EISENSTEIN SERIES FOR RANK ONE UNITARY GROUPS AND SOME COHOMOLOGICAL APPLICATIONS

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Abstract. Let $U/\mathbb{Q}$ be a unitary group of $\mathbb{Q}$-rank one so that the group of real points $U(\mathbb{R}) \cong U(n,1)$. The group U is only quasi-split over $\mathbb{Q}$ if and only if $n = 1, 2$. The cohomology of a congruence subgroup of U is closely related to the theory of automorphic forms. This relation is best captured in the so-called automorphic cohomology spaces $H^*(U,\mathbb{C})$, a natural module under the action of the group $U(\mathbb{A}_f)$. This paper gives a structural account of the $U(\mathbb{A}_f)$-module structure of that part of the cohomology which is generated by residues or derivatives of Eisenstein series. In particular, we determine a set of arithmetic conditions, mainly given in terms of partial automorphic L-functions, subject to which residues of Eisenstein series may give rise to non-vanishing cohomology classes. The main task is, although the usual method due to Langlands-Shahidi is not applicable, to analyze the analytic behavior of suitable Eisenstein series and to determine the location of their possible poles.

Introduction

0.1. Let $F/\mathbb{Q}$ be an imaginary quadratic extension of the field of rational numbers, and let (V,h) be a non-degenerate hermitian space over F of dimension $n+1$, endowed with the hermitian form h. Suppose that (V,h) is not anisotropic, and that the conjugate of h under the conjugation of F with respect to $\mathbb{Q}$ is of signature $(n,1)$. We denote by $U = U(V,h)$ the unitary group attached to the hermitian space (V,h). It is a connected reductive algebraic group defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank one.[1] The group U is quasi-split over $\mathbb{Q}$ if and only if $n = 1$ or $n = 2$. We fix a good maximal compact subgroup $K \subset U(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of $\mathbb{Q}$, decomposed in its archimedean component K_∞ and its non-archimedean component K_f. Given an open compact subgroup $C \subset U(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the subring of finite adèles in $\mathbb{A}$, the deRham cohomology $H^*(Y_C,\mathbb{C})$ of the orbit space $Y_C := U(\mathbb{Q})\backslash U(\mathbb{A})/K_\infty C$ is defined. Passing over to the inductive limit

$$H^*(U,\mathbb{C}) := \mathrm{colim}_C H^*(Y_C,\mathbb{C})$$

over all open compact subgroups C of $U(\mathbb{A}_f)$ defines a natural object to study the cohomology of congruence subgroups of G. Indeed, the cohomology $H^*(U,\mathbb{C})$ is a $U(\mathbb{A}_f)$-module in a natural way, and the cohomology of the congruence subgroup $\Gamma = U(\mathbb{Q}) \cap C$ is obtained by taking the C-invariants, that is, $H^*(\Gamma,\mathbb{C}) = H^*(U,\mathbb{C})^C$. Thus, we are in the realm of the study of Picard modular varieties and their arithmetic.

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[1] By the classification of unitary groups over global fields this construction exhausts all unitary groups over $\mathbb{Q}$ of $\mathbb{Q}$-rank one.

The cohomology groups $H^*(U,\mathbb{C})$ are closely related to the theory of automorphic forms. Let $\mathcal{J}$ be the annihilator of the (conjugate dual) of the trivial representation in the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{g}_\infty$, where $\mathfrak{g}_\infty$ is the Lie algebra of $U(\mathbb{R}) \cong U(n,1)$. Then, $\mathcal{J}$ is an ideal of finite codimension in $\mathcal{Z}$, and we denote by

$$\mathcal{A} = \mathcal{A}_{\mathcal{J}}(U(\mathbb{Q})\backslash U(\mathbb{A}))$$

the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module of all smooth K-finite complex-valued functions of uniform moderate growth on the orbit space $U(\mathbb{Q})\backslash U(\mathbb{A})$ which are annihilated by some power of $\mathcal{J}$ (see [3] resp. [27, Sect. I.2.3]). Due to Franke [6], there is an isomorphism

$$H^*(U,\mathbb{C}) \xrightarrow{\sim} H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A})$$

with the relative Lie algebra cohomology attached to the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module $\mathcal{A}$.

