}}(G, E) = 0 \text{ if } j \notin [q_0(G(\mathbb{R})), q_0(G(\mathbb{R})) + \ell_0(G(\mathbb{R}))] \cap \mathbb{Z}.$$

In particular, if k is totally real, and $G = Res_{k/\mathbb{Q}}H$ with $H = GL(4)/k$, then

$$H^j_{\mathrm{cusp}}(G, E) = 0 \text{ if } j \notin [4[k : \mathbb{Q}], 5[k : \mathbb{Q}]] \cap \mathbb{Z},$$

where $[k : \mathbb{Q}]$ is the degree of extension $k/\mathbb{Q}$.

Proof. We consider a summand $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{G\},\phi}) \otimes_\mathbb{C} E$ corresponding to the associate class ϕ of a cuspidal automorphic representation π of $G(\mathbb{A})$. We may view π as a cuspidal automorphic representation of $GL_n(\mathbb{A}_k)$. By the Künneth rule, since $G(\mathbb{R}) \cong \prod_{v \in V_\infty} GL_n(k_v)$ with $k_v = \mathbb{R}$ or $k_v = \mathbb{C}$, the corresponding representation π_v must have non-trivial cohomology for all $v \in V_\infty$.

On the other hand, π is cuspidal automorphic representation of $GL_n(\mathbb{A}_k)$, and hence generic. Thus, the local components π_v are generic as well. As they are also unitary, comparing the classification of generic unitary dual for $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ of Vogan [37] to the classification of cohomological unitary representations as in [39], it is not too difficult to see that π_v is necessarily tempered. For a detailed discussion of this fact see [27, Section 3.5] or [29, Chapter 6]. The degrees where these specific tempered representations have non-vanishing cohomology give the required bounds. If k is totally real, and $G = Res_{k/\mathbb{Q}}H$ with $H = GL(4)/k$, the vanishing result follows by the Künneth rule from the vanishing outside $[4, 5] \cap \mathbb{Z}$ for $GL_4(\mathbb{R})$ mentioned before the statement of the Theorem. □

By the construction of cuspidal cohomology classes for congruence subgroups of GL_n/k with respect to suitable coefficient systems (ν, E) as pursued in [20] this bound $q_0(G(\mathbb{R}))$ is sharp (at least if we vary the choice of the base field k).

In the case of the trivial representation $E = \mathbb{C}$, there is up to equivalence exactly one unitary representation of $GL_n(\mathbb{R})$ which is generic and has non-trivial cohomology with respect to $E = \mathbb{C}$. It is tempered, and given as the fully induced representation from a square-integrable representation of the Levi factor L_{Q_ρ} of the parabolic subgroup Q_ρ with $\rho = (r_1, \ldots, r_m)$, where $m = \left[\frac{n}{2}\right]$ if n is even, and $m = \left[\frac{n}{2}\right] + 1$ if n is odd, and $r_i = 2$ for $i = 1, \ldots, \left[\frac{n}{2}\right]$, and if n is odd $r_m = 1$.

The square-integrable representation of L_{Q_ρ} is the tensor product of discrete series representations D_{n-2i+2} of $GL_2(\mathbb{R})$ of lowest $O(2)$-type $n-2i+2$ for $i = 1, \dots, \left[\frac{n}{2}\right]$ (see Section 2.2 in Chapter II), and if n is odd a trivial character of $\mathbb{R}^\times$.

In particular, if $n = 4$, we obtain that this representation is fully induced from the tensor product $D_4 \otimes D_2$, where D_i is the discrete series representation of lowest $O(2)$-type i (see Section 2.2 in Chapter II), of the Levi factor $L_{Q_2}(\mathbb{R}) \cong GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$.

