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ENDOSCOPIC TRANSFER AND AUTOMORPHIC L-FUNCTIONS: THE CASE OF THE GENERAL SPIN GROUP AND THE TWISTED SYMMETRIC AND EXTERIOR SQUARE L-FUNCTIONS

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Dedicated to Marko Tadić on the occasion of his 70th birthday

Abstract. The endoscopic classification and the Langlands spectral theory are two approaches to the discrete spectrum of the group of adèlic points of a reductive linear algebraic group defined over a number field. The two points of view on the same object yield interesting consequences. In this paper, the case of the general spin group is considered. In that case, it is shown how the comparison of the two approaches implies that the twisted symmetric and exterior square complete automorphic L-functions associated to a cuspidal automorphic representation of the general linear group are holomorphic in the critical strip.

1. Introduction

It happens quite often in mathematics that different points of view on the same object yield the most fruitful ideas and new insights. Examples of this phenomenon pop out in each and every part of mathematics. It is very common that this kind of ideas are considered to be mathematical jewels – the most beautiful parts of a mathematical theory. We begin this paper with an incomplete and very personal choice of evidence for this claim.

The representations of groups on vector spaces make a perfect basic example. The group is a non-linear object, often difficult to grasp. Its representation on a vector space realizes the elements of the group as linear operators, while at the same time resembling the group structure and its symmetries. However, the full power of this approach lies in the study and comparison of as many as possible different representations of the given group. Different

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representations may be viewed as different points of view on the same group, thus providing the first evidence for our opening claim.

This paper is dedicated to Marko Tadić. In his work, the above phenomenon can be seen, for instance, in his applications of Jacquet modules in the study of parabolically induced representations of p-adic groups, see [36], [42], [43], [44], [45], [46], among other papers. The idea is to apply as many Jacquet modules as possible to a given parabolically induced representation, in order to get results about its structure and reducibility through a comparison of different Jacquet modules. In a sense, different Jacquet modules provide different points of view on the same induced representation. This idea of Tadić bears the name *Tadić philosophy*, coined by Corinne Blondel in [8].

Another prominent example, within the scope of our own research interests, is the study of cohomology of arithmetic groups. Let Γ be an arithmetic subgroup of a connected semisimple linear algebraic group G defined over the field $\mathbb{Q}$ of rational numbers. Then Γ can be viewed as a discrete subgroup of the group $G(\mathbb{R})$ of real points of G. Assume that Γ is a torsion free congruence subgroup. The cohomology of Γ can be expressed in two substantially different ways.

The first point of view on cohomology of Γ is geometric. Viewing Γ as a discrete subgroup of the real Lie group $G(\mathbb{R})$, its cohomology can be expressed in terms of differential forms as the de Rham cohomology of the locally symmetric space $\Gamma\backslash G(\mathbb{R})/K_{\mathbb{R}}$, where $K_{\mathbb{R}}$ is a maximal compact subgroup of $G(\mathbb{R})$. We refer to [11] for more details. The other point of view on the cohomology of Γ is from the arithmetic perspective of the theory of automorphic forms. According to the regularization theorems of Borel [9] and the result of Franke [12], it can be expressed as the relative Lie algebra cohomology of the space of automorphic forms on $G(\mathbb{R})$ with respect to Γ.

The two complementary points of view on the cohomology of arithmetic groups allow the flow of ideas in both directions. For example, geometric constructions of non-trivial cohomology classes imply the existence of automorphic forms with certain properties. On the other hand, the understanding of the structure of spaces of automorphic forms provides an approach to explicit calculation of cohomology. The problem is that the structure of spaces of automorphic forms depends on deep arithmetic conditions given in terms of analytic properties of Eisenstein series and automorphic L-functions. However, the combination of the geometric and arithmetic point of view often yields best results. The geometric necessary non-vanishing conditions in cohomology reduce the problem to the understanding of the structure of spaces of automorphic forms with quite regular cuspidal support, which turns out to be feasible in many cases. This line of research is followed in the collaboration of the author with Joachim Schwermer [17], [18], [19], [20], [21], and with Harald Grobner [16]. See [15] for a survey of this work.

We are now slowly moving towards the scope of the present paper. The Arthur trace formula [1], [3] is another superb example of the phenomenon mentioned in the opening paragraph. It is the formula that combines the geometric and spectral expressions of the same object – the trace of the right regular representation on the relevant space of automorphic forms on a reductive group. The applications of the trace formula compare not only the two sides of the formula, but also the trace formulas for different groups and certain other objects related to them. All these points of view on the trace, result in the theory of endoscopy, which provides constructions of functorial transfers of automorphic representations between different groups, as predicted by the Langlands program. The endoscopic classification of automorphic representations of classical groups is obtained in that way [4], [35], [51], subject to certain technical issues related to the trace formula. The endoscopic transfer mentioned in the title is the functorial transfer of automorphic representations of the general spin group to the automorphic representations of the general linear group, formulated without proof in terms of Arthur parameters in [2].

The last evidence for the opening claim of this paper, at least in our incomplete list, arises from the comparison of the endoscopic classification and the Langlands theory of spectral decomposition of spaces of automorphic forms on reductive groups [31], [34]. Given a reductive linear algebraic group G defined over a number field, let $L^2_{\mathrm{disc}}(G,\chi)$ denote the discrete part of the L^2 space of square-integrable automorphic forms on the group $G(\mathbb{A})$ of adèlic points of G with central character χ. For precise definition of these notions see [10]. There are two different points of view on the space $L^2_{\mathrm{disc}}(G,\chi)$.

The first point of view is the endoscopic classification, relying on the Arthur trace formula, which yields the description of $L^2_{\mathrm{disc}}(G,\chi)$ in terms of Arthur parameters, which can be reformulated in terms of the discrete spectrum of the general linear groups. The second point of view is the Langlands spectral theory, which describes the constituents of $L^2_{\mathrm{disc}}(G,\chi)$ in terms of the analytic properties of Eisenstein series and the automorphic L-functions appearing in their constant terms via the Langlands–Shahidi method [34], [40]. The comparison of the two descriptions of $L^2_{\mathrm{disc}}(G,\chi)$ can be used to prove certain analytic properties of the automorphic L-functions in the constant term of the Eisenstein series. This idea was successfully applied in the case of the symplectic and special orthogonal groups to show how endoscopic classification implies the holomorphy in the critical strip of the symmetric and exterior square automorphic L-functions associated to a cuspidal automorphic representation of the general linear group [13], and in the case of the quasi-split unitary groups implies the holomorphy of the Asai automorphic L-functions [22]. See [14] for an overview of these results.