The decomposition of the module $\mathcal{A}$ with respect to the cuspidal support of the automorphic representations in question, as obtained in [20] in general, gives rise to a corresponding decomposition in cohomology. In this case, since there is just one associate class of proper parabolic $\mathbb{Q}$-subgroups in U, say represented by P_0 with Levi component M_0, it takes the following form:

$$\begin{aligned} H^*(U,\mathbb{C}) &\cong H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathrm{cusp}}) \oplus \bigoplus_\pi H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \\ &\cong H^*_{\mathrm{cusp}}(U,\mathbb{C}) \oplus H^*_{\mathrm{Eis}}(U,\mathbb{C}). \end{aligned}$$

The first summand, called the cuspidal cohomology, is the cohomological counterpart of the submodule $\mathcal{A}_{\mathrm{cusp}} \subset \mathcal{A}$ of cuspidal automorphic forms. The second summand accounts for the modules $\mathcal{A}_\pi$ of automorphic forms with cuspidal support in the associate class of a (not necessarily unitary) cuspidal representation π of $M_0(\mathbb{A})$. Roughly spoken, the module $\mathcal{A}_\pi$ is spanned by all possible residues or derivatives (with respect to a one-dimensional complex parameter) of Eisenstein series attached to cuspidal automorphic forms of type π at values in the positive Weyl chamber defined by P_0 for which the infinitesimal character of the trivial representation is matched. Thus, as usual, we call this part the Eisenstein cohomology of U, denoted $H^*_{\mathrm{Eis}}(U,\mathbb{C})$. In this case of a $\mathbb{Q}$-group of rank one, residues of Eisenstein series are necessarily square-integrable automorphic forms, thus, their very existence leads to non-vanishing square-integrable classes in $H^*(U,\mathbb{C})$ which are not cuspidal.

This paper is concerned with the structure of the cohomology spaces $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ as a $U(\mathbb{A}_f)$-module. Aside from the study of the Franke filtration of the spaces $\mathcal{A}_\pi$ and an analysis of certain necessary conditions so that the cohomology spaces $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ are possibly non-vanishing, the main task is to analyze the analytic behavior of Eisenstein series, in particular, to determine the location of their poles. Note, since the group U is not quasi-split over $\mathbb{Q}$ for $n > 2$, the usual methodological approach due to Langlands-Shahidi is not at hand.

The results of this paper can be easily generalized to a more general setting, but for simplicity of exposition and to avoid technicalities which would not bring any new ideas, the paper is written over $\mathbb{Q}$. The analytic properties of the Eisenstein series hold, with the very same proof, for unitary groups of relative rank one defined with respect to arbitrary imaginary quadratic extension of algebraic number fields. The calculation of cohomology in the more general setting is slightly more involved. If the unitary group of relative rank one is defined with respect to an imaginary quadratic extension of a totally real algebraic number field, one may easily combine the results of this paper with the Küneth rule to describe Eisenstein cohomology of such unitary groups.

Similarly, the results may be generalized to the case of cohomology with respect to any non-trivial local system of coefficients arising from a finite-dimensional algebraic representation of the unitary group U. The finite-dimensional representation is given by its highest weight, which is zero for the trivial representation. If the highest weight is non-zero, i.e., the local system of coefficients is non-trivial, there is a slight modification of the necessary conditions for non-vanishing of cohomology in Sect. 2.1, and the parametrization of cohomological representations of $U(n,1)$ in Sect. 3. Otherwise, the computation of cohomology remains the same, and the analytic properties of the Eisenstein series typically become simpler to determine. For instance, if the highest weight is regular, it is proved in [30] that the point of evaluation of the Eisenstein series which matters for cohomology is always in the domain of convergence of the Eisenstein series, and thus, the Eisenstein series is holomorphic. In a sense, the case of the trivial coefficient system, which is considered in this paper, is the most complicated case from the point of view of the analytic properties of the Eisenstein series.

0.2. Analytic behavior of Eisenstein series – normalizing factor. The minimal parabolic subgroup P_0 has a Levi decomposition $P_0 = M_0N_0$ where M_0 is the Levi subgoup and N_0 its unipotent radical. The group M_0 splits into a direct product

$$M_0 \cong Res_{F/\mathbb{Q}}GL_1 \times U',$$

where $Res_{F/\mathbb{Q}}GL_1$ is the algebraic group over $\mathbb{Q}$ obtained from GL_1 over F by restriction of scalars, and U' a unitary group with $U'(\mathbb{R}) \cong U(n-1)$, and (for a non-archimedean place) with local groups $U'(\mathbb{Q}_p)$ of a form analogous to the case U.