2.2 The summands $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\}} \otimes_\mathbb{C} E)$, P maximal parabolic

Let $\{P\}$ be one of the two associate classes $\{P_1\}$ and $\{P_2\}$ of maximal parabolic $\mathbb{Q}$-subgroups of the reductive $\mathbb{Q}$-group G. As explained in Section 3.2 in Chapter I, given $\phi \in \Phi_{E,\{P\}}$, there is a natural two step filtration

$$\mathcal{L}_{E,\{P\},\phi} \subset \mathcal{A}_{E,\{P\},\phi},$$

of the space $\mathcal{A}_{E,\{P\},\phi}$ of automorphic forms, where $\mathcal{L}_{E,\{P\},\phi}$ is the subspace of $\mathcal{A}_{E,\{P\},\phi}$ consisting of square integrable automorphic forms. The space $\mathcal{L}_{E,\{P\},\phi}$ is spanned by the residues at $s > 0$ of the Eisenstein series attached to functions $f \in W_\pi$, where $\pi \cong \sigma \otimes \sigma' \in \phi_P$ with σ and σ' cuspidal automorphic representations of the general linear groups appearing in the Levi factor L_P of P (see Section 3.3 in Chapter I). Those residues are square-integrable automorphic forms [24, Section I.4.11]. The quotient $\mathcal{A}_{E,\{P\},\phi}/\mathcal{L}_{E,\{P\},\phi}$ is in both cases spanned by the principal value of the derivatives of such Eisenstein series at $Re(s) \geq 0$.

By the description of the residual spectrum of $H(\mathbb{A}_k) \cong GL_4(\mathbb{A}_k)$ (cf. [23], recalled in Section 3.3 in Chapter II), which is the same as $G(\mathbb{A})$, the space $\mathcal{L}_{E,\{P_1\},\phi} = (0)$ for every associate class ϕ. In the other case, $\mathcal{L}_{E,\{P_2\},\phi} \neq (0)$ if and only if ϕ is the associate class of a cuspidal representation $\pi \cong \sigma \otimes \sigma$, i.e., $\sigma' \cong \sigma$. Then we have the following result in our case of interest which is the main theorem obtained in the case $Res_{k/\mathbb{Q}} GL_n$ in [10, Section 5.6].

Theorem 2.2. *Let $\{P\}$ be an associate class of maximal parabolic $\mathbb{Q}$-subgroups of the group G, and let $\phi \in \Phi_{E,\{P\}}$ be an associate class of irreducible cuspidal automorphic representations of the Levi components of elements in $\{P\}$.*

(1) *If $\{P\} = \{P_1\}$, that is, the elements in $\{P\}$ are not conjugate to their opposite P^-, then*

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_1\}} \otimes_\mathbb{C} E) = (0).$$

Consequently, $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P_1\}} \otimes_\mathbb{C} E) = (0)$, and thus the whole space $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P_1\}} \otimes_\mathbb{C} E)$ is generated by so called regular Eisenstein cohomology classes.

(2) *If $\{P\} = \{P_2\}$, that is, the elements in $\{P\}$ are conjugate to their opposite P^-, then*

  (a) *if ϕ is the associate class of $\pi \cong \sigma \otimes \sigma'$ with $\sigma \ncong \sigma'$, then*

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_2\},\phi} \otimes_\mathbb{C} E) = (0).$$

Consequently, $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P_2\},\phi} \otimes_{\mathbb{C}} E) = (0)$*, and thus the space* $H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P_1\},\phi} \otimes_{\mathbb{C}} E)$ *is generated by so called regular Eisenstein cohomology classes.*

(b) *Otherwise, i.e., in the case* $\pi \cong \sigma \otimes \sigma$*, the space*

$$H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_2\},\phi} \otimes_{\mathbb{C}} E)$$

is non-trivial. It consists of square integrable cohomology classes represented by residues of suitable Eisenstein series attached to π*. In particular, if* $k/\mathbb{Q}$ *is totally real of degree* $[k : \mathbb{Q}]$*, then*

$$H^q\,(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_2\},\phi} \otimes_{\mathbb{C}} E) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \begin{array}{l} \text{if } q \in \big[3[k:\mathbb{Q}], 6[k:\mathbb{Q}]\big] \cap (3\mathbb{Z}) \text{ and} \\ \Pi_v \cong J_v(P_2(\mathbb{R}), D_3 \otimes D_3, (1/2, 1/2)) \text{ for } v \in V_\infty, \end{array} \\ 0, & \text{otherwise,} \end{cases}$$

where Π_v *is the local component at the place* v *of* $\mathcal{L}_{E,\{P_2\},\phi}$*, and* J_v *stands for the Langlands quotient.*