The present paper considers the case of the general spin groups and the twisted symmetric and exterior square automorphic L-functions associated to a cuspidal automorphic representation of the general linear group, which appear in the constant term of the Eisenstein series on the general spin groups. The Langlands spectral theory is applied in this case by Mahdi Asgari [5], and the endoscopic classification is formulated without proof by James Arthur in [2]. The main result is that the endoscopic classification of the automorphic representations in $L^2_{\mathrm{disc}}(G,\chi)$ in the case of the general spin group G would imply that the twisted symmetric and exterior square complete automorphic L-functions are holomorphic at the values of their complex parameter in the critical strip.

There are other approaches to the study of these automorphic L-functions. In particular, the theory of integral representations also implies the analytic properties of twisted symmetric and exterior square partial automorphic L-functions [47], [48], [27], [7], but not the complete ones, because it avoids the finite number of bad local places. However, the Langlands–Shahidi method and the theory of integral representations for automorphic L-functions provide yet another example of two points of view on the same object, thus providing more evidence for the opening claim of this paper.

The motivation and inspiration for writing this paper arose almost ten years ago, at the Marko Tadić 60th birthday conference,[1] from a question by Guy Henniart, and the subsequent discussion with Colette Mœglin. After my talk on the analytic properties of the symmetric and exterior square automorphic L-functions, Henniart asked whether these results could be generalized to the twisted L-functions. The case of the general spin groups treated here provides the affirmative answer.

It is really great how this special volume, dedicated to the occasion of Marko Tadić's 70th birthday, popped out these ideas from the back of my mind and made me write these lines... I would like to express my gratitude to Guy Henniart for raising the question and Colette Mœglin for the useful discussion. I am thankful to Ivan Matić for answering my questions regarding the structure of general spin groups. I am indebted to Goran Muić for the invitation to contribute to the special volume. Thanks are also due to the referee for careful reading of the manuscript and pointing out the missing details which substantially improved the exposition.

Finally, it is my great pleasure to acknowledge my gratitude to Marko Tadić for leading the representation theory research group in Zagreb, including researchers from other Croatian universities such as myself, for so many years, handling the administration of a series of research projects during which we all grew up into what could be now referred to as the Zagreb or Croatian school of representation theory and automorphic forms. I look forward to future collaboration, cooperation and companionship in many years to come. Happy birthday, Marko!

[1] Conference *Representations of p-adic Groups; A Conference Dedicated to Marko Tadić on his 60th Birthday*, held at the Department of Mathematics, University of Zagreb, Croatia, in June 2014.

2. Preliminaries

Let F be an algebraic number field. For a place v of F, let F_v denote the completion of F at v. Let $\mathbb{A}$ denote the ring of adèles, and $\mathbb{I}$ the group of idèles of F.

Let n be a positive integer, and $r = \lfloor \frac{n}{2} \rfloor$ the largest integer not greater than $\frac{n}{2}$. We assume throughout the paper that $r \geq 2$, in order to avoid trivial cases. Let $G_n = GSpin_n$ be the F-split general spin group of F-rank r defined over F. It is a split reductive linear algebraic group over F with the root system of type B_r if $n = 2r + 1$ is odd, and of type D_r if $n = 2r$ is even. The derived group of G_n is the spin group $Spin_n$, which is a double covering of the split special orthogonal group SO_n, viewed as algebraic groups. In this paper, we do not require a complete description of the structure of G_n, and refer to [5] or [23] for a detailed account. See also [32] for the closely related case of the split spin groups.

Let P be a parabolic F-subgroup of G_n with the Levi decomposition $P = MN$, where the Levi factor M is isomorphic to

$$M \cong GL_r \times GL_1,$$

and N denotes the unipotent radical. Following [23], we refer to such parabolic subgroup as the Siegel parabolic subgroup. In the case of $n = 2r + 1$ odd, the Siegel parabolic subgroup is unique up to conjugacy, and thus, self-associate.

However, in the case of $n = 2r$ even, there are two non-conjugate such parabolic subgroups with isomorphic, but not conjugate, Levi factors [6, page 143], [29]. Since the Levi factors are not conjugate, both parabolic subgroups are self-associate. These two parabolic subgroups are obtained by removing either the last or the second-to-last simple root from the set of simple roots of G_{2r}, which form the root system of type D_r. In what follows, we choose the Siegel parabolic subgroup P of G_{2r} to be the parabolic subgroup that is obtained by removing the last simple root from the set of simple roots.

Let W, resp. W_M, be the Weyl group of G_n, resp. M. Let w_0 be the unique non-trivial Weyl group element of G_n of minimal length in its coset $w_0 W_M$ such that $w_0 M w_0^{-1}$ is the Levi factor of a standard parabolic subgroup. Note that implicit in these considerations is the choice of a minimal parabolic F-subgroup of G_n and its maximal F-split torus.

Let $\widehat{G}_n$ denote the complex dual group of G_n. It is the similitude symplectic group $\widehat{G}_n = GSp_{2r}(\mathbb{C})$ of rank r if $n = 2r+1$ is odd, and the similitude special orthogonal group $\widehat{G}_n = GSO_{2r}(\mathbb{C})$ of rank r if $n = 2r$ is even. The complex dual group of the Levi factor M of the Siegel parabolic subgroup is $\widehat{M} = GL_r(\mathbb{C}) \times GL_1(\mathbb{C})$. In the case of the Siegel parabolic subgroup, the action R of the complex dual group $\widehat{M}$ on the Lie algebra of the complex dual group of the unipotent radical N is irreducible [5], [23], [30].