Write $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s_0}$, where π^u is a unitary cuspidal automorphic representation of $M_0(\mathbb{A})$, and s_0 is a real number. Changing the representative π if necessary, we may always assume $s_0 \geq 0$. Let f_s be a section of the induced representations

$$I(s,\pi^u) = \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})}\left(\pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s}\right)$$

as in [7]. The Eisenstein series associated to π^u is defined as the analytic continuation from the cone of absolute convergence of the series

$$E(f_s,g) = \sum_{\gamma\in P_0(\mathbb{Q})\backslash U(\mathbb{Q})} f_s(\gamma g).$$

It is meromorphic in s with only finitely many poles with $s > 0$ and all other poles lie in the half-plane $Re(s) < 0$. Consider the Eisenstein series $E(f_s,g)$ evaluated at $s = s_0 > 0$. The possible pole at $s = s_0$ is at most simple. In any case, there exists a polynomial $F(s)$ such that $F(s)E(f_s,g)$ is holomorphic at $s = s_0$ for every section f_s. Thus, we may write the Taylor expansion of $F(s)E(f_s,g)$ around $s = s_0$. Then the space $\mathcal{A}_\pi$ is the span of all Taylor coefficients in that expansion. If the Eisenstein series $E(f_s,g)$ has a pole at $s = s_0 > 0$, then the residues span a $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-submodule $\mathcal{L}_\pi$ of $\mathcal{A}_\pi$ consisting of all square-integrable automorphic forms with the cuspidal support in (the associate class of) π. It follows that the Franke filtration of the spaces $\mathcal{A}_\pi$ is (at most) a two-step filtration, namely, $\mathcal{L}_\pi \subset \mathcal{A}_\pi$. The successive quotients of the filtration may be described as induced representations.

Let $\pi^u \cong \chi\otimes\sigma$ be a cuspidal automorphic representation of the Levi factor $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of the group of idèles $\mathbb{I}_F$, and σ a cuspidal automorphic representation of $U'(\mathbb{A})$. Let $E(f_s, g)$ be the Eisenstein series associated to a section f_s of the induced representation

$$I(s,\pi^u) = \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})}\left(\pi^u \otimes |\cdot|_{\mathbb{I}_F}^s\right) = \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})}\left(\chi|\cdot|_{\mathbb{I}_F}^s \otimes \sigma\right).$$

Our aim is to determine the poles of $E(f_s, g)$ at s such that $Re(s) > 0$, whose residues may possibly contribute to $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$. Necessarily, this is carried through under certain simplifying assumptions on π^u which are aligned with the demand of cohomological relevance.

The poles of the Eisenstein series $E(f_s, g)$ are the same as the poles of its constant term $E(f_s, g)_{P_0}$ along the parabolic subgroup P_0. The constant term can be expressed as

$$E(f_s, g)_{P_0} = \int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} E(f_s, ng)dn = f_s(g) + M(s, \pi^u, w)f_s(g),$$

where dn is the appropriate measure on the unipotent radical of P_0 and $M(s, \pi^u, w)$ is the standard intertwining operator on the induced representation $I(s, \pi^u)$, with w the unique non-trivial element of the relative Weyl group of the unitary group U. Thus, the poles of $E(f_s, g)$ coincide with the poles of the standard intertwining operator $M(s, \pi^u, w)$.

Let S be the finite set of places of $\mathbb{Q}$, containing the archimedean place, and such that, for a non-archimedean place p of $\mathbb{Q}$, we have $p \notin S$ if and only if the following three assertions hold

- the extension $F/\mathbb{Q}$ is not ramified over p,
- the group U, viewed as an algebraic group over $\mathbb{Q}_p$, is quasi-split over $\mathbb{Q}_p$,
- the representation π^u is unramified at p.

Now we can formulate the first main result which relates the possible poles $E(f_s, g)$ to the ones of a certain normalizing factor precisely defined in Section 5. Given as a product of local factors

$$r^S(s, \pi^u, w) = \prod_{p \notin S} r(s, \pi_p^u, w),$$

it involves partial automorphic L-functions, some of Rankin-Selberg type, or linked with (twisted) Asai L-functions attached to χ.

