

AN ALGORITHM FOR EXPLICIT CALCULATION OF THE FRANKE FILTRATION FOR THE GENERAL LINEAR GROUP*

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ABSTRACT. The Franke filtration is a descending filtration of spaces of automorphic forms on adelic reductive groups. The consecutive quotients of the filtration can be described in terms of parabolically induced representations. Although such representation theoretic structural description of spaces of automorphic forms has found many applications, its explicit calculation is often required. This is a difficult task, not only because of demanding combinatorics, but also because it depends on fine arithmetic information related to Eisenstein series and automorphic L-functions. The goal of this paper is to present an algorithm for explicit calculation of the Franke filtration in the case of the general linear group and its implementation on a computer.

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1. INTRODUCTION

The Franke filtration is a descending filtration of spaces of automorphic forms on adèlic reductive groups, defined by Jens Franke in his famous paper [6]. Its main advantage is that the quotients of the filtration can be described as parabolically induced representations. Thus, the Franke filtration is a way to access the internal structure of spaces of automorphic forms in terms of representation theory.

Already in its most general form, the Franke filtration has found several applications. In the original paper [6], Franke proved that every non-cuspidal automorphic form is a derivative of an Eisenstein series or its residue. This fact was previously known only in the function field case by

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Mœglin–Waldspurger [22, App. II]. He also proved a generalization of Borel’s regularization theorem [2] that cohomology of congruence subgroups can be determined from the automorphic cohomology of the ambient group, that is, the relative Lie algebra cohomology of the space of automorphic forms on the adèlic points of the ambient group. This claim is sometimes referred to as the conjecture of Borel and Harder [3], [19]. Finally, Franke proved certain trace formula for the Hecke operators on full cohomology, not only L^2 . As a consequence of the Franke filtration, Franke–Schwermer proved in [7] the rationality in cohomology of the decomposition along the cuspidal support for the general linear group, generalizing Clozel’s result in the cuspidal case [5]. This was generalized to inner forms of the general linear group by Grobner–Raghuram in [18]. Recently, Su proved in [24] that the coherent cohomology of Shimura varieties can be determined from the (\mathfrak{p}_h, K_h) -cohomology of the space of automorphic forms, paving the way for possible applications of the Franke filtration.

However, for explicit calculations of automorphic cohomology and cohomology of arithmetic groups, the general form of the Franke filtration is not sufficient. A more detailed description of the filtration is obtained for spaces of automorphic forms with a fixed cuspidal support as in [7], [8]. See also [9] for more details. The quotients of the filtration are described in terms of parabolic induction from explicit inducing data, at least in principle. Such refined Franke filtration leads to further applications. In a series of papers [12], [17], [15], [16], [14], explicit calculations and constructions of non-trivial classes in automorphic cohomology and cohomology of arithmetic groups are carried out in different cases and settings. Explicit description of the Franke filtration for the exceptional group G_2 was used by Mundy [23] to construct non-trivial elements in the Bloch–Kato Selmer group, in accordance with the Bloch–Kato conjecture. Very recently, the first named author has developed a new method for proving holomorphy of Eisenstein series based on the Franke filtration [11].

The problem with all these applications is that the explicit description of the Franke filtration is combinatorially demanding and also depends on fine arithmetic information related to Eisenstein series and automorphic L -functions in its constant term including their analytic properties and special values. Therefore, our goal in this paper is to present an algorithm for explicit calculation of the Franke filtration for the spaces of automorphic forms on the general linear group with a fixed cuspidal support together with its implementation on a computer. The input in the algorithm is the cuspidal support, and the output are the inducing data for the parabolic induction describing the quotients of the filtration.

This goal is achievable in the case of the general linear group due to the Mœglin–Waldspurger classification of the residual spectrum [21], which does not depend on any unknown properties of L -functions. The reason for that is the fact that only the Rankin–Selberg automorphic L -functions of pairs appear in the constant term of Eisenstein series on the general linear group.

Our original implementation of the algorithm is written in Python programming language, and is available at the GitHub repository of the second named author.¹ The implementation is also part of *The Langlands Programs* initiative.² *The Langlands Programs* is a project of Petar Bakić, Thomas Lanard and Elad Zelingher designated to provide “Computational tools for representation theory and the Langlands program. Explore concepts in representation theory without writing a single line of code.” as they explain on the title webpage of the project. They kindly offered to port our implementation of the algorithm to TypeScript in such a way that it now has a user-friendly

¹<https://github.com/LovroGreganic/Franke-filtration>

²<https://langlandsprograms.com/>

graphical interface, and is offered among the tools of *The Langlands Programs*. We are grateful for all their efforts and including our algorithm in *The Langlands Programs*.

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2. THE GENERAL LINEAR GROUP

Let F be an algebraic number field with the ring of adèles \mathbb{A} , and the subring of finite adèles \mathbb{A}_f . For a given place v of F , let F_v denote the completion of F at v . If v is non-archimedean, we denote by \mathfrak{o}_v the ring of integers of F_v .

Let GL_n be the general linear group viewed as an algebraic group over F . For a given F -algebra \mathcal{R} , the group $GL_n(\mathcal{R})$ of \mathcal{R} -points of GL_n is the group of invertible elements in the matrix algebra of $n \times n$ matrices with entries in \mathcal{R} .