Let $\pi \otimes \chi$ be a cuspidal automorphic representation of the Levi factor $M(\mathbb{A}) \cong GL_r(\mathbb{A}) \times GL_1(\mathbb{A})$. It is the exterior tensor product of a cuspidal automorphic representation π of $GL_r(\mathbb{A})$ and the unitary Hecke character χ of the group of idèles $\mathbb{I} = GL_1(\mathbb{A})$. In this case, since every cuspidal automorphic representation of $GL_r(\mathbb{A})$ is globally generic (with respect to any fixed non-trivial additive character), the automorphic L-functions associated to $\pi \otimes \chi$ and the irreducible representation R of the complex dual group $\widehat{M}$ can be defined by the Langlands–Shahidi method [40], [39]. If $n = 2r + 1$ is odd, the L-function associated to $\pi \otimes \chi$ and R is the twisted symmetric square automorphic L-function

$$L(s, \pi, Sym^2 \otimes \chi),$$

and if $n = 2r$ is even, the L-function associated to $\pi \otimes \chi$ and R is the twisted exterior square automorphic L-function

$$L(s, \pi, \wedge^2 \otimes \chi).$$

These two L-functions are the main objects of concern in this paper.

3. Langlands spectral theory

In this section we construct certain non-trivial constituents of the discrete spectrum $L^2_{\text{disc}}(G_n, \chi)$ of the general spin group $G_n(\mathbb{A})$, where χ is the central character, subject to the analytic properties of the twisted symmetric and exterior square automorphic L-function associated to a cuspidal automorphic representation of the Levi factor $M(\mathbb{A})$ of the Siegel parabolic subgroup P in G_n.

Let $\pi \otimes \chi$ be a cuspidal automorphic representation of the Levi factor $M(\mathbb{A}) \cong GL_r(\mathbb{A}) \times GL_1(\mathbb{A})$, where π is a cuspidal automorphic representation of $GL_r(\mathbb{A})$, and χ a unitary Hecke character of $\mathbb{I} \cong GL_1(\mathbb{A})$. We always assume that cuspidal automorphic representations of the general linear group are irreducible and unitary. We also make the following convention.

Convention. *In what follows, it is always assumed that π and χ are normalized in such a way that the possible poles of the Eisenstein series on $G_n(\mathbb{A})$ associated to $\pi \otimes \chi$, and the automorphic L-functions in their constant terms, are all real. This assumption is not restrictive, as it is just a convenient choice of coordinates. It is achieved by an appropriate twist by a unitary character. See [28, p. 121, Sect. 4.1] for more details.*

Given $s \in \mathbb{C}$, we may form the parabolically induced representation

$$I(s, \pi \otimes \chi) = \operatorname{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})} \left(\pi |\det|^s \otimes \chi\right),$$

where, as usual, the parabolic induction is normalized by the half-sum of positive roots in the unipotent radical N.

The Eisenstein series on $G_n(\mathbb{A})$ associated to $\pi \otimes \chi$ are constructed using sections f_s of the induced representations $I(s, \pi \otimes \chi)$. They are defined as the analytic continuation from the domain of convergence of the sum

$$E(f_s, g) = \sum_{\gamma \in P(F)\backslash G_n(F)} f_s(\gamma g),$$

where $g \in G_n(\mathbb{A})$. The sum converges absolutely for s in some right half-plane, and its analytic continuation is a meromorphic function on the whole complex plane. See [34], [31] for more details.

In the case of the general spin group, the Langlands–Shahidi method [40], [38] implies that the constant term $E_P(f_s, g)$ along P of the Eisenstein series $E(f_s, g)$ can be expressed in terms of the twisted symmetric and exterior square automorphic L-functions as

$$E_P(f_s, g) = f_s(g) + \frac{L(2s, \pi, R)}{\varepsilon(2s, \pi, R) L(1 + 2s, \pi, R)} N(s, \pi \otimes \chi, w_0) f_s(g),$$

where $R = Sym^2 \otimes \chi$, resp. $R = \wedge^2 \otimes \chi$, if $n = 2r + 1$ is odd, resp. $n = 2r$ is even. The operator $N(s, \pi \otimes \chi, w_0)$ is the normalized intertwining operator associated to π and the Weyl group element w_0, as in [6].

Our first task is to determine the poles of the Eisenstein series $E(f_s, g)$ for the values s of its complex parameter such that $Re(s) \geq 0$. Our Convention made in Section 3 regarding the normalization of π implies that these poles are real. By the general theory of Eisenstein series [34], the possible poles of the Eisenstein series coincide with the poles of its constant term. Hence, we should study the poles of the second summand in the expression for the constant term. The first step is the following lemma which shows that the pole is determined by the L-functions, and not by the normalized intertwining operator.

Lemma 3.1. *In the notation as above, the normalized intertwining operator $N(s, \pi \otimes \chi, w_0)$ is holomorphic and not identically vanishing at the values s of its complex parameter such that $Re(s) \geq 0$.*

Proof. This lemma is essentially Proposition 3.6 in [6] with a slight improvement due to the fact that we deal with the Siegel parabolic subgroup. Our statement for the global normalized intertwining operator $N(s, \pi \otimes \chi, w_0)$ is reduced to the local result for the local normalized intertwining operators $N(s, \pi_v \otimes \chi_v, w_0)$, where $\pi \cong \otimes_v \pi_v$ is the restricted tensor product decomposition and $\chi = \prod_v \chi_v$, as follows.

Let $f_s = \otimes_v f_{s,v}$ be a decomposable section, where $f_{s,v}$ are sections of the local induced representations

$$I(s, \pi_v \otimes \chi_v) = \mathrm{Ind}_{P(F_v)}^{G_n(F_v)} \left(\pi_v |\det|_v^s \otimes \chi_v \right).$$

Then, there is a finite set of places S_f, which contains all archimedean places, such that the local component π_v of π is unramified and $f_{s,v} = f_{s,v}^{\circ}$ for all places $v \notin S_f$, where $f^\circ_{s,v}$ is the unramified vector in $I(s, \pi_v \otimes \chi_v)$ normalized by the condition that $f^\circ_{s,v}$ takes value one on the identity element of $G_n(F_v)$. The action of the global normalized intertwining operator decomposes as

$$\begin{aligned} N(s, \pi \otimes \chi, w_0) f_s &= \otimes_v N(s, \pi_v \otimes \chi_v, w_0) f_{s,v} \\ &= \Big(\otimes_{v \in S_f} N(s, \pi_v \otimes \chi_v, w_0) f_{s,v} \Big) \otimes \Big(\otimes_{v \notin S_f} \widetilde{f}^\circ_{-s,v} \Big), \end{aligned}$$

where $\widetilde{f}^\circ_{-s,v}$ is the unramified vector in the representation $I(-s, w_0(\pi_v))$ normalized by the condition that it takes value one on the identity element of $G_n(F_v)$. This decomposition implies that the holomorphy and non-vanishing of the global intertwining operator is reduced to the same result for the local normalized intertwining operator at every place v.