Theorem 0.1. *Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ is cohomological at the archimedean place. Suppose that σ_p is tempered for all non-archimedean places $p \in S$. Then, the poles of the Eisenstein series $E(f_s, g)$ associated to π^u at s such that $Re(s) > 0$, coincide with the poles of the normalizing factor $r^S(s, \pi^u, w)$.*

The same conclusion regarding the poles of Eisenstein series may be deduced under a certain assumption on a weak base change of σ. Although this assumption is of a different nature, in fact it implies that σ_p is tempered for all non-archimedean places $p \in S$. Thus we can prove

Corollary 0.2. *Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ_p is cohomological at **the archimedean place** p. Suppose that U', viewed as an algebraic $\mathbb{Q}_p$-group, is quasi-split over $\mathbb{Q}_p$ for all **non-archimedean** $p \in S$, and that a weak base change of σ, as constructed by Shin* [5, Thm. A.1]*, is a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. Then, the poles of the Eisenstein series $E(f_s, g)$ associated to π^u at s such that $Re(s) > 0$, coincide with the poles of the normalizing factor $r^S(s, \pi^u, w)$.*

In a different approach, using a unitarity argument, we prove a variant under weaker conditions, as it is neither assumed that the group U' is quasi-split at non-archimedean places $p \in S$, nor the representation π^u is assumed to be tempered at $p \in S$.

Theorem 0.3. *Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ is cohomological at the archimedean place. Suppose that a weak base change of σ, constructed in* [5, Thm. A.1], *is a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. Then the Eisenstein series $E(f_s, g)$, associated to π^u, is holomorphic at s with $Re(s) \geq 3/2$.*

0.3. Analytic behavior of Eisenstein series – automorphic L-functions. These results permit to determine the location of poles in terms of the analytic properties of the automorphic L-functions appearing in the normalizing factor. Since in the case when χ is not conjugate self-dual, the Eisenstein series is holomorphic in the region $Re(s) > 0$ (see Theorem 5.5 for details), we assume now that χ *is conjugate self-dual*, that is, trivial on the norm subgroup $N_{F/\mathbb{Q}}(\mathbb{I}_F)$. Thus, χ restricted to $\mathbb{I}$ is either trivial or the quadratic character $\delta_{F/\mathbb{Q}}$ of $\mathbb{I}$ attached to the extension $F/\mathbb{Q}$ by class field theory.

Theorem 0.4. *Let $n \geq 3$. Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ which is conjugate self-dual, and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ is cohomological at the archimedean place. Suppose that σ_p is tempered for all non-archimedean places $p \in S$. Then, the Eisenstein series $E(f_s, g)$, associated to π^u, is holomorphic for s such that $Re(s) > 0$, except for possible simple poles at $s \in \{1/2, 1, 3/2, \ldots, n/2\}$.*

The pole at $s = 1/2$ occurs if and only if

- *either condition $\mathcal{C}_{\text{even}}$, given by*

$$\mathcal{C}_{\text{even}} \equiv \begin{cases} n+1 \textit{ is even}, \\ \textit{the restriction of } \chi \textit{ to } \mathbb{I} \textit{ is trivial}, \\ L^S(1/2, \chi \otimes \sigma, r_1) \neq 0, \end{cases}$$

- *or condition $\mathcal{C}_{\text{odd}}$, given by*

$$\mathcal{C}_{\text{odd}} \equiv \begin{cases} n+1 \textit{ is odd}, \\ \textit{the restriction of } \chi \textit{ to } \mathbb{I} \textit{ is the quadratic character } \delta_{F/\mathbb{Q}} \textit{ of } \mathbb{I} \\ \qquad \textit{attached to the extension } F/\mathbb{Q} \textit{ by class field theory}, \\ L^S(1/2, \chi \otimes \sigma, r_1') \neq 0, \end{cases}$$

is satisfied. The pole at $s = \frac{m+1}{2}$ with $1 \leq m \leq n-1$ an integer occurs if and only if the weak local lift Σ of σ to $GL_{n-1}(\mathbb{A}_F)$ contains as a summand in the isobaric sum the discrete spectrum representation of $GL_m(\mathbb{A}_F)$ isomorphic to the unique irreducible quotient $J(m, \chi^c)$ of the induced representation

$$\mathrm{Ind}_{B_m(\mathbb{A}_F)}^{GL_m(\mathbb{A}_F)} \left(\chi^c | \, |_{\mathbb{I}_F}^{\frac{m-1}{2}} \otimes \chi^c | \, |_{\mathbb{I}_F}^{\frac{m-3}{2}} \otimes \cdots \otimes \chi^c | \, |_{\mathbb{I}_F}^{-\frac{m-1}{2}} \right),$$

where B_m is a Borel subgroup of GL_m, and χ^c is the conjugate of χ by the non-trivial Galois automorphism c.

We treat the case $n \geq 3$ separately, since in the low rank cases $n = 1$ and $n = 2$ we can provide a more precise description due to some simplifications in the conditions (see Theorems 5.7, 5.8).

Finally, in the general case, we deal with the case of the trivial representation of $M_0(\mathbb{A})$.