Observe that in the case ($2b$) of the Theorem, the square-integrable cohomology space

$$H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P_2\},\phi} \otimes_{\mathbb{C}} E)$$

is not given. It is still an open problem to determine this space. However, it is proved in [10] that the square-integrable cohomology classes are separated from the regular ones by the degree in which they may occur.

2.3 The summand $H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P_0\}} \otimes_{\mathbb{C}} E)$

Let $\{P_0\}$ be the associate class of the fixed minimal parabolic $\mathbb{Q}$-subgroup P_0 of the reductive $\mathbb{Q}$-group $Res_{k/\mathbb{Q}} GL(n)$ obtained from the k-group $GL(n)/k$ by the restriction of scalars. Let $\phi \in \Phi_{E,\{P_0\}}$ be an associate class of cuspidal automorphic representations of $L_0(\mathbb{A})$. By the description of the residual spectrum of $GL_n(\mathbb{A}_k)$ by Mœglin and Waldspurger [23], the space of square-integrable automorphic forms $\mathcal{L}_{E,\{P_0\},\phi}$ inside $\mathcal{A}_{E,\{P_0\},\phi}$ is trivial unless ϕ_{P_0} contains a character of $L_0(\mathbb{A})$ of the form $\chi \otimes \chi \otimes \cdots \otimes \chi$, where χ is a unitary character of $k^\times \backslash \mathbb{I}_k$. If this necessary condition is satisfied, then $\mathcal{L}_{E,\{P_0\},\phi}$ is one-dimensional and isomorphic to $\chi \circ \det$.

In any case, by Franke's filtration [8, Theorem 14, Section 6], the quotient $\mathcal{A}_{E,\{P_0\},\phi}/\mathcal{L}_{E,\{P_0\},\phi}$ is spanned by principal values of the derivatives of all the Eisenstein series, attached to either residual automorphic representation, supported in $\chi \otimes \chi \otimes \cdots \otimes \chi$, of the Levi factor of a proper parabolic subgroup which is not minimal, or a cuspidal automorphic representation $\chi \otimes \chi \otimes \cdots \otimes \chi$ of $L_0(\mathbb{A})$. Then, in the case of interest of this paper, we have the following result in cohomology. The proof follows directly from the above discussion, except for the degrees of non-vanishing cohomology which is a consequence of the Künneth rule and Section 1.2.

Theorem 2.3. *Let $\{P_0\}$ be the associate class of the minimal parabolic $\mathbb{Q}$-subgroup P_0 of the group G, and let $\phi \in \Phi_{E,\{P_0\}}$ be an associate class of irreducible cuspidal automorphic representations of the Levi components of elements in $\{P_0\}$. Let $\chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4 \in \phi_{P_0}$ be a unitary character of $L_0(\mathbb{A})$, where χ_i is a unitary character of $k^\times \backslash \mathbb{I}_k$ for $i = 1, 2, 3, 4$.*

(1) *If there are $i \neq j$ such that $\chi_i \neq \chi_j$, then*

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_0\},\phi} \otimes_\mathbb{C} E) = (0).$$

Consequently, $H^*_{(sq)}(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P_0\},\phi} \otimes_\mathbb{C} E) = (0)$.