The holomorphy of $N(s, \pi_v \otimes \chi_v, w_0)$ for the values s of the complex parameter such that $Re(s) \geq 0$ now follows in the same way as in the proof of Proposition 3.6 in [6]. The better bound is obtained, because in the Levi factor of the Siegel parabolic subgroup there is no general spin group of smaller rank. For convenience of the reader we sketch the proof below.

Since π_v is a local component of a cuspidal automorphic representation of $GL_r(\mathbb{A})$, it is unitary and generic. According to the classification of irreducible unitary representations of the general linear group in [41] over a p-adic field and in [50] over an archimedean field, and the standard module conjecture for the general linear group in [52] over a p-adic field and in [49] over an archimedean field, the representation π_v is isomorphic to the fully induced representation

$$\pi_v \cong \mathrm{Ind}_{Q(F_v)}^{GL_r(F_v)} \left(\delta_1 |\det|_v^{\alpha_1} \otimes \cdots \otimes \delta_k |\det|_v^{\alpha_k} \right),$$

where Q is an appropriate parabolic subgroup of GL_r with the Levi factor isomorphic to a product of general linear groups of smaller rank, the representations δ_j are unitary square-integrable representations of the smaller general linear groups in the Levi factor of Q, and α_j are real numbers such that $|\alpha_j| < 1/2$.

As in the proof of Proposition 3.6 in [6], by the decomposition of the intertwining operators and induction in stages, it is sufficient to check the holomorphy of each factor viewed as an intertwining operator for a maximal parabolic subgroup in a certain smaller group. In the case of the Siegel parabolic subgroup, the factors are simpler than in the general case. They are all either of the form

$$N(2s + \alpha_i + \alpha_j, \delta_i \otimes \widetilde{\delta}_j),$$

where $i < j$ and $\widetilde{\delta}_j$ is the contragredient of δ_j, which is the normalized intertwining operator associated to a square-integrable representation of a maximal parabolic subgroup in the general linear group, or of the form

$$N(s + \alpha_i, \delta_i),$$

which is the normalized intertwining operator associated to a square-integrable representation of the Levi factor of the Siegel parabolic subgroup of a smaller general spin group. The former are holomorphic for the values of the complex parameter s such that

$$Re(2s + \alpha_i + \alpha_j) > -1$$

by [33], and the latter are holomorphic for the values of s such that

$$Re(s + \alpha_i) > -1/2$$

by [53]. The assumptions in [53] are all proved in the case of general spin groups, as explained in [6]. The above inequalities are all satisfied by the values of s such that $Re(s) \geq 0$, since $|\alpha_i| < 1/2$. Thus, all the factors are holomorphic for such s, as required.

The non-vanishing of $N(s, \pi_v \otimes \chi_v, w_0)$ is a consequence of holomorphy by Zhang's lemma [53]. ∎

We are now ready to prove the main result of this section, which relates the poles of the ratio of the twisted symmetric and exterior square automorphic L-function to the constituents of the discrete spectrum $L^2_{\text{disc}}(G_n, \chi)$ of $G_n(\mathbb{A})$.

THEOREM 3.2. *Let π be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ and χ a unitary Hecke character, normalized as in Convention made in Section 3. Let either $n = 2r + 1$ or $n = 2r$. Let $E(f_s, g)$ be the Eisenstein series on the general spin group $G_n(\mathbb{A})$ associated to the cuspidal automorphic representation $\pi \otimes \chi$ of the Levi factor $M(\mathbb{A})$ of the Siegel parabolic subgroup P of G_n. Then, the poles of the Eisentstein series $E(f_s, g)$ for the values of the complex parameter s in the region $Re(s) > 0$ coincide with the poles of the ratio of the automorphic L-functions*

$$\frac{L(z, \pi, R)}{L(1 + z, \pi, R)}$$

at the value $z = 2s$ of the complex parameter, where $R = Sym^2 \otimes \chi$ if $n = 2r + 1$, and $R = \wedge^2 \otimes \chi$ if $n = 2r$. In particular, if $z_0 > 0$ is a positive real number such that the ratio above has a pole at $z = z_0$, then there exists a non-trivial summand Π in the discrete spectrum $L^2_{\text{disc}}(G_n, \chi)$ of the general spin group $G_n(\mathbb{A})$ such that $\Pi \cong \otimes_v \Pi_v$ and Π_v is a quotient of the parabolically induced representation

$$I(z_0/2, \pi_v \otimes \chi_v)$$

for every place v.

PROOF. According to Lemma 3.1, the poles of $E(f_s, g)$ at the values s of its complex parameter such that $Re(s) > 0$ coincide with the poles of the normalizing factor

$$\frac{L(2s, \pi, R)}{\varepsilon(2s, \pi, R) L(1 + 2s, \pi, R)},$$

where R is as in the statement of the theorem. Since the ε-factor is entire and non-zero, the first claim of the theorem follows.

Suppose now that $z_0 > 0$ is the pole of the ratio

$$\frac{L(z, \pi, R)}{L(1 + z, \pi, R)},$$

then the Eisenstein series $E(f_s, g)$ has a pole at $2s = z_0$, i.e., $s = z_0/2$. By the general theory of Eisenstein series [34], the residues of $E(f_s, g)$ at the pole $s = z_0/2 > 0$ span a residual representation Π of $G_n(\mathbb{A})$, that is, Π is a summand in $L^2_{\mathrm{disc}}(G_n, \chi)$. Let $\Pi \cong \otimes_v \Pi_v$ be the decomposition of Π into a restricted tensor product. Then, Π_v is isomorphic to the image of the normalized intertwining operator $N(s, \pi_v \otimes \chi_v, w_0)$ at $s = z_0/2$. Since this intertwining operator intertwines the induced representation $I(z_0/2, \pi_v \otimes \chi_v)$, its image is isomorphic to a quotient of that induced representation, as claimed. □

4. Holomorphy of L-functions in the non-self-dual case

In this section we treat the case in which the analytic properties of the twisted symmetric and exterior square automorphic L-functions are obtained independently of the endoscopic classification.