Let B be the Borel subgroup of GL_n consisting of upper-triangular matrices, which is fixed once and for all. The standard parabolic subgroups of GL_n defined over F are those parabolic F -subgroups of GL_n that contain B . They are in bijection with ordered partitions of n into positive integers. Given such a partition $\underline{n} = (n_1, \dots, n_k)$, the corresponding parabolic subgroup $P_{\underline{n}}$ is the subgroup of GL_n consisting of all block-upper-triangular matrices in GL_n with blocks of sizes n_1, n_2, \dots, n_k along the diagonal. Let $P_{\underline{n}} = L_{\underline{n}}N_{\underline{n}}$ be a Levi decomposition of $P_{\underline{n}}$, where $L_{\underline{n}}$ is the Levi factor, and $N_{\underline{n}}$ the unipotent radical. Then, the Levi factor is isomorphic to

$$L_{\underline{n}} \cong GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k},$$

and consists of all block-diagonal matrices with blocks of sizes n_1, n_2, \dots, n_k along the diagonal.

In this paper, all parabolic subgroups are standard, unless specified otherwise. We often suppress \underline{n} from the notation, if it is clear from the context. Thus, we write $P = LN$ for the Levi decomposition of the parabolic subgroup $P = P_{\underline{n}}$.

For a given parabolic subgroup P , corresponding to the ordered partition \underline{n} as above, let

$$\check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $X^*(P)$ denotes the \mathbb{Z} -module of all F -rational characters of P , and let $\check{\mathfrak{a}}_{P, \mathbb{C}}$ be its complexification. Then, $\check{\mathfrak{a}}_{P, \mathbb{C}}$ is a k -dimensional complex vector space. We fix an isomorphism to \mathbb{C}^k in such a way that $\underline{s} = (s_1, s_2, \dots, s_k) \in \mathbb{C}^k$ corresponds to the character $\nu_{\underline{s}}$ of the Levi factor $L(\mathbb{A})$ given by the assignment

$$\nu_{\underline{s}}(l) = |\det l_1|^{s_1} |\det l_2|^{s_2} \cdots |\det l_k|^{s_k},$$

where $l = (l_1, l_2, \dots, l_k) \in L(\mathbb{A})$ with $l_j \in GL_{n_j}(\mathbb{A})$ for all $j = 1, 2, \dots, k$.

Let $\check{\mathfrak{a}}_P^{GL_n}$ and $\check{\mathfrak{a}}_{P, \mathbb{C}}^{GL_n}$ denote the subspace of $\check{\mathfrak{a}}_P$ and $\check{\mathfrak{a}}_{P, \mathbb{C}}$, respectively, corresponding to characters trivial on the center. These subspaces are given by the condition

$$n_1 s_1 + n_2 s_2 + \cdots + n_k s_k = 0,$$

and thus $(k-1)$ -dimensional. The positive Weyl chamber in $\check{\mathfrak{a}}_P^{GL_n}$ is given by the condition $s_1 > s_2 > \cdots > s_k$, and its closure by $s_1 \geq s_2 \geq \cdots \geq s_k$.

Restriction of characters gives rise to an embedding ι of $\check{\mathfrak{a}}_P^{GL_n}$ into $\check{\mathfrak{a}}_B^{GL_n}$, and $\check{\mathfrak{a}}_P$ into $\check{\mathfrak{a}}_B$. It is given in coordinates by the assignment

$$\iota(\underline{s}) = \iota((s_1, s_2, \dots, s_k)) = (s_1, \dots, s_1, s_2, \dots, s_2, \dots, s_k, \dots, s_k),$$

where $\underline{s} = (s_1, \dots, s_k)$ is an element of $\check{\mathfrak{a}}_P^{GL_n}$ or $\check{\mathfrak{a}}_P$, and coordinate s_j occurs n_j times on the right-hand side. Abusing the notation, we use ι to denote such embedding of $\check{\mathfrak{a}}_R^{GL_n}$ and $\check{\mathfrak{a}}_R$ for any parabolic subgroup R .

Let W be the Weyl group of GL_n with respect to a maximal split torus in B . It is isomorphic to the symmetric group \mathfrak{S}_n of permutations of n letters. The Weyl group W_L of the Levi factor L is isomorphic to a direct product

$$W_L \cong \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \dots \times \mathfrak{S}_{n_k}$$

of symmetric groups. Let W^P be the set of Kostant representatives for P , that is, the set of minimal length representatives for the right cosets of W_L in W . As in [22, p. 13], we denote by $W(P)$ the set of Kostant representatives $w \in W^P$ such that wLw^{-1} is again a standard Levi factor. In our case, the elements $w \in W(P)$ can be identified with permutations of k letters. In this identification $w \in W(P)$ corresponds to a permutation of diagonal blocks in the Levi factor L .

We fix throughout the paper a maximal compact subgroup K of $GL_n(\mathbb{A})$ as a product of maximal compact subgroups K_v of $GL_n(F_v)$ over all places of F . At non-archimedean places $K_v = GL_n(\mathfrak{o}_v)$. The archimedean part of K is denoted by K_∞ .

Let $\mathfrak{gl}_{n,\infty}$ denote the complexification of the real Lie algebra of the archimedean component of $GL_n(\mathbb{A})$, that is, of the Lie group given as a product of $G(F_v)$ over archimedean places of F .

3. AUTOMORPHIC FORMS AND THEIR CUSPIDAL SUPPORT

Let $\mathcal{A} = \mathcal{A}(GL_n, \omega)$ denote the space of smooth K -finite automorphic forms on $GL_n(\mathbb{A})$ of central character ω , as defined in [4]. Although it carries only the structure of a $(\mathfrak{gl}_{n,\infty}, K_\infty; GL_n(\mathbb{A}_f))$ -module, abusing the language, we refer to such a module as an automorphic representation of $GL_n(\mathbb{A})$, if it can be realized on a constituent of the space \mathcal{A} . Irreducible summands in the subspace of cuspidal automorphic forms in \mathcal{A} are referred to as (irreducible) cuspidal automorphic representations. Such a representation is called unitary if it is isomorphic to the space of smooth K -finite elements in an irreducible constituent of the cuspidal part of the spectral decomposition of the space of L^2 automorphic forms on $GL_n(\mathbb{A})$.