(2) *Otherwise, i.e., in the case $\chi_1 = \chi_2 = \chi_3 = \chi_4$, we denote this character by χ. Then, the space $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_0\},\phi} \otimes_\mathbb{C} E)$ is non-trivial. It consists of square integrable cohomology classes represented by residues of suitable Eisenstein series. In particular, if $k/\mathbb{Q}$ is totally real of degree $[k : \mathbb{Q}]$, and t_0 the number of places $v \in V_\infty$ such that $\chi_v = \mathrm{sgn}$, then*

$$H^q(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_0\},\phi} \otimes_\mathbb{C} E) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \text{if } q \in \big[4t_0, 5[k:\mathbb{Q}] + 4t_0\big] \cap (5\mathbb{Z} + 4t_0) \text{ and } \\ & \chi_v = \mathbf{1} \text{ or } \chi_v = \mathrm{sgn} \text{ at all places } v \in V_\infty, \\ 0, & \text{otherwise.} \end{cases}$$

The problem of determining the image $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_0\},\phi} \otimes_\mathbb{C} E)$ was studied for the minimal parabolic subgroup in the case $SL(n)/\mathbb{Q}$ by Franke in [9].

2.4 The summand $H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P_{\{\alpha_1\}}\}} \otimes_\mathbb{C} E)$

Let $\{P_{\{\alpha_1\}}\}$ be the associate class of intermediate parabolic $\mathbb{Q}$-subgroups of the reductive $\mathbb{Q}$-group G, i.e., the class consisting of proper parabolic $\mathbb{Q}$-subgroups which are neither minimal, nor maximal. By Section 1.1 in Chapter II, there is only one such associate class. Again, the results of Mœglin and Waldspurger [23], show that there are no square-integrable automorphic forms in the space $\mathcal{A}_{E,\{P_{\{\alpha_1\}}\}}$. Thus in this case the space $\mathcal{L}_{E,\{P_{\{\alpha_1\}}\},\phi}$ of square-integrable automorphic forms inside $\mathcal{A}_{E,\{P_{\{\alpha_1\}}\},\phi}$ is trivial for any associate class $\phi \in \Phi_{E,\{P_{\{\alpha_1\}}\}}$. Thus, the same holds in cohomology, i.e.,

$$H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{L}_{E,\{P_{\{\alpha_1\}}\}} \otimes_\mathbb{C} E) = (0).$$

Consequently, the space $H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P_{\{\alpha_1\}}\}} \otimes_\mathbb{C} E) = (0)$.

By Franke [8, Theorem 14, Section 6], the space $\mathcal{A}_{E,\{P_{\{\alpha_1\}}\}}$ itself is spanned by the principal values of the derivatives of all the Eisenstein series attached to residual automorphic representations, supported in $\{P_{\{\alpha_1\}}\}$, of the Levi factor $L_{P_2}(\mathbb{A})$ of the parabolic subgroup P_2, and the cuspidal Eisenstein series supported in $\{P_{\{\alpha_1\}}\}$. More precisely, if ϕ is the associate class of a cuspidal automorphic representation $\pi \cong \sigma \otimes \chi_1 \otimes \chi_2$ of $L_{P_{\{\alpha_1\}}}(\mathbb{A})$, with $\chi_1 = \chi_2$ denoted by χ, then the Franke filtration of $\mathcal{A}_{E,\{P_{\{\alpha_1\}}\},\phi}$ is a two-step filtration

$$\mathcal{A}'_{E,\{P_{\{\alpha_1\}}\},\phi} \subset \mathcal{A}_{E,\{P_{\{\alpha_1\}}\},\phi},$$

where $\mathcal{A}'_{E,\{P_{\{\alpha_1\}}\},\phi}$ is spanned by the holomorphic values of all the derivatives of the Eisenstein series attached to the representation $\sigma \otimes \chi \circ \det_2$ of $L_{P_2}(\mathbb{A})$ at an appropriate evaluation point. The quotient is spanned by main values of the derivatives of the Eisenstein series attached to π itself at an appropriate evaluation point. Otherwise, if $\chi_1 \neq \chi_2$, then $\mathcal{A}'_{E,\{P_{\{\alpha_1\}}\},\phi} = (0)$.