Let χ be a unitary Hecke character of the group of idèles $\mathbb{I}$. A cuspidal automorphic representation π of $GL_m(\mathbb{A})$ is called χ-self-dual (or twisted self-dual), if it satisfies

$$\pi \cong \widetilde{\pi} \otimes \chi,$$

where $\widetilde{\pi}$ is the contragredient representation of π. Recall Convention made in Section 3, which says that it is always assumed that cuspidal automorphic representations of $GL_m(\mathbb{A})$ are normalized in such a way that the poles of the associated Eisenstein series and the automorphic L-functions are real.

In this section, we consider the case of a cuspidal automorphic representation π of $GL_r(\mathbb{A})$ which is not χ-self-dual. We begin with the lemma that describes the analytic properties of Eisenstein series associated to such π and χ.

Lemma 4.1. *Let π be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ which is not χ-self-dual, that is, $\pi \not\cong \widetilde{\pi} \otimes \chi$. Let either $n = 2r+1$ or $n = 2r$. Let $E(f_s, g)$ be the Eisenstein series on the general spin group $G_n(\mathbb{A})$ associated to the cuspidal automorphic representation $\pi \otimes \chi$ of the Levi factor $M(\mathbb{A})$ of the Siegel parabolic subgroup P of G_n. Then, the Eisenstein series $E(f_s, g)$ is holomorphic for the value s of its complex parameter such that $Re(s) \geq 0$.*

Proof. According to the general theory of Eisenstein series, the necessary condition for the existence of the pole of $E(f_s, g)$ in the right half-plane $Re(s) \geq 0$ is that P is self-associate and π is χ-self-dual. See [34, Sect. IV.3.12] for more details. Since we are now dealing with the case of π not χ-self-dual, the Eisenstein series $E(f_s, g)$ is holomorphic in the right half-plane $Re(s) \geq 0$. ∎

THEOREM 4.2. *Let π be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ and χ a unitary Hecke character of the group of idèles $\mathbb{I}$. Suppose that π is not χ-self-dual, that is, $\pi \not\cong \widetilde{\pi} \otimes \chi$. Then, the twisted symmetric and exterior automorphic L-functions*

$$L(z, \pi, Sym^2 \otimes \chi) \quad \textit{and} \quad L(z, \pi, \wedge^2 \otimes \chi)$$

are entire as functions of their complex parameter z, and non-zero for the values of z such that $Re(z) \leq 0$ and $Re(z) \geq 1$.

PROOF. Consider the Eisenstein series $E(f_s, g)$ on the general spin group $G_n(\mathbb{A})$ associated to the cuspidal automorphic representation $\pi \otimes \chi$ of the Levi factor $M(\mathbb{A})$ of the Siegel parabolic subgroup P of G_n, where either $n = 2r+1$ or $n = 2r$. According to Lemma 4.1, the Eisenstein series $E(f_s, g)$ is holomorphic for the values s of their complex parameter such that $Re(s) \geq 0$.

For the non-vanishing of the automorphic L-functions, we follow the method of Shahidi and consider the non-constant Fourier coefficient $E(f_s, g)_\psi$ of the Eisenstein series $E(f_s, g)$ with respect to a non-trivial continuous additive character ψ of $F\backslash\mathbb{A}$. Its evaluation at the identity element e of $G_n(\mathbb{A})$ can be written as

$$E(f_s, e)_\psi = \frac{1}{L^S(1 + 2s, \pi, R)} \prod_{v \in S} W_v(e_v),$$

where $R = Sym^2 \otimes \chi$ if $n = 2r + 1$ and $R = \wedge^2 \otimes \chi$ if $n = 2r$, and S is a finite set of places, containing all archimedean places, and such that the local components π_v, χ_v and ψ_v are unramified for all $v \notin S$. The function $L^S(1 + 2s, \pi, R)$ is the partial L-function associated to $\pi \otimes \chi$ and R, defined as the product of the unramified local L-functions over all places $v \notin S$. For $v \in S$, the function $W_v(e_v)$ is the ψ_v-Whittaker function associated to the section $f_{s,v}$ and e_v is the identity element of $G_n(F_v)$. See [37] and [40, Sect. 7] for more details.

As in [40, Sect. 7.2], there is a choice of the local data such that $W_v(e_v)$ is non-zero for every $v \in S$. The Fourier coefficient $E(f_s, g)_\psi$ is holomorphic at the values of s such that $Re(s) \geq 0$, because the same holds for the Eisenstein series $E(f_s, g)$. Hence, the partial L-function $L^S(1 + 2s, \pi, R)$ in the denominator must be non-zero for the values of s such that $Re(s) \geq 0$. It follows that the partial L-function $L^S(z, \pi, R)$ is non-zero for the values z of its complex parameter such that $Re(z) \geq 1$. Since the local L-functions at the remaining places are non-vanishing, the same holds for the complete L-function $L(z, \pi, R)$. The non-vanishing in the left half-plane $Re(z) \leq 0$ follows by the functional equation for automorphic L-functions. This proves the claim regarding the non-vanishing of $L(z, \pi, R)$.

For the holomorphy, consider first the Rankin–Selberg automorphic L-function $L(z, \pi \times (\pi \otimes \chi))$ of the pair of cuspidal automorphic representations π and $\pi \otimes \chi$ of $GL_r(\mathbb{A})$. It can be decomposed into the product

$$L(z, \pi \times (\pi \otimes \chi)) = L(z, \pi, Sym^2 \otimes \chi) L(z, \pi, \wedge^2 \otimes \chi),$$

of the twisted symmetric and exterior square automorphic L-functions. This follows from the direct sum decomposition of the twisted tensor product of the standard representations of $GL_r(\mathbb{C})$ into a direct sum of the twisted symmetric and exterior square of the standard representation of $GL_r(\mathbb{C})$. See [23, Sect. 2.2] for more details.

The analytic properties of the Rankin–Selberg L-functions of pairs are well-known from [25], [26], [24], [33]. Since π is not χ-self-dual, the Rankin–Selberg L-function $L(z, \pi \times (\pi \otimes \chi))$ is entire. We already proved that the twisted symmetric and exterior square L-functions on the right-hand side are non-zero for the values z such that $Re(z) \geq 1$. It follows that they are both holomorphic for z such that $Re(z) \geq 1$. Otherwise, the pole at z such that $Re(z) \geq 1$ of any of the L-functions $L(z, \pi, R)$ would produce the pole at z with $Re(z) \geq 1$ of the Rankin–Selberg L-function, which would be a contradiction. The holomorphy in the left half-plane $Re(s) \leq 0$ is now a consequence of the functional equation.