Let $P = LN$ be the parabolic subgroup of GL_n corresponding to the ordered partition $\underline{n} = (n_1, n_2, \dots, n_k)$, as in Section 2. Let

$$\pi^u \cong \pi_1 \otimes \pi_2 \otimes \dots \otimes \pi_k$$

be an irreducible unitary cuspidal automorphic representation of $L(\mathbb{A})$, where π_j is a unitary cuspidal automorphic representation of $GL_{n_j}(\mathbb{A})$. We assume that the product of central characters of π_j match the central character ω . Fix an element $\underline{s} = (s_1, s_2, \dots, s_k) \in \check{\mathfrak{a}}_P$. Then,

$$\pi \cong \pi^u \otimes \nu_{\underline{s}} \cong \pi_1 |\det|^{s_1} \otimes \pi_2 |\det|^{s_2} \otimes \dots \otimes \pi_k |\det|^{s_k}$$

is a cuspidal automorphic representation of $L(\mathbb{A})$ that serves as representative of a fixed cuspidal support for automorphic forms in \mathcal{A} . Without loss of generality, we will assume that \underline{s} is in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P$, that is, $s_1 \geq s_2 \geq \dots \geq s_k$. This can be achieved by replacing P and π by appropriate representatives in their associate classes.

For future use, we introduce some notation. Let

$$\mathcal{E}(\pi) = \{s_1, s_2, \dots, s_k\}$$

be the multiset of exponents appearing in the cuspidal support, that is, $\mathcal{E}(\pi)$ is the sequence $\underline{s} = (s_1, s_2, \dots, s_k)$ viewed as a multiset. Given a unitary cuspidal representation σ of $GL_m(\mathbb{A})$, where m is a positive integer, let $\mathcal{E}_\sigma(\pi)$ be the submultiset of $\mathcal{E}(\pi)$ containing all exponents s_j such that the corresponding π_j is isomorphic to σ . Then, $\mathcal{E}(\pi)$ is the multiset sum

$$\mathcal{E}(\pi) = \bigsqcup_{\sigma} \mathcal{E}_\sigma(\pi).$$

Since $\mathcal{E}(\pi)$ is finite, there is only finitely many representations σ such that $\mathcal{E}_\sigma(\pi)$ is non-empty.

Let $(\{P\}, \varphi(\pi))$ be the fixed cuspidal support as above. We denote by $\mathcal{A}_{\{P\}, \varphi(\pi)}$ the space of all automorphic forms in \mathcal{A} with cuspidal support in the associate class $(\{P\}, \varphi(\pi))$. It is an automorphic representation of $GL_n(\mathbb{A})$, in the sense introduced above. The main goal of this paper is to present an algorithm that explicitly computes the Franke filtration of $\mathcal{A}_{\{P\}, \varphi(\pi)}$.

4. RESIDUAL SPECTRUM OF THE GENERAL LINEAR GROUP

The residual spectrum of the general linear group is determined by Mœglin and Waldspurger [21]. We recall it here and introduce the formalism of Bernstein–Zelevinsky segments [1], [25] in that context.

Let σ be a unitary cuspidal automorphic representation of $GL_m(\mathbb{A})$, for some positive integer m . Let $k > 1$ be a positive integer such that $mk = n$. Then, the parabolically induced representation

$$\text{Ind}_{P_{(m, m, \dots, m)}(\mathbb{A})}^{GL_n(\mathbb{A})} \left(\sigma | \det |^{\frac{k-1}{2}} \otimes \sigma | \det |^{\frac{k-3}{2}} \otimes \dots \otimes \sigma | \det |^{-\frac{k-1}{2}} \right)$$

admits a unique irreducible quotient, denoted by $J(\sigma, k)$, that belongs to the residual spectrum of $GL_n(\mathbb{A})$. Conversely, all constituents of the residual spectrum of $GL_n(\mathbb{A})$ are obtained in this way for appropriate σ and $k > 1$. The case of $k = 1$ trivially gives all unitary cuspidal automorphic representations of $GL_n(\mathbb{A})$, because $J(\sigma, 1) = \sigma$.

A segment of cuspidal automorphic representations of $GL_m(\mathbb{A})$ is any tensor product of the form

$$\Delta = \Delta(\sigma, [a, b]) = \sigma | \det |^b \otimes \sigma | \det |^{b-1} \otimes \dots \otimes \sigma | \det |^a,$$

where a and b are real numbers such that $b - a + 1$ is a positive integer called the length of the segment. The segment $\Delta(\sigma, [a, b])$ is the cuspidal support of the twisted discrete spectrum representation

$$J(\sigma, b - a + 1) | \det |^{\frac{a+b}{2}}.$$

All such twisted discrete spectrum representations of $GL_n(\mathbb{A})$ for all $n \geq 1$ are in one-to-one correspondence with the set of all segments. More precisely, the twisted discrete spectrum representation

$$J(\sigma, k) | \det |^s$$

has the cuspidal support represented by the segment

$$\Delta \left(\sigma, \left[s - \frac{k-1}{2}, s + \frac{k-1}{2} \right] \right).$$

This correspondence allows the study of discrete spectrum in terms of Bernstein–Zelevinsky segments.