Theorem 0.5. *Let $\pi^u \cong \mathbf{1}_{\mathbb{I}_F} \otimes \mathbf{1}_{U'(\mathbb{A})}$ be the trivial representation of the Levi factor $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where $\mathbf{1}_{\mathbb{I}_F}$ is the trivial character of $\mathbb{I}_F$ and $\mathbf{1}_{U'(\mathbb{A})}$ is the trivial representation of $U'(\mathbb{A})$. Then, the Eisenstein series $E(f_s, g)$, associated to π^u, has a simple pole at $s = n/2$. The space $\mathcal{L}_\pi$ spanned by the residues, where $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{n/2}$, is isomorphic to the trivial representation $\mathbf{1}_{U(\mathbb{A})}$ of $U(\mathbb{A})$.*

0.4. Eisenstein cohomology – preliminary considerations. In a first step, given the real Lie group $U(n,1)$, the archimedean component of $U(\mathbb{A})$, we have to recall the known parametrization (by θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$), of unitary representations of $U(n,1)$ with non-zero cohomology (with respect to trivial coefficients). Then it is necessary to adjust this parametrization to the needs of the actual construction of Eisenstein cohomology classes, as exhibited in [29]. In this work the notion of a cuspidal representations of type (π, w), where w ranges through the set W^{P_0} of minimal coset representatives for the right cosets $W_{P_0}\backslash W$, with W the absolute Weyl group of U, W_{P_0} the absolute Weyl group of M_0, plays a decisive role. Given an irreducible unitary representation $A_\mathfrak{q}$, there is a corresponding minimal coset representative $w_{k,l} \in W^{P_0}$, uniquely determined by a pair of integers (k,l) such that $1 \le k, l \le n+1$ and $k > l$. In turn, this procedure permits to determine the Langlands data of $A_\mathfrak{q}$. We may rewrite the cohomology of $A_\mathfrak{q}$ in these terms. Let $A_\mathfrak{q}$ correspond to the Weyl group element $w_{k,l} \in W^{P_0}$ with $1 \le l < k \le n+1$. Then,

$$H^*(\mathfrak{g}, K; A_\mathfrak{q}) = \begin{cases} \mathbb{C}, & \text{if } q = \ell(w_{k,l}) + 1 - 2j \text{ with } 0 \le j \le \ell(w_{k,l}) + 1 - n, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the set of degrees in which the contribution is non-trivial is symmetric around the middle degree $q = n$.

Note that, given a cuspidal representations of type $(\pi, w_{k,l})$, $w_{k,l} \in W^{P_0}$, the evaluation point $s_{w_{k,l}}$ and the highest weight $\mu_{w_{k,l}}$, associated with the corresponding Eisenstein series (see [29, Sects. 3 and 4], or [22, Sect. 3]), are given by the formulas

$$s_{w_{k,l}} = -w_{k,l}(\rho)\big|_{\check{\mathfrak{a}}}, \qquad \mu_{w_{k,l}} = w_{k,l}(\rho) - \rho,$$

where $\check{\mathfrak{a}}$ is the dual of the Lie algebra of a maximal split torus in the center of M, and ρ is the half-sum of positive roots for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The second equation tells us that the infinitesimal character χ_{π_∞} of the archimedean component of π is equal to $\chi_{-w_{k,l}(\rho)}$.

In a second step (see Section 4), we have to take into account certain compatibility conditions for the cuspidal support which assure the possible non-vanishing of the Eisenstein cohomology spaces $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$. These conditions afford the link to the parametrization of unitary representations of $U(n,1)$ with non-zero cohomology (with respect to trivial coefficients) discussed in the first step.

0.5. Eisenstein cohomology – final results. In Section 6, using the results concerning the analytic properties of Eisenstein series, we finally determine in a precise way the structure of the spaces $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ as a $U(\mathbb{A}_f)$-module. The final results depend on arithmetic data as laid out in the statements regarding the location of poles of the Eisenstein series to be considered. The proofs use the long exact sequence in cohomology induced by the Franke filtration $\mathcal{L}_\pi \subset \mathcal{A}_\pi$.

In Theorem 6.2 we deal with the contributions $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ that are related to the point of evaluation $s_0 = 1/2$ and that may contain residual Eisenstein cohomology classes. The existence of the latter classes depends on the existence of cuspidal automorphic representations $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{1/2} \otimes \sigma$ of $M_0(\mathbb{A})$, where χ is a unitary conjugate self-dual Hecke character of $\mathbb{I}_F$ and σ a unitary cuspidal automorphic representation of $U'(\mathbb{A})$, with σ_p tempered for all non-archimedean places $p \in S$ such that the necessary cohomological conditions and one of the two conditions $\mathcal{C}_{even}$ and $\mathcal{C}_{odd}$ in Theorem 0.4 are satisfied.