It remains to prove the holomorphy of $L(z, \pi, R)$ at the values z such that $0 < Re(z) < 1$. Suppose that there is z_0 such that $0 < Re(z_0) < 1$ and the automorphic L-function $L(z, \pi, R)$ has a pole at $z = z_0$. Then, the ratio of L-functions

$$\frac{L(2s, \pi, R)}{L(1 + 2s, \pi, R)}$$

has a pole at the value of s such that $2s = z_0$, because in the denominator $Re(1 + 2s) = Re(1 + z_0) > 1$ and we already proved the holomorphy of $L(z, \pi, R)$ in that region. Invoking Theorem 3.2 implies that the Eisenstein series $E(f_s, g)$ associated to $\pi \otimes \chi$ has a pole at $s = z_0/2$ with $Re(s) > 0$. This is a contradiction to Lemma 4.1, so that the theorem is proved. □

5. Endoscopic classification

The endoscopic classification of the discrete spectrum $L^2_{\mathrm{disc}}(G_n, \chi)$ of the general spin group $G_n(\mathbb{A})$ in terms of Arthur parameters is formulated, without proof, in [2], at least in the case of odd n. However, the case of even n is analogous. In this section, we recall the definition of Arthur parameters for the general spin groups, and explain how endoscopic transfer would provide the classification in terms of discrete spectra of general linear groups.

Lemma 5.1. *Let π be a χ-self-dual cuspidal automorphic representation of $GL_m(\mathbb{A})$, that is, $\pi \cong \widetilde{\pi} \otimes \chi$, normalized as in Convention made in Section 3. Then, exactly one of the automorphic L-functions $L(z, \pi, Sym^2 \otimes \chi)$ and*

$L(z,\pi,\wedge^2\otimes\chi)$ has a pole of order one at $z=0$ and $z=1$, and the other L-function is holomorphic at $z=0$ and $z=1$.

Proof. Consider the Rankin–Selberg automorphic L-function $L(z,\pi\times(\pi\otimes\chi))$ of the pair of cuspidal automorphic representations π and $\pi\otimes\chi$ of $GL_r(\mathbb{A})$, which we already used in the proof of Theorem 4.2. In the case of χ-self-dual π, it is holomorphic, except for the poles of order one at $z=0$ and $z=1$. These facts are the result of the theory of integral representations of L-functions, developed in this case in [25], [26], [24], see also [33].

Consider again the decomposition

$$L(z,\pi\times(\pi\otimes\chi)) = L(z,\pi,Sym^2\otimes\chi)L(z,\pi,\wedge^2\otimes\chi),$$

as in the proof of Theorem 4.2. According to [37], [39], the L-functions on the right-hand side are non-zero at $z=0$ and $z=1$. Hence, in the case of χ-self-dual π, exactly one of the L-functions on the right-hand side has a pole at $z=0$ and $z=1$, and that pole is of order one. □

The lemma provides a way to make a distinction between two types of χ-self-dual cuspidal automorphic representations of $GL_m(\mathbb{A})$, depending on the L-function that has a pole at $z=1$. These two types are called the orthogonal and symplectic type, because they are precisely the functorial transfers from similitude orthogonal and symplectic groups.

Definition 5.2. *The χ-self-dual cuspidal automorphic representation π of $GL_m(\mathbb{A})$ is orthogonal, resp. symplectic, if the automorphic L-function $L(z,\pi,Sym^2\otimes\chi)$, resp. $L(z,\pi,\wedge^2\otimes\chi)$, associated to π, has a pole at $z=1$.*

Consider now χ as a unitary Hecke character which is the similitude character of G_n. The set $\Psi_2(G_n,\chi)$ of Arthur parameters for $L^2_{\mathrm{disc}}(G_n,\chi)$ is defined as follows.

Definition 5.3. *An Arthur parameter for $L^2_{\mathrm{disc}}(G_n,\chi)$ is an unordered formal sum*

$$\psi = (\mu_1\boxtimes\nu_1)\boxplus\cdots\boxplus(\mu_\ell\boxtimes\nu_\ell)$$

of formal tensor products $\mu_i\boxtimes\nu_i$, such that

- *(i) the positive integers $m_1,\dots,m_\ell$ and $n_1,\dots,n_\ell$ are such that $m_1n_1+\cdots+m_\ell n_\ell=2r$,*
- *(ii) μ_i is a χ-self-dual cuspidal automorphic representation of $GL_{m_i}(\mathbb{A})$, for $i=1,\dots,\ell$,*
- *(iii) ν_i is the unique irreducible algebraic representation of $SL_2(\mathbb{C})$ of dimension n_i, for $i=1,\dots,\ell$,*
- *(iv) $\mu_i\boxtimes\nu_i$ are all distinct, i.e., for $i\neq j$, we have $\mu_i\ncong\mu_j$ or $n_i\neq n_j$,*
- *(v) the central characters ω_{μ_i} of μ_i, for $i=1,\dots,\ell$, and χ satisfy a compatibility condition $\omega_{\mu_i}^{n_i}=\chi$,*
- *(vi) the type of representations μ_i, for $i=1,\dots,\ell$, is determined as follows:*

- *if* $n = 2r+1$ *is odd, then* $\begin{array}{l}\mu_i \textit{ is of symplectic type, if } n_i \textit{ is odd,}\\ \mu_i \textit{ is of orthogonal type, if } n_i \textit{ is even,}\end{array}$
- *if* $n = 2r$ *is even, then* $\begin{array}{l}\mu_i \textit{ is of orthogonal type, if } n_i \textit{ is odd,}\\ \mu_i \textit{ is of symplectic type, if } n_i \textit{ is even.}\end{array}$

The set of such Arthur parameters is denoted by $\Psi_2(G_n, \chi)$*. Note that the adjective unordered means that two formal sums that differ only by the order of summands are considered equal.*

The Arthur packet Π_ψ of automorphic representations of $G_n(\mathbb{A})$ with the similitude character χ is associated to each Arthur parameter $\psi \in \Psi_2(G_n, \chi)$ as in [3]. The members of Arthur packets are candidates for the constituents of the discrete spectrum $L^2_{\mathrm{disc}}(G_n, \chi)$. The endoscopic classification of the discrete spectrum $L^2_{\mathrm{disc}}(G_n, \chi)$ is then given by a multiplicity formula for members of Arthur packets, which singles out those members of Arthur packets which contribute with non-zero multiplicity. The multiplicity formula amounts to a parity condition precisely defined in the discussion following [4, Thm. 1.5.2].