3 The automorphic cohomology of $G' = Res_{k/\mathbb{Q}} GL(2, D)$

We consider now the automorphic cohomology of the $\mathbb{Q}$-group G' which is obtained from the k-group $H' = GL(2, D)$ by restriction of scalars, where D is a quaternion division algebra central over k. Since there is a unique conjugacy (and associate) class of proper $\mathbb{Q}$-parabolic subgroups of G', given by $P' = Res_{k/\mathbb{Q}} Q'$, the decomposition of the automorphic cohomology $H^*(G', E)$ with respect to associate classes of parabolic subgroups consists of two summands

$$H^*(G', E) = H^*_{\mathrm{cusp}}(G', E) \bigoplus H^*_{\mathrm{Eis}}(G', E),$$

where

$$H^*_{\mathrm{Eis}}(G', E) = H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{A}_{E,\{P'\}} \otimes_{\mathbb{C}} E).$$

We are primarily interested into square-integrable cohomology, namely

$$H^*_{(\mathrm{sq})}(G', E) = H^*_{\mathrm{cusp}}(G', E) \bigoplus H^*_{\mathrm{Eis},(\mathrm{sq})}(G', E)$$

We study each summand separately using the global Jacquet-Langlands correspondence (see Chapter II, Section 3) to make a comparison with the case of $G = Res_{k/\mathbb{Q}} H$, where $H = GL(4)/k$, considered in Section 2.

3.1 Global Jacquet-Langlands correspondence in cohomology

The injective map Ξ of the global Jacquet-Langlands correspondence between $H'(\mathbb{A}_k)$ and $H(\mathbb{A}_k)$, defined in Section 3.2 in Chapter II, is in fact also an injective map from the discrete spectrum representations of $G'(\mathbb{A})$ into the discrete spectrum representations of $G(\mathbb{A})$. This map gives rise to a map, also denoted by Ξ, between the cohomology spaces

$$\Xi : H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{R'\},\phi'} \otimes_{\mathbb{C}} E) \to H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \Xi(\mathcal{L}_{E,\{R'\},\phi'}) \otimes_{\mathbb{C}} E),$$

where either $R' = P'$ or $R' = G'$, and $\Xi(\mathcal{L}_{E,\{R'\},\phi'})$ is defined as follows. Let $\pi' \in \phi'_{R'}$ be a cuspidal automorphic representation of the Levi factor $L_{R'}(\mathbb{A})$. Observe that we allow here $R' = G'$, and thus $L_{R'} = G'$. Then, by the global Jacquet-Langlands correspondence, there is a discrete spectrum representation $\Xi(\pi')$ of the corresponding Levi factor in G. Hence, $\Xi(\pi')$ belongs to $\mathcal{L}_{E,\{P\},\phi}$ for uniquely determined parabolic $\mathbb{Q}$-subgroup P of G, and an associate class ϕ of a cuspidal automorphic representation of its Levi factor which is the support of $\Xi(\pi')$. Then we define

$$\Xi(\mathcal{L}_{E,\{R'\},\phi'}) = \mathcal{L}_{E,\{P\},\phi}.$$

In accordance to the terminology regarding the global Jacquet-Langlands correspondence, introduced in Section 3.2 in Chapter II, we say that a cohomology space

$$H^*\left(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{L}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E\right)$$

for G is D-compatible if it is among the spaces in the range of Ξ. This is equivalent to D-compatibility of $\mathcal{L}_{E,\{P\},\phi}$. The properties of the map Ξ in cohomology are given in the following theorem.

Theorem 3.1. *In the notation as above, the cohomology space*

$$H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{R'\},\phi'} \otimes_{\mathbb{C}} E)$$

is non-trivial if and only if

$$H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \Xi(\mathcal{L}_{E,\{R'\},\phi'}) \otimes_{\mathbb{C}} E)$$

is non-trivial.