Theorem 6.3 concerns the general case, that is, the points of evaluation are at $s = \frac{m+1}{2}$ with $1 \leq m \leq n-1$.

Finally, as proved in Theorem 6.4, if $\pi \cong |\cdot|_{\mathbb{I}_F}^{n/2} \otimes \mathbf{1}_{U'(\mathbb{A})}$ is the trivial representation of the Levi factor $M_0(\mathbb{A})$ twisted by the character $|\cdot|_{\mathbb{I}_F}^{n/2}$, then the summand in the Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} \mathbf{1}_{U(\mathbb{A}_f)}, & \text{if } q = 0, 2, \ldots, 2n-2, \\ \text{non-trivial submodule of } I_{\text{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})}), & \text{if } q = 2n-1, \\ \text{either } \mathbf{1}_{U(\mathbb{A}_f)}, \text{ or } 0, & \text{if } q = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $H^{2n}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \mathbf{1}_{U(\mathbb{A}_f)}$ if and only if $H^{2n-1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong I_{\text{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})})$, and if $H^{2n}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is trivial, then $H^{2n-1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is the submodule of $I_{\text{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})})$ for which the quotient is the trivial representation.

We conclude Section 6 with a complete description of the Eisenstein cohomology of relative rank one unitary groups in two and three variables. Recall that these groups are quasi-split over $\mathbb{Q}$. Our approach provides a different proof of results of Harder [15].

1. Unitary groups of rank one

Let F be an imaginary quadratic extension of the field $\mathbb{Q}$ of rational numbers, with Galois group $Gal(F/\mathbb{Q})$. We denote by c the non-trivial element of $Gal(F/\mathbb{Q})$. Given an archimedean or non-archimedean place p, let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at p. For the archimedean place $p = \infty$, we have $\mathbb{Q}_\infty \cong \mathbb{R}$. For a non-archimedean place p, let $\mathbb{Z}_p$ be the ring of integers in $\mathbb{Q}_p$. Let $\mathbb{A}$, resp. $\mathbb{A}_F$, be the ring of adèles, and $\mathbb{I}$, resp. $\mathbb{I}_F$, the group of idèles of $\mathbb{Q}$, resp. F. The subring of finite adèles in $\mathbb{A}$ is denoted $\mathbb{A}_f$.

We now introduce the unitary groups of relative rank one we are concerned with in this paper. Let n be a positive integer. Let V be a non-degenerate hermitian space over F of dimension $n+1$ with the hermitian form h. We assume that V is not anisotropic, and that the conjugate of V by the complex embedding given by the archimedean place is of signature $(n, 1)$. Let

$$U = U(V)$$

be the unitary group preserving the hermitian form on V, viewed as an algebraic group defined over $\mathbb{Q}$. It is of $\mathbb{Q}$-rank one. Observe that our setting is not restrictive, because all unitary groups over $\mathbb{Q}$ of $\mathbb{Q}$-rank one, arise in this way. This is clear from the classification of unitary groups over local and global fields (of characteristic zero), which we now recall. See [25, Chap. I] for more details about classification of classical groups.

By our assumption, at the archimedean place $p = \infty$, we have $U(\mathbb{R}) \cong U(n, 1)$ is the unitary group of signature $(n, 1)$. The Hasse invariant $\epsilon(U(n, 1))$ attached to $U(n, 1)$ is

$$\epsilon(U(n,1)) = \begin{cases} 1, & \text{if } n+1 \text{ is odd,} \\ (-1)^{\frac{n-1}{2}}, & \text{if } n+1 \text{ is even.} \end{cases}$$

If p is a non-archimedean place that splits in F, then $U(\mathbb{Q}_p) \cong GL_{n+1}(\mathbb{Q}_p)$. If p does not split in F, let $\mathfrak{P}$ be the unique place of F lying above p. Then, for $n+1$ odd, $U(\mathbb{Q}_p)$ is the quasi-split unitary group in $n+1$ variables given by the quadratic extension $F_{\mathfrak{P}}/\mathbb{Q}_p$. The Hasse invariant in this case is $\epsilon(U(\mathbb{Q}_p)) = 1$. However, for $n+1$ even, there are two non-isomorphic possibilities for $U(\mathbb{Q}_p)$. Either $U(\mathbb{Q}_p)$ is the quasi-split unitary group in $n+1$ variables given by the quadratic extension $F_{\mathfrak{P}}/\mathbb{Q}_p$, or the conjugate of V at place $\mathfrak{P}$ has an anisotropic subspace of dimension two and $U(\mathbb{Q}_p)$ is the non-quasi-split unitary group preserving the hermitian form. In the former case, the Hasse invariant is $\epsilon(U(\mathbb{Q}_p)) = 1$, while in the latter case, it is $\epsilon(U(\mathbb{Q}_p)) = -1$. By the classification of unitary groups over a number field, given $U = U(V)$, the group $U(\mathbb{Q}_p)$ is quasi-split at all but finitely many places, and the product of the Hasse invariants over all places of $\mathbb{Q}$ that do not split in F, including the archimedean place, equals one.