5. FRANKE FILTRATION FOR THE GENERAL LINEAR GROUP IN TERMS OF SEGMENTS

We now define the Franke filtration of the space $\mathcal{A}_{\{P\},\varphi(\pi)}$ of automorphic forms on $GL_n(\mathbb{A})$, with cuspidal support in the associate class represented by the cuspidal automorphic representation π of the Levi factor $L(\mathbb{A})$ of the parabolic subgroup P , which is introduced in Section 3. Recall that

$$\pi \cong \pi_1 |\det|^{s_1} \otimes \cdots \otimes \pi_k |\det|^{s_k},$$

where π_j is a unitary cuspidal automorphic representation of $GL_{n_j}(\mathbb{A})$, and $\underline{s} = (s_1, \dots, s_k) \in \check{\mathfrak{a}}_P$ is in the closure of the positive Weyl chamber, that is, $s_1 \geq s_2 \geq \cdots \geq s_k$. From this point on, we further assume that the considered space of automorphic forms is normalized in such a way that $\underline{s} = (s_1, \dots, s_k) \in \check{\mathfrak{a}}_P^{GL_n}$, i.e., $n_1 s_1 + \cdots + n_k s_k = 0$. This assumption is just a convenient choice of coordinates, with no loss in generality. It can always be achieved using a twist by an appropriate character.

The Franke filtration is defined in terms of certain groupoid of triples as follows. Let $\mathcal{M} = \mathcal{M}_{\{P\},\varphi(\pi)}$ be the groupoid in which the objects are the triples (R, Π, \underline{z}) , where

- R is a parabolic subgroup that contains an element of the associate class $\{P\}$ represented by P ,
- Π is a unitary discrete spectrum representation of the Levi factor $L_R(\mathbb{A})$,
- \underline{z} is an element in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$,

such that $\Pi \otimes \nu_{\underline{z}}$ has cuspidal support in $(\{P\}, \varphi(\pi))$. The set of isomorphisms from a triple (R, Π, \underline{z}) to a (possibly equal) triple $(R', \Pi', \underline{z}')$ consists of all $w \in W(R)$ such that w conjugates R to R' , Π to Π' , and \underline{z} to \underline{z}' .

In the case of GL_n , the groupoid \mathcal{M} can be described in terms of Bernstein–Zelevinsky segments introduced in Section 4. This is the subject of the Grbac–Grobner Lemma [13, Lemma 3.1], which says that the set of triples in \mathcal{M} is in finite-to-one correspondence with the set of all partitions of the cuspidal support π in segments. For convenience of the reader we now recall its statement.

Lemma 5.1 (Grbac–Grobner [13, Lemma 3.1]). *Let P and π be as above. Let*

$$\Delta_t = \Delta(\sigma_t, [a_t, b_t]), \quad t = 1, 2, \dots, l,$$

be a partition of the cuspidal support π in segments, where σ_t is a unitary cuspidal automorphic representation of $GL_{m_t}(\mathbb{A})$. Assume that Δ_t are ordered in such a way that

$$\frac{a_t + b_t}{2} \geq \frac{a_{t+1} + b_{t+1}}{2}, \quad t = 1, 2, \dots, l-1.$$

Then, one of the triples in \mathcal{M} that belongs to the finite set corresponding to the given partition Δ_t , $t = 1, \dots, l$, is the triple $(R_0, \Pi_0, \underline{z}_0)$ given as follows. The parabolic subgroup R_0 corresponds to the partition

$$((b_1 - a_1 + 1)m_1, \dots, (b_l - a_l + 1)m_l),$$

the discrete spectrum representation Π_0 of $L_{R_0}(\mathbb{A})$ is the tensor product

$$\Pi_0 \cong J(b_1 - a_1 + 1, \sigma_1) \otimes \cdots \otimes J(b_l - a_l + 1, \sigma_l),$$

and the element \underline{z}_0 in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_{R_0}^{GL_n}$ is

$$\underline{z}_0 = \left(\frac{a_1 + b_1}{2}, \dots, \frac{a_l + b_l}{2} \right).$$

Other triples (R, Π, \underline{z}) in the finite set corresponding to the given partition Δ_t , $t = 1, \dots, l$, are obtained from $(R_0, \Pi_0, \underline{z}_0)$ by permutations of blocks with equal value of $\frac{a_t + b_t}{2}$.

There is a functor $M = M_{\{P\}, \varphi(\pi)}$ from the groupoid \mathcal{M} to the category of $(\mathfrak{gl}_{n, \infty}, K_\infty; GL_n(\mathbb{A}_f))$ -modules, given on objects by the assignment

$$M((R, \Pi, \underline{z})) = \text{Ind}_{R(\mathbb{A})}^{GL_n(\mathbb{A})} (\Pi \otimes \nu_{\underline{z}}) \otimes S(\check{\mathfrak{a}}_{R, \mathbb{C}}^{GL_n})$$

where (R, Π, \underline{z}) is a triple in \mathcal{M} , and $S(\check{\mathfrak{a}}_{R, \mathbb{C}}^{GL_n})$ is the symmetric algebra on $\check{\mathfrak{a}}_{R, \mathbb{C}}^{GL_n}$. We omit here a precise definition of the functor M on isomorphisms in \mathcal{M} . It is given in terms of standard intertwining operators as in [6, p. 234].

The Franke filtration depends on the choice of an integer-valued function $T = T_{\{P\}, \varphi(\pi)}$ on the set $\mathcal{S} = \mathcal{S}_{\{P\}, \varphi(\pi)}$ of all inclusions $\iota(\underline{z})$ of third entries of triples in \mathcal{M} , such that T preserves the partial order on \mathcal{S} defined by $\iota(\underline{z}) \succ \iota(\underline{z}')$ if and only if $\iota(\underline{z}) \neq \iota(\underline{z}')$ and the partial sums for $j = 1, \dots, n$ satisfy

$$\zeta_1 + \dots + \zeta_j \leq \zeta'_1 + \dots + \zeta'_j,$$

where $\iota(\underline{z}) = (\zeta_1, \dots, \zeta_n)$ and $\iota(\underline{z}') = (\zeta'_1, \dots, \zeta'_n)$ are written in coordinates on $\check{\mathfrak{a}}_B$. We always fix, as we may, the choice of T such that its image is the set $\{0, 1, \dots, \ell\}$ of consecutive integers for an appropriate ℓ .