All representations in the Arthur packet Π_ψ associated to ψ are nearly equivalent to each other, that is, their local components are isomorphic at all but finitely many places. More precisely, given an Arthur parameter $\psi \in \Psi_2(G_n, \chi)$ as a formal sum

$$\psi = (\mu_1 \boxtimes \nu_1) \boxplus \cdots \boxplus (\mu_\ell \boxtimes \nu_\ell),$$

then, for every representation in the Arthur packet Π_ψ, its local component at all but finitely many places is the unramified representation of $G_n(F_v)$ given by the Satake parameter

$$c(\psi_v) = \bigoplus_{i=1}^{\ell} \left(c(\mu_{i,v}) \otimes c(\nu_i) \right),$$

where $c(\mu_{i,v})$ is the Satake parameter of the unramified representation $\mu_{i,v}$ of $GL_{m_i}(F_v)$ and

$$c(\nu_i) = \mathrm{diag}\left(q_v^{\frac{n_i-1}{2}}, q_v^{\frac{n_i-3}{2}}, \ldots, q_v^{-\frac{n_i-1}{2}} \right),$$

with q_v the cardinality of the residual field of F_v and n_i the dimension of ν_i.

6. Holomorphy of L-functions in the self-dual case

So far, in Section 4, we proved the analytic properties of the twisted symmetric and exterior square automorphic L-functions of a cuspidal automorphic representation π of $GL_r(\mathbb{A})$ in the case of π that is not χ-self-dual. Those results are independent of the endoscopic classification for general spin groups.

In this section, we consider the case of a cuspidal automorphic representation π of $GL_r(\mathbb{A})$ which is χ-self-dual, that is, $\pi \cong \widetilde{\pi} \otimes \chi$. In what follows we make the assumption, as we may, that π and χ are normalized as in Convention made in Section 3. In the χ-self-dual case, the results below depend on the eventual complete proof of the endoscopic classification of the discrete spectrum $L^2_{\text{disc}}(G_n, \chi)$ of the general spin groups. In other words, we freely use the description elaborated in Section 5.

However, observe that the analytic behavior at $z = 0$ and $z = 1$ of the twisted symmetric and exterior square automorphic L-function in the case of χ-self-dual π is described in Lemma 5.1. Note that this result is also independent of the endoscopic classification. What remains is to determine the analytic properties of the twisted symmetric and exterior square automorphic L-functions $L(z, \pi, Sym^2 \otimes \chi)$ and $L(z, \pi, \wedge^2 \otimes \chi)$ away from the vertical lines $Re(z) = 0$ and $Re(z) = 1$.

We begin with a lemma regarding the analytic properties of the Eisenstein series $E(f_s, g)$ on the general spin group $G_n(\mathbb{A})$ associated to a χ-self-dual cuspidal automorphic representation π of $GL_r(\mathbb{A})$, as defined in Section 3.

LEMMA 6.1. *Let π be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ which is χ-self-dual, that is, $\pi \cong \widetilde{\pi} \otimes \chi$, normalized as in Convention of Section 3. Let either $n = 2r+1$ or $n = 2r$. Let $E(f_s, g)$ be the Eisenstein series on the general spin group $G_n(\mathbb{A})$ associated to a cuspidal automorphic representation $\pi \otimes \chi$ of the Levi factor $M(\mathbb{A})$ of the Siegel parabolic subgroup P of G_n. Then, it would follow from the endoscopic classification of the discrete spectrum of general spin groups that the Eisenstein series $E(f_s, g)$ is holomorphic at the values s of its complex parameter such that $Re(s) \geq 0$ and $s \neq 1/2$.*

PROOF. First of all, by the general theory of Eisenstein series, they are holomorphic on the imaginary axis, so that it is sufficient to give the proof for $Re(s) > 0$. Suppose that the Eisenstein series $E(f_s, g)$ has a pole at $s = s_0$ such that $Re(s_0) > 0$. Then, the residues span a residual representation of $G_n(\mathbb{A})$, which is a direct summand in the discrete spectrum $L^2_{\text{disc}}(G_n, \chi)$. According to Theorem 3.2, it is isomorphic to $\Pi \cong \otimes_v \Pi_v$, where Π_v is a quotient of the parabolically induced representation

$$I(s_0, \pi_v \otimes \chi_v) = \operatorname{Ind}_{P(F_v)}^{G_n(F_v)} \left(\pi_v |\det|_v^{s_0} \otimes \chi_v \right).$$

For the places v at which Π_v is unramified, we have that π_v and χ_v are unramified, so that Π_v is the unramified constituent of the induced representation above. Hence, the Satake parameter of Π_v is

$$c(\Pi_v) = c(\pi_v) \otimes \operatorname{diag}\left(q_v^{s_0}, q_v^{-s_0}\right)$$

at all but finitely many places.

By the endoscopic classification of $L^2_{\mathrm{disc}}(G_n,\chi)$, the representation Π belongs to the Arthur packet Π_ψ, where the parameter ψ is such that

$$c(\psi_v) = c(\Pi_v)$$

at all but finitely many places. Hence, due to the strong multiplicity one for the general linear group [26, Thm. 4.4], the Arthur parameter ψ is necessarily of the form

$$\psi = \pi \boxtimes \nu,$$

where ν should be the unique irreducible representation of $SL_2(\mathbb{C})$ of dimension $2s_0 + 1$, so that the exponents of q_v in the Satake parameters match. But the dimensions should satisfy the condition (i) in Definition 5.3 of Arthur parameters, so that

$$r(2s_0 + 1) = 2r.$$

This implies that $s_0 = 1/2$ as claimed, because otherwise the Arthur parameter required to parameterize Π does not exist. □

We are now ready to prove the main result of this section regarding analytic properties of twisted symmetric and exterior square automorphic L-functions in the case of χ-self-dual cuspidal automorphic representation of $GL_r(\mathbb{A})$.