The proof follows directly from the description of the local Jacquet-Langlands correspondence at a real place in Section 2.2 in Chapter II. Namely, it shows that the representations with non-zero cohomology of $GL_4(\mathbb{R})$ correspond to such representations for $GL_2(\mathbb{H})$. In what follows, we refine the map Ξ by the degrees of cohomology. However, this will be done considering case by case below.

Although we have now defined a map Ξ between the cohomology of the spaces of square-integrable automorphic forms, it is not clear how is this related to the summands in the decomposition along the cuspidal support of the square-integrable automorphic cohomology. More precisely, we would like to understand the following diagram

$$\begin{array}{ccc} H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{R'\},\phi'} \otimes_{\mathbb{C}} E) & \xrightarrow{\Xi} & H^*(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{L}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E) \\ \downarrow & & \downarrow \\ H^*_{(\mathrm{sq})}(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{A}_{E,\{R'\},\phi'} \otimes_{\mathbb{C}} E) & \xrightarrow{\Xi_{(\mathrm{sq})}} & H^*_{(\mathrm{sq})}(\mathfrak{m}_G, K_{\mathbb{R}}; \mathcal{A}_{E,\{P\},\phi} \otimes_{\mathbb{C}} E) \end{array} \tag{3.1}$$

where $\mathcal{L}_{E,\{P\},\phi} = \Xi(\mathcal{L}_{E,\{R'\},\phi'})$, the vertical maps are induced from the inclusion of the space of square-integrable ones into the space of automorphic forms. The map $\Xi_{(\mathrm{sq})}$ is to be considered case by case below.

3.2 Cuspidal cohomology

The cuspidal cohomology decomposes into

$$H^*_{\mathrm{cusp}}(G', E) = \bigoplus_{\phi' \in \Phi_{E,\{G'\}}} H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{G'\},\phi'} \otimes_{\mathbb{C}} E),$$

where the sum ranges over all associate classes of cuspidal automorphic representations of $G'(\mathbb{A})$ which are represented by a representation $\pi' \in \phi_{G'}$ with non-trivial cohomology with respect to E. Hence, any cuspidal automorphic representation π' of $G'(\mathbb{A})$ whose infinite component is cohomological has non-trivial automorphic cohomology in the same degrees as its infinite component as a Harish-Chandra module. We have thus the following result.

Theorem 3.2. *Assume k is a totally real number field of degree $[k:\mathbb{Q}]$. Let t denote the number of archimedean places of k at which D does not split. Then, $H^q_{\mathrm{cusp}}(G',E)$ vanishes if*

$$q \notin \Big(\big[4[k:\mathbb{Q}]-2t, 5[k:\mathbb{Q}]-2t\big] \cap \mathbb{Z}\Big) \cup \Big(\big[3[k:\mathbb{Q}]-2t, 6[k:\mathbb{Q}]-2t\big] \cap (3\mathbb{Z}+t)\Big).$$

In particular, if $k=\mathbb{Q}$ and D is non-split at the real place of $\mathbb{Q}$, then $H^q_{\mathrm{cusp}}(G',E)$ vanishes in the degrees $q=0$ and $q \geq 5$.

More precisely, let ϕ' be the associate class of a cuspidal automorphic representation π' of $G'(\mathbb{A})$. If π' is such that $\Xi(\pi')$ is cuspidal (see Section 3.4 in Chapter II), then

$$H^q\left(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{G'\},\phi'} \otimes_{\mathbb{C}} E\right) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \begin{array}{l} \textit{if } q \in \big[4[k:\mathbb{Q}]-2t, 5[k:\mathbb{Q}]-2t\big] \cap \mathbb{Z} \textit{ and} \\ \pi'_v \cong \begin{cases} Ind_{P_2(\mathbb{R})}^{GL_4(\mathbb{R})}(D_2 \otimes D_4), & v \in V_\infty \setminus V_D, \\ Ind_{P'(\mathbb{H})}^{GL_2(k_v)(\mathbb{H})}(\mathbf{1}_{\mathbb{H}^\times} \otimes D'_4), & v \in V_\infty \cap V_D. \end{cases} \end{array} \\ 0, & \textit{otherwise.} \end{cases}$$