For a fixed choice of function T as above, the Franke filtration of the space $\mathcal{A}_{\{P\}, \varphi(\pi)}$ is the descending filtration

$$\mathcal{A}_{\{P\}, \varphi(\pi)} = \mathcal{A}_{\{P\}, \varphi(\pi)}^0 \supsetneq \mathcal{A}_{\{P\}, \varphi(\pi)}^1 \supsetneq \dots \supsetneq \mathcal{A}_{\{P\}, \varphi(\pi)}^\ell \supsetneq \mathcal{A}_{\{P\}, \varphi(\pi)}^{\ell+1} = \{0\},$$

where the consecutive quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}_{\{P\}, \varphi(\pi)}^i / \mathcal{A}_{\{P\}, \varphi(\pi)}^{i+1} &\cong \text{colim}_{\substack{(R, \Pi, \underline{z}) \in \mathcal{M} \\ T(\iota(\underline{z})) = i}} M((R, \Pi, \underline{z})) \\ &\cong \text{colim}_{\substack{(R, \Pi, \underline{z}) \in \mathcal{M} \\ T(\iota(\underline{z})) = i}} \text{Ind}_{R(\mathbb{A})}^{GL_n(\mathbb{A})} (\Pi \otimes \nu_{\underline{z}}) \otimes S(\check{\mathfrak{a}}_{R, \mathbb{C}}^{GL_n}), \end{aligned}$$

as $(\mathfrak{gl}_{n, \infty}, K_\infty; GL_n(\mathbb{A}_f))$ -modules, for $i = 0, 1, \dots, \ell$, where colim stands for the colimit of the functor M on \mathcal{M} , as in [20]. This is essentially the main result of [6, Sect. 6], refined by fixing the cuspidal support as in [7], and then describing the quotients of the filtration in terms of explicit parabolically induced representations. The latter approach is carried out for the first time in [8], see also [10] for more details, and applied in explicit calculations of automorphic cohomology and cohomology of arithmetic groups in [12], [17], [15], [16], [14], [23].

In the case of GL_n , the colimits above can be calculated explicitly as in the appendix of [10]. The following abstract lemma shows that the assumptions of [10, Thm. A.2] are met for any groupoid in which there is an isomorphism between every pair of objects.

Lemma 5.2. *Let \mathcal{N} be a groupoid with finitely many objects X_0, X_1, \dots, X_m . Let $W_{i,j}$ denote the set of isomorphisms from X_i to X_j in \mathcal{N} , and write $W_i := W_{i,i}$ for the set of automorphisms of X_i . Suppose that $W_{i,j}$ is not empty for any pair (i, j) . For each pair (i, j) , fix an isomorphism $w_{i,j} \in W_{i,j}$ in such a way that $w_{j,i} = w_{i,j}^{-1}$, and $w_i := w_{i,i}$ is the identity on X_i . Then,*

$$W_{i,j} = w_{i,j} W_i = W_j w_{i,j},$$

and in particular, W_i and W_j are conjugate by $w_{i,j}$.

Proof. It is obvious that the isomorphisms that belong to $w_{i,j}W_i$ and $W_jw_{i,j}$ are in $W_{i,j}$. Conversely, let $w \in W_{i,j}$ be arbitrary. Then, $w_{i,j}^{-1}w$ is clearly in W_i , and $w w_{i,j}^{-1}$ in W_j , so that $w \in w_{i,j}W_i$ and $w \in W_jw_{i,j}$. \square

As a consequence, we obtain the following explicit description of the quotients of the Franke filtration in the case of GL_n .

Theorem 5.3. *The quotients of the Franke filtration of the space $\mathcal{A}_{\{P\},\varphi(\pi)}$ of automorphic forms on $GL_n(\mathbb{A})$ with cuspidal support in the associate class $(\{P\}, \varphi(\pi))$ are isomorphic to*

$$\begin{aligned} \mathcal{A}_{\{P\},\varphi(\pi)}^i / \mathcal{A}_{\{P\},\varphi(\pi)}^{i+1} &\cong \bigoplus_{\substack{\{\Delta_t\} \\ T(\iota(\underline{z}_0))=i}} M((R_0, \Pi_0, \underline{z}_0))^{W_0} \\ &\cong \bigoplus_{\substack{\{\Delta_t\} \\ T(\iota(\underline{z}_0))=i}} \left(\text{Ind}_{R_0(\mathbb{A})}^{GL_n(\mathbb{A})} (\Pi_0 \otimes \nu_{\underline{z}_0}) \otimes S(\check{\mathfrak{a}}_{R_0, \mathbb{C}}^{GL_n}) \right)^{W_0} \end{aligned}$$

as $(\mathfrak{gl}_{n,\infty}, K_\infty; GL_n(\mathbb{A}_f))$ -modules, for $i = 0, 1, \dots, \ell$. The direct sum ranges over all partitions $\{\Delta_t\}$ of the cuspidal support π in segments such that $T(\iota(\underline{z}_0)) = i$, where $(R_0, \Pi_0, \underline{z}_0)$ is a triple corresponding to $\{\Delta_t\}$ as in Lemma 5.1. The exponent W_0 stands for the invariants under all intertwining operators $M(w)$ with $w \in W_0$, where W_0 is the group of automorphisms of the triple $(R_0, \Pi_0, \underline{z}_0)$. It is isomorphic to a direct product

$$W_0 \cong \mathfrak{S}_{t_1} \times \mathfrak{S}_{t_2} \times \dots \times \mathfrak{S}_{t_r}$$

of symmetric groups, where t_1, t_2, \dots, t_r are the multiplicities in the partition $\{\Delta_t\}$ viewed as a multiset of segments. In particular $t_1 + \dots + t_r = t$, but since \mathfrak{S}_1 is the trivial group, we can omit all factors associated with $t_j = 1$.