We fix, once and for all, a maximal compact subgroup K of $U(\mathbb{A})$ of the form $K = \prod_p K_p$, where K_p is a maximal compact subgroup of $U(\mathbb{Q}_p)$, and we have $K_p \cong U(\mathbb{Z}_p)$ for almost all p, and $K_\infty \cong U(n) \times U(1)$.

Since the algebraic group U is of $\mathbb{Q}$-rank one, a maximal isotropic subspace of V is one-dimensional. Its stabilizer is a representative of the unique conjugacy class of proper parabolic $\mathbb{Q}$-subgroups of U. We fix, once and for all, one such representative P_0. We assume, as we may, that P_0 is in good position with respect to the fixed maximal compact subgroup K (cf. [27, Sect. I.1.4]).

If x is a non-zero vector in the isotropic space defining P_0, let y be another isotropic vector such that $h(x, y) \neq 0$. Let V' be the orthogonal complement of the span of x and y. It is a hermitian space over F, of dimension $n-1$, with the hermitian form obtained from h by restriction. We denote by

$$U' = U(V')$$

the unitary group preserving the hermitian form on V'. Then, for the archimedean place $p = \infty$, we have $U'(\mathbb{R}) = U(n-1)$ is the compact unitary group in $n-1$ variables. For non-archimedean places p, the local groups $U'(\mathbb{Q}_p)$ are of the same form as in the case of the unitary group U above.

The parabolic subgroup P_0 has a Levi decomposition $P_0 = M_0 N_0$, where M_0 is the Levi factor stabilizing the hermitian space V' and the one-dimensional isotropic subspaces spanned by x and y, and N_0 is the unipotent radical. Then

$$M_0 \cong Res_{F/\mathbb{Q}} GL_1 \times U',$$

where $Res_{F/\mathbb{Q}} GL_1$ is the algebraic group over $\mathbb{Q}$ obtained from GL_1 over F by restriction of scalars.

2. Automorphic cohomology, Eisenstein series, and the Franke filtration

In this section, we introduce the main objects of concern in this paper. We are quite brief, and for more details we urge the reader to follow the references in the text.

2.1. Automorphic cohomology. Let $\mathcal{J}$ be the annihilator of the (conjugate dual) of the trivial representation in the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{g}_\infty$, where $\mathfrak{g}_\infty$ is the Lie algebra of $U(\mathbb{R}) \cong U(n, 1)$. Then, $\mathcal{J}$ is an ideal of finite codimension in $\mathcal{Z}$, and we denote by

$$\mathcal{A} = \mathcal{A}_{\mathcal{J}}(U(\mathbb{Q}) \backslash U(\mathbb{A}))$$

the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module of all automorphic forms on $U(\mathbb{A})$, in the sense of [3], which are annihilated by a power of $\mathcal{J}$. Then, the automorphic cohomology $H^*(U, \mathbb{C})$ of U, with respect to the trivial coefficient system, may be defined as the relative Lie algebra cohomology

$$H^*(U,\mathbb{C}) = H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}).$$

Observe that this cohomology space is in fact the same as the relative Lie algebra cohomology of the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module of all automorphic forms on $U(\mathbb{A})$, but due to Wigner's lemma [4, Sect. I.4], only automorphic forms that match the infinitesimal character of the trivial representation may contribute non-trivially to the automorphic cohomology.

This cohomology space is obtained as a direct limit over all open compact subgroups of $U(\mathbb{A}_f)$ of the de Rham cohomology of certain locally symmetric spaces. Since there is a $U(\mathbb{A}_f)$-action on the directed system, the resulting automorphic cohomology $H^*(U,\mathbb{C})$ carries the structure of a $U(\mathbb{A}_f)$-module. As explained in the introduction, one should have in mind that this object captures the cohomology of congruence arithmetic subgroups of U, as follows from [4], [2], [6]. See [22, Sect. 2] for more details.