If π' is such that $\Xi(\pi')$ is residual (see Section 3.4 in Chapter II), then

$$H^q\left(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{G'\},\phi'} \otimes_{\mathbb{C}} E\right) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \begin{array}{l} \textit{if } q \in \big[3[k:\mathbb{Q}]-2t, 6[k:\mathbb{Q}]-2t\big] \cap (3\mathbb{Z}+t) \textit{ and} \\ \pi'_v \cong \begin{cases} J_v(P_2(\mathbb{R}), D_3 \otimes D_3, (1/2,1/2)), & v \in V_\infty \setminus V_D, \\ J'_v(P'(\mathbb{H}), D'_3 \otimes D'_3, (1/2,-1/2)), & v \in V_\infty \cap V_D. \end{cases} \end{array} \\ 0, & \textit{otherwise,} \end{cases}$$

where J_v and J'_v denote the Langlands quotients in $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$, respectively.

Proof. It suffices to show the second claim of the theorem, since it implies the vanishing result for the cuspidal cohomology $H^*_{\mathrm{cusp}}(G',E)$. Viewing π' as a cuspidal automorphic representation of $H'(\mathbb{A}_k)$, and applying the Künneth rule, the theorem reduces to the consideration of local components at archimedean places. These were studied in Section 1.3. □

Note that the non-vanishing of $H^q_{\mathrm{cusp}}(G',E)$ in the degrees where the theorem does not give a vanishing result depends only on the existence of cuspidal automorphic π' with the required archimedean components. In the case where

$\Xi(\pi')$ is residual, the non-vanishing reduces to the existence of a cuspidal automorphic representation of $GL_2(\mathbb{A}_k)$ with given square-integrable representations at archimedean places, and at least one non-split non-archimedean place where the local component is not square-integrable. In the other case where $\Xi(\pi')$ is cuspidal, the non-vanishing is equivalent to the existence of a cuspidal automorphic representation of $GL_4(\mathbb{A}_k)$ with the given tempered representation at all archimedean places, and the local components at non-split non-archimedean places as in part (1) of Proposition 3.5 in Chapter II.

We consider finally the diagram (3.1). If $\Xi(\pi')$ is cuspidal, both vertical maps are isomorphisms because the target space is a part of cuspidal cohomology. Thus, in this case $\Xi_{(\mathrm{sq})}$ can be identified with Ξ. In the other case, i.e., $\Xi(\pi')$ is residual, only the left hand side vertical map is an isomorphism. Thus, $\Xi_{(\mathrm{sq})}$ is not determined by Ξ, although Ξ gives the possible range of $\Xi_{(\mathrm{sq})}$. Finally note that when applying Ξ there is a shift in the degrees in which the cohomology space is non-zero. In the case of a totally real number field k this shift equals $2t$, where t is the number of non-split places of D.

3.3 Eisenstein cohomology

In the case of Eisenstein cohomology, the space of square-integrable automorphic forms $\mathcal{L}_{E,\{P'\},\phi'}$ is a proper subspace of $\mathcal{A}_{E,\{P'\},\phi'}$. Hence, the cohomology space

$$H^*(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{P'\},\phi'} \otimes_{\mathbb{C}} E)$$

just gives possible non-trivial classes in the corresponding square-integrable cohomology space

$$H^*_{(\mathrm{sq})}(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{A}_{E,\{P'\},\phi'} \otimes_{\mathbb{C}} E).$$

The problem of determining which classes are indeed non-trivial in the latter space is subtle and out of the scope of our consideration in this paper. However, the former space can be described via the global Jacquet-Langlands correspondence.