THEOREM 6.2. *Let χ be a unitary Hecke character of the group of idèles $\mathbb{I}$. Let π be a cuspidal automorphic representation of $GL_r(\mathbb{A})$ which is χ-self-dual, that is, $\pi \cong \widetilde{\pi} \otimes \chi$. Then, it would follow from the endoscopic classification for the general spin groups, that the twisted symmetric and exterior square automorphic L-functions*

$$L(z, \pi, Sym^2 \otimes \chi) \quad \textit{and} \quad L(z, \pi, \wedge^2 \otimes \chi)$$

associated to π are holomorphic at the values z of their complex parameter such that $Re(z) \neq 0$ and $Re(z) \neq 1$, and non-zero at the values of z such that $Re(z) \geq 1$ and $Re(z) \leq 0$.

PROOF. It is sufficient to prove the theorem for π normalized as in Convention made in Section 3, so that the possible poles are real.

We begin with the proof of non-vanishing. As in the proof of Theorem 4.2, we use the method of Shahidi [37], [40, Sect. 7], and consider the non-constant Fourier coefficient of the Eisenstein series $E(f_s, g)$ associated to $\pi \otimes \chi$. The same argument as in the proof of Theorem 4.2, relying on Lemma 6.1, implies that the automorphic L-functions $L(z, \pi, R)$ are non-zero at the values z of the complex parameter such that $Re(z) \geq 1$ and $z \neq 2$. The possible zero at $z = 2$ is a consequence of the possible pole of the Eisenstein series $E(f_s, g)$ at $s = 1/2$, obtained in Lemma 6.1.

In order to deal with the remaining issue at $z = 2$, we consider again the decomposition

$$L(z, \pi \times (\pi \otimes \chi)) = L(z, \pi, Sym^2 \otimes \chi) L(z, \pi, \wedge^2 \otimes \chi)$$

of the Rankin–Selberg L-function of pairs, as in the proof of Theorem 4.2 and Lemma 5.1. According to [24], [25], [26], [33], the Rankin–Selberg L-function is holomorphic and non-zero at the values z of its complex parameter such that $Re(z) > 1$. Hence, if one of the automorphic L-functions on the right-hand side has a zero at $z = 2$, then the other one would have a pole at $z = 2$. Suppose that $L(z, \pi, \wedge^2 \otimes \chi)$ has a zero at $z = 2$ and that $L(z, \pi, Sym^2 \otimes \chi)$ has a pole at $z = 2$. The other possibility is treated in the same way.

Consider the Eisenstein series $E(f_s, g)$ on the general spin group $G_{2r+1}(\mathbb{A})$ associated to $\pi \otimes \chi$. According to Theorem 3.2, if the ratio

$$\frac{L(z, \pi, Sym^2 \otimes \chi)}{L(1 + z, \pi, Sym^2 \otimes \chi)}$$

of automorphic L-functions had a pole at $z = z_0 > 0$, then the Eisenstein series $E(f_s, g)$ would have a pole at $s = z_0/2 > 0$. By Lemma 6.1, the Eisenstein series $E(f_s, g)$ is holomorphic at $s = 1$, which implies that the above ratio of automorphic L-functions is holomorphic at $z = 2$. Since the numerator has a pole at $z = 2$, it must be cancelled by the pole of the denominator at $z = 2$. This means that the L-function $L(z, \pi, Sym^2 \otimes \chi)$ has a pole at $z = 3$. But the Rankin–Selberg L-function in the product decomposition above is holomorphic at $z = 3$. Thus, the pole of $L(z, \pi, Sym^2 \otimes \chi)$ must be cancelled by a zero of $L(z, \pi, \wedge^2 \otimes \chi)$ at $z = 3$. However, we already proved that $L(z, \pi, \wedge^2 \otimes \chi)$ is non-zero at $z = 3$, so that we obtain a contradiction which proves that $L(z, \pi, \wedge^2 \otimes \chi)$ is non-zero at $z = 2$ as well.

This proves that the twisted symmetric and exterior square automorphic L-functions $L(z, \pi, R)$ are non-zero in the right half-plane $Re(z) \geq 1$. The non-vanishing in the left half-plane $Re(s) \leq 0$ follows by the functional equation.

For the holomorphy of $L(z, \pi, R)$, we first consider the right half-plane $Re(z) > 1$. In this half-plane, the Rankin–Selberg L-function $L(z, \pi \times (\pi \otimes \chi))$ is holomorphic and non-zero according to [24], [25], [26], [33]. We already proved that the twisted symmetric and exterior square automorphic L-functions are non-zero in the same right half-plane. It follows that both L-functions are holomorphic at values z such that $Re(z) > 1$, because otherwise the pole would produce a pole of the Rankin–Selberg L-function.

It remains to prove the holomorphy of $L(z, \pi, R)$ in the critical strip, that is, at the values z such that $0 < Re(z) < 1$. Suppose that $L(z, \pi, R)$ has a pole at $z = z_0$ such that $0 < z_0 < 1$. Since we already proved that $L(z, \pi, R)$

is holomorphic at z such that $Re(z) > 1$, the ratio of automorphic L-functions

$$\frac{L(z, \pi, R)}{L(1 + z, \pi, R)}$$

has a pole at $z = z_0$. According to Theorem 3.2, it follows that the Eisenstein series $E(f_s, g)$ on the general spin group $G_n(\mathbb{A})$ associated to $\pi \otimes \chi$ has a pole at $s = z_0/2$, where $n = 2r + 1$ if $R = Sym^2 \otimes \chi$ and $n = 2r$ if $R = \wedge^2 \otimes \chi$. This is a contradiction to Lemma 6.1, because $0 < s < 1/2$. □

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Endoskopski transfer i automorfne L-funkcije: slučaj opće spinorne grupe i L-funkcija zakrenutog simetričnog i vanjskog kvadrata

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Sažetak. Endoskopska klasifikacija i Langlandsova spektralna teorija su dva pristupa diskretnom spektru grupe adeličkih točaka reduktivne linearne algebarske grupe definirane nad poljem algebarskih brojeva. Ova dva pogleda na isti objekt imaju zanimljive posljedice. U ovom radu, promatra se slučaj opće spinorne grupe. U tom slučaju, pokazano je kako usporedba dvaju pristupa implicira da su potpune automorfne L-funkcije zakrenutog simetričnog i vanjskog kvadrata pridružene kuspidalnoj automorfnoj reprezentaciji opće linearne grupe holomorfne u kritičnoj pruzi.

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