There is a decomposition of the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module $\mathcal{A}$ along the cuspidal support. More precisely, let $\mathcal{A}_{\mathrm{cusp}}$ denote the submodule of all cuspidal forms in $\mathcal{A}$. The complement of $\mathcal{A}_{\mathrm{cusp}}$ in $\mathcal{A}$ exhibits a decomposition with respect to (associate classes) of cuspidal automorphic representations of the Levi factor $M_0(\mathbb{A})$. Let

$$\mathcal{A} \cong \mathcal{A}_{\mathrm{cusp}} \oplus \bigoplus\nolimits_\pi \mathcal{A}_\pi$$

be the decomposition of $\mathcal{A}$ along the cuspidal support, where $\mathcal{A}_\pi$ is the $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-module of automorphic forms with cuspidal support in the associate class of a (not necessarily unitary) cuspidal automorphic representation π of $M_0(\mathbb{A})$, as defined in [27, Sect. III.2.6] and, in an equivalent way, in [7, Sect. 3.1]. The sum is actually only over those π which are compatible with the trivial coefficient system (cf. [22, Sect. 1.3]).

The decomposition of the space $\mathcal{A}$ gives rise to the corresponding decomposition in cohomology. Namely,

$$\begin{aligned} H^*(U,\mathbb{C}) &\cong H^*_{\mathrm{cusp}}(U,\mathbb{C}) \oplus \bigoplus_\pi H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \\ &\cong H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathrm{cusp}}) \oplus H^*_{\mathrm{Eis}}(U,\mathbb{C}), \end{aligned}$$

where the cuspidal cohomology $H^*_{\mathrm{cusp}}(U,\mathbb{C})$ is the cohomology of the module $\mathcal{A}_{\mathrm{cusp}}$, and its natural complement $H^*_{\mathrm{Eis}}(U,\mathbb{C})$ is the Eisenstein cohomology, which is the main concern of this paper.

There are certain necessary conditions for non-vanishing of the individual summands in the Eisenstein cohomology. In other words, the cohomology space $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ vanishes, except possibly if these necessary conditions are satisfied. In order to state them, we need more notation.

Let W be the absolute Weyl group of U, and let W_{P_0} be the absolute Weyl group of the Levi factor M_0. Every right coset in $W_{P_0}\backslash W$ has a unique representative of minimal length in its right coset. Let W^{P_0} be the set of such minimal coset representatives, also called the Kostant representatives (cf. [17]).

Write $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s_0}$, where π^u is a unitary cuspidal automorphic representation of $M_0(\mathbb{A})$, and s_0 is a real number. Changing the representative π if necessary, we may always assume $s_0 \geq 0$.

Now the necessary non-vanishing conditions for $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ say that there exists $w \in W^{P_0}$ such that

NV1 s_0 corresponds to $s_w = -w(\rho)\big|_{\check{\mathfrak{a}}_\infty}$ (see Sect. 3),

NV2 the highest weight of the archimedean component π^u_∞ of π^u is $\mu_w = w(\rho) - \rho$, where $\check{\mathfrak{a}}_\infty$ is the dual of the Lie algebra of a maximal split torus in the center of $M_0(\mathbb{R})$, and ρ is the half-sum of positive roots in the complexification of $\mathfrak{g}_\infty$. We make these conditions explicit in Sect. 3.4. Note that, for a given π, there is at most one $w \in W^{P_0}$ such that the two non-vanishing conditions are satisfied. In the terminology of [22], the cohomology classes in a non-trivial summand $H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ originate with classes of type (π, w) (in the sense of [29]), where $w \in W^{P_0}$ is such that the necessary non-vanishing conditions are satisfied for π.

2.2. Eisenstein series and the Franke filtration. Let $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s_0}$, with $s_0 > 0$, be a cuspidal automorphic representation of $M_0(\mathbb{A})$, as in Sect. 2.1. Let

$$I(s, \pi^u) = \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})} \left(\pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s} \right)$$

be the induced representation, where $s \in \mathbb{C}$. Note that the induction is normalized, and we always assume that π^u is trivial on the connected component of the real points of the maximal split torus in M_0. The latter is not restricting and assures that the poles of the Eisenstein series will be real.

Let f_s be a section of the induced representations $I(s, \pi^u)$ as in [7]. The Eisenstein series associated to π^u is defined as the analytic continuation from the cone of absolute convergence of the series

$$E(f_s, g) = \sum_{\gamma \in P_0(\mathbb{Q}) \backslash U(\mathbb