Proof. Recall that an isomorphism $w \in W(R)$ from a triple (R, Π, \underline{z}) to a triple $(R', \Pi', \underline{z}')$ can be identified with a permutation of blocks. More precisely, it permutes the diagonal blocks of the Levi factor of R , the corresponding factors in the tensor product of Π , and the corresponding coordinates of \underline{z} . Since both \underline{z} and \underline{z}' belong to the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$ and $\check{\mathfrak{a}}_{R'}^{GL_n}$, respectively, the permutation should not change the non-increasing property of the sequence \underline{z} . Hence, $w \in W(R)$ may permute only the equal coordinates in \underline{z} , and thus $\underline{z} = \underline{z}'$ as l -tuples of real numbers.

In particular, given a partition $\{\Delta_t\}$ of the cuspidal support in segments, the corresponding finite set of triples as in Lemma 5.1 is given by all permutations of blocks in the triple $(R_0, \Pi_0, \underline{z}_0)$ such that \underline{z}_0 is preserved as an l -tuple of real numbers. Hence, there is an isomorphism between any pair of triples in the finite set corresponding to a given partition.

We claim that there are no isomorphisms between the triples that correspond to different partitions of the cuspidal support in segments. This follows from the fact that the partition in segments completely determines the factors in the tensor product of Π and the corresponding coordinates in \underline{z} . Therefore, if these can be obtained from each other by a permutation, it would follow that the partitions in segments are obtained as permutations of each other, that is, they arise from the same partition.

To summarize, we proved that the partition of the groupoid \mathcal{M} of triples in finite sets corresponding to partitions of the cuspidal support in segments has the property that any pair of triples

in the same finite set admits an isomorphism as in Lemma 5.2, and there are no isomorphisms between triples from different finite sets. Observe also that the triples in the same finite set have equal \underline{z} , so that $T(\iota(\underline{z}))$ are equal and thus contribute to the same quotient of the filtration.

According to [20, Thm. V.3.1], since there are no isomorphisms between finite sets corresponding to different partitions of the cuspidal support in segments, the colimit becomes a direct sum of colimits for the finite sets of triples corresponding to different partitions. This gives the direct sum over partitions in segments as in the theorem.

It remains to compute the colimits over the finite sets of triples that correspond to a partition of the cuspidal support in segments. These groupoids satisfy the assumptions of Lemma 5.2, with $X_0 = (R_0, \Pi_0, \underline{z}_0)$. Hence, we may apply [10, Thm. A.2] to obtain the description of summands in terms of invariants under intertwining operators associated with $w \in W_0$, where W_0 is the group of automorphisms of $X_0 = (R_0, \Pi_0, \underline{z}_0)$.

Finally, the group of automorphisms W_0 can be identified with the group of permutations of blocks of $(R_0, \Pi_0, \underline{z}_0)$ that preserve R_0 , Π_0 and \underline{z}_0 . Since these are determined by the segments in the partition of the cuspidal support, this is equivalent to permutation that preserves the order of segments in the partition. Such permutation may only permute equal segments. Hence, W_0 is isomorphic to the product of symmetric groups \mathfrak{S}_{t_j} , where t_j are the multiplicities of segments in the partition. \square

At the end of this section, we state the following elementary lemma, which implies that the restriction on the central character does not affect the partition in segments and partial order required for the Franke filtration. This allows us to ignore the condition on the central character in the implementation of the algorithm.

Lemma 5.4. *Let $(\{P\}, \varphi(\pi))$ be the fixed cuspidal support as above, represented by a cuspidal automorphic representation*

$$\pi \cong \pi_1 |\det|^{s_1} \otimes \pi_2 |\det|^{s_2} \otimes \cdots \otimes \pi_k |\det|^{s_k}$$

of the Levi factor $L(\mathbb{A})$, where π_j is a unitary cuspidal automorphic representation of $GL_{n_j}(\mathbb{A})$, and s_j are real numbers such that

$$s_1 \geq s_2 \geq \cdots \geq s_k \quad \text{and} \quad n_1 s_1 + n_2 s_2 + \cdots + n_k s_k = 0.$$

Let $\pi' \cong \pi \nu_{\underline{s}'}$ be the twist of π by the character corresponding to $\underline{s}' = (s, s, \dots, s)$, where $s \in \mathbb{R}$, that is,

$$\pi' \cong \pi_1 |\det|^{s_1+s} \otimes \pi_2 |\det|^{s_2+s} \otimes \cdots \otimes \pi_k |\det|^{s_k+s}.$$

Then, the multisets of exponents satisfy

$$\mathcal{E}(\pi') = \mathcal{E}(\pi) + s \quad \text{and} \quad \mathcal{E}_\sigma(\pi') = \mathcal{E}_\sigma(\pi) + s$$

for every unitary cuspidal automorphic representation σ of $GL_m(\mathbb{A})$.

The partitions in segments of π are in natural one-to-one correspondence with partitions in segments of π' given by twisting all the factors in the segments in a partition of π by $|\det|^s$. Given two pairs of corresponding partitions, let $(R_0, \Pi_0, \underline{z}_0)$ and $(R'_0, \Pi'_0, \underline{z}'_0)$ be the triples associated as in Lemma 5.1 with one pair of partitions, and $(S_0, \Sigma_0, \underline{\zeta}_0)$ and $(S'_0, \Sigma'_0, \underline{\zeta}'_0)$ the triples associated with the other pair. Then,

$$\iota(\underline{z}_0) \succ \iota(\underline{\zeta}_0) \quad \text{if and only if} \quad \iota(\underline{z}'_0) \succ \iota(\underline{\zeta}'_0)$$

in the partial order required for the Franke filtration.