Theorem 3.3. *Assume k is a totally real number field of degree $[k:\mathbb{Q}]$. Let t denote the number of archimedean places of k at which D does not split. Let ϕ' be the associate class of a cuspidal automorphic representation $\pi'_1 \otimes \pi'_2$ of the Levi factor $L'(\mathbb{A})$ of P'. If $\pi'_1 \neq \pi'_2$, then $H^*_{\mathrm{Eis},(\mathrm{sq})}(G', E)$ is trivial.*

Suppose that $\pi'_1 = \pi'_2 = \pi'$. If π' is not one-dimensional, then

$$H^q\,(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{P'\},\phi'} \otimes_{\mathbb{C}} E) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \begin{array}{l} \text{if } q \in \big[3[k:\mathbb{Q}]-2t, 6[k:\mathbb{Q}]-2t\big] \cap (3\mathbb{Z}+t) \text{ and} \\ \Pi'_v \cong \begin{cases} J_v(P_2(\mathbb{R}), D_3 \otimes D_3, (1/2,1/2)), & v \in V_\infty \setminus V_D, \\ J'_v(P'(\mathbb{H}), D'_3 \otimes D'_3, (1/2,-1/2)), & v \in V_\infty \cap V_D. \end{cases} \end{array} \\ 0, & \text{otherwise,} \end{cases}$$

where Π'_v is the local component at the place v of $\mathcal{L}_{E,\{P'\},\phi'}$, and J_v and J'_v denote the Langlands quotient in $GL_4(\mathbb{R})$ and $GL_2(\mathbb{H})$, respectively. If $\pi' \cong \chi \circ \mathrm{nrd}$, where

χ is a unitary character of $k^\times\backslash\mathbb{I}_k$, let t_0' denote the number of split archimedean places where $\chi_v = sgn$. Then

$$H^q\left(\mathfrak{m}_{G'}, K'_{\mathbb{R}}; \mathcal{L}_{E,\{P'\},\phi'} \otimes_{\mathbb{C}} E\right) = \begin{cases} \mathbb{C}^{[k:\mathbb{Q}]}, & \text{if } q \in \left[4t_0', 5[k:\mathbb{Q}] + 4t_0'\right] \cap (5\mathbb{Z} + 4t_0') \text{ and} \\ & \chi_v = \mathbf{1} \text{ or } \chi_v = sgn \text{ at all places } v \in V_\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Theorem 3.2, the Künneth rule reduces the proof to local considerations of Section 1.2 and Section 1.3. □

For the Eisenstein cohomology spaces the diagram (3.1) is out of reach of the methods of this paper. The vertical arrows both may not be isomorphisms. Thus, the relation between Ξ and $\Xi_{(\mathrm{sq})}$ is not clear at all. The shift in the degrees when applying Ξ for the case of totally real number field depends on whether the cuspidal support π' is one-dimensional or not. If not the shift is again $2t$, where t is the number of non-split archimedean places of D. If $\pi' = \chi \circ \mathrm{nrd}$ is one-dimensional, the shift in degrees is $4(t_0 - t_0')$, where $t_0 - t_0'$ equals the number of archimedean places at which D does not split and $\chi_v = \mathrm{sgn}$.

We remark at the end that one could try to follow the original approach of Langlands to determine the spaces of residues of the Eisenstein series for $G'(\mathbb{A})$, instead of using the Jacquet-Langlands correspondence of Badulescu resp. Badulescu and Renard which relies on the trace formula. The difficulty in applying that approach is in the fact that the Langlands-Shahidi method for normalization of intertwining operators (cf. [31]) is not available for groups which are not quasi-split. To overcome this difficulty one should find a way to compare the normalizing factors between the inner and split form of the group. This can not be done in general. Nevertheless, this approach was used in the thesis [19], as well as in the Appendix of [1] where a substantial ingredient was already established by the trace formula in the body of the paper. For inner forms of some split classical groups the same approach is pursued in [12], [13], [14], [15].

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