Proof. The claim about sets of exponents is clear. Adding s to all exponents implies that in partitions of π and π' in segments the only difference is the added s in the exponents. This is exactly what twist by $|\det|^s$ takes care of, and thus we obtain the one-to-one correspondence as claimed.

The partial order is defined in terms of partial sums of coordinates. Hence, we compare the partial sums for $(R_0, \Pi_0, \underline{z}_0)$ and $(R'_0, \Pi'_0, \underline{z}'_0)$ that are associated with a pair of corresponding partitions of π and π' . Let $\{\Delta_t\}$, $t = 1, \dots, l$, be the partition of π from which $(R_0, \Pi_0, \underline{z}_0)$ arises. Write

$$\Delta_t = \Delta(\sigma_t, [a_t, b_t]).$$

Then the corresponding partition of π' is $\{\Delta'_t\}$, $t = 1, \dots, l$, where

$$\Delta'_t = \Delta(\sigma_t, [a_t + s, b_t + s]).$$

Since

$$\frac{(a_t + s) + (b_t + s)}{2} = \frac{a_t + b_t}{2} + s \quad \text{and} \quad (b_t + s) - (a_t + s) + 1 = b_t - a_t + 1,$$

it follows that

$$\underline{z}'_0 = \underline{z}_0 + \underline{s}',$$

and the same for the inclusions

$$\iota(\underline{z}'_0) = \iota(\underline{z}_0) + \iota(\underline{s}'),$$

where $\iota(\underline{s}') = (s, s, \dots, s)$. This shows that j -th partial sum of coordinates of $\iota(\underline{z}'_0)$ differs from the j -th partial sum of coordinates of $\iota(\underline{z}_0)$ by js . Since this holds for any pair of triples, the claim on the partial order is proved. \square

6. THE ALGORITHM AND ITS IMPLEMENTATION

We begin with a high level description of the algorithm for explicit calculation of the Franke filtration of the space $\mathcal{A}_{\{P\}, \varphi(\pi)}$ of automorphic forms supported in the associate class $(\{P\}, \varphi(\pi))$. The algorithm relies on the results of Section 5.

Input: The user input to the algorithm is the cuspidal support $(\{P\}, \varphi(\pi))$, that is, the sizes of diagonal blocks of the Levi factor L of P , and for each unitary cuspidal automorphic representation σ of $GL_m(\mathbb{A})$ that appears in π the multiset of exponents $\mathcal{E}_\sigma(\pi)$.

Step A: Given a multiset of exponents $\mathcal{E}_\sigma(\pi)$ associated with a unitary cuspidal automorphic representation σ that appears in the tensor product for π , the algorithm first determines all partitions of $\mathcal{E}_\sigma(\pi)$ in subsets of the form $\{a, a + 1, \dots, b\}$, where $a, b \in \mathbb{R}$ and $b - a + 1$ is a positive integer. Since exponents belonging to $\mathcal{E}_\sigma(\pi)$ for different σ cannot be combined in a segment, this is sufficient to obtain all partitions of π in segments.

Step B: The definition of the Franke filtration is based on the partial order on the set of partitions of $\mathcal{E}_\sigma(\pi)$ for all σ that appear in π . The partial order can be completely understood from the segments forming the partitions, as follows from Lemma 5.1 and Theorem 5.3. The second step of the algorithm is the calculation of the partial order. Iteratively taking the maximal elements in the partial order constructs the Franke filtration, up to taking invariants of intertwining operators arising from the colimits that are explicitly given in terms of segments as part of Theorem 5.3.

Output: Following Theorem 5.3, the main output of the algorithm is the inducing data for the summands in the consecutive quotients $\mathcal{A}_{\{P\},\varphi(\pi)}^i / \mathcal{A}_{\{P\},\varphi(\pi)}^{i+1}$ of the Franke filtration, and an indicator in case of summands for which the invariants under certain intertwining operators should be taken. Besides that, the user can see the list of partitions in segments of the cuspidal support, as well as all the values of \underline{z} and $\iota(\underline{z})$.

In the rest of the section we explain the details in our implementation of the above algorithm. The complete Python code is available at the GitHub repository of the second named author.³

6.1. Input. The first input from the user is the number k^b of non-isomorphic unitary cuspidal automorphic representations in the representative

$$\pi \cong \pi_1 | \det |^{s_1} \otimes \pi_2 | \det |^{s_2} \otimes \cdots \otimes \pi_k | \det |^{s_k}$$

of the cuspidal support $(\{P\}, \varphi(\pi))$. Let ρ_j , $j = 1, \dots, k^b$, denote these representations, that is, unitary cuspidal automorphic representations of some $GL_{m_j}(\mathbb{A})$ such that $\mathcal{E}_{\rho_j}(\pi)$ is non-empty. The next input from the user is, for each representation ρ_j , the size m_j of the corresponding general linear group, and the multiset $\mathcal{E}_{\rho_j}(\pi)$ as a comma-separated list. The allowed exponents in $\mathcal{E}_{\rho_j}(\pi)$ are rational numbers. The input data are represented in a computer as a list of lists, i.e., a list of length k^b in which the j -th entry is the list of exponents in $\mathcal{E}_{\rho_j}(\pi)$ with m_j added at the end.

6.2. Step A. In this step all possible partitions in segments of the cuspidal support represented by π are obtained. Clearly, the problem of partitions in segments reduces to the partition of submultisets of $\mathcal{E}_{\rho_j}(\pi)$ given by intersection with classes in the additive group \mathbb{R}/\mathbb{Z} . This follows from the fact that the segment cannot be formed with different ρ_j , and from exponents in different classes of \mathbb{R}/\mathbb{Z} .

Consider the multiset of exponents in $\mathcal{E}_{\rho_j}(\pi)$ intersected with a class in \mathbb{R}/\mathbb{Z} represented by $y \in \mathbb{R}$. Subtracting y from all the exponents does not change their differences, but transforms all of them to integers. Hence, we may construct the partition in segments of the multiset of integers, and adding back y get the partition of the original multiset. This is achieved by a recursive algorithm that we now describe.

0. Let X be the input consisting of a multiset of integers, let \mathcal{P} be the empty set of segments, and x_{\max} a maximal integer in X .
1. Consider all segments that exist in X such that x_{\max} is the maximal element of the segment, and that are not longer than any of the segments with x_{\max} as the maximal element already in \mathcal{P} .
2. Take a segment S among segments considered in Step 1, put it in \mathcal{P} , replace X with $X \setminus S$, and replace x_{\max} with a new maximal element in the new set X . Do this step separately for each segment considered in Step 1.
3. With the new X , \mathcal{P} and x_{\max} defined in Step 2, go back to Step 1, unless X is empty. If X is empty, the algorithm finishes and \mathcal{P} is one of the partitions.

As a result of this step, we obtain all partitions of π in segments as lists in which each entry is a list representing a segment of the partition. More precisely, given a partition Δ_t , $t = 1, \dots, l$, where

$$\Delta_t = \Delta(\sigma_t, [a_t, b_t]),$$

³<https://github.com/LovroGreganic/Franke-filtration>

as in Lemma 5.1, the corresponding list is of length l , and the t -th entry is the list

$$b_t, b_t - 1, \dots, a_t + 1, a_t, j, m_j,$$

where j is the integer for which $\sigma_t \cong \rho_j$, and m_j is the corresponding size of the general linear group as above.

6.3. Step B. In this step, we first transform partitions in segments of π in triples of groupoid \mathcal{M} . This is a straightforward calculation using Lemma 5.1. Given a partition Δ_t , $t = 1, \dots, l$, as above, the corresponding triple is represented as a list of length l in which the t -th entry is the three-element list

$$\frac{a_t + b_t}{2}, j, m_j(b_t - a_t + 1),$$

where $\frac{a_t + b_t}{2}$ is the value of coordinate z_t in \underline{z} , the integer j is such that $\sigma_t \cong \rho_j$, and $m_j(b_t - a_t + 1)$ is the size of the corresponding diagonal block in L_R . These information completely determine the triple corresponding to the given partition.

The partial order required for the Franke filtration is determined using the NetworkX python package designed for network analysis.⁴ The partially ordered sets are implemented as directed graphs of their transitive reduction, that is, with edges representing order relations only between those comparable elements that cannot be reconstructed from the transitivity property of the partial order. In our implementation, we define the finite partially ordered set as the set of third entries \underline{z} of all triples obtained from partitions in segments of π , with the partial order defined by the conditions on partial sums of $\iota(\underline{z})$ given in Section 5. By searching the graph for nodes of out-degree zero, we repeatedly determine the triples that are maximal among the remaining triples. In that way, the triples are partitioned in classes that can be assigned the same value of function T , and thus they are partitioned according to the quotient of the filtration to which they contribute.

6.4. Output. Let $\ell \geq 0$ be the integer such that the range of the function T in the definition of the Franke filtration of $\mathcal{A}_{\{P\}, \varphi(\pi)}$ is the set $\{0, 1, \dots, \ell\}$. In other words, taking repeatedly the maximal elements in the partial order in Step B results with $\ell + 1$ classes of triples.

The triples that contribute to the i -th quotient $\mathcal{A}_{\{P\}, \varphi(\pi)}^i / \mathcal{A}_{\{P\}, \varphi(\pi)}^{i+1}$ of the filtration are taken as maximal elements in $(\ell + 1 - i)$ -th repeated use of the method for maximal elements in Step B. The output is then the list of quotients $\mathcal{A}_{\{P\}, \varphi(\pi)}^i / \mathcal{A}_{\{P\}, \varphi(\pi)}^{i+1}$ of the filtration given as a direct sum of contributions arising from the appropriate triples. These are given in terms of partitions in segments with the indicator for summands that require taking the invariants arising from the colimits.

More precisely, if the triple (R, Π, \underline{z}) that contributes to the quotient $\mathcal{A}_{\{P\}, \varphi(\pi)}^i / \mathcal{A}_{\{P\}, \varphi(\pi)}^{i+1}$ corresponds to the partition

$$\Delta_t = \Delta(\sigma_t, [a_t, b_t]), \quad t = 1, \dots, l,$$

then the summand in the output for that quotient is of the form

$$\operatorname{colim}_{[t_1, \dots, t_r]} \bigotimes_{t=1}^l \rho_j[a_t, b_t],$$

where j is the integer such that $\sigma_t \cong \rho_j$, and the colimit indicates that one should take invariants under intertwining operators associated with Weyl group elements given by permutations of equal

⁴<https://networkx.org/documentation/stable/index.html>

segments in the partition. The integers t_1, \dots, t_r stand for the multiplicities larger than one of segments in the partition.

For convenience, the output also contains some additional information obtained during computation. In particular, the list of third entries \underline{z} and their inclusions $\iota(\underline{z})$ of all triples in \mathcal{M} are listed, together with the quotient of the filtration to which they contribute and the corresponding partition in segments.

On the other hand, to simplify the output, we omit the symmetric algebras that should appear in every summand of every quotient of the filtration. The reason is that they can be easily recovered from the form of the summands, and thus it is not necessary to be specified in the output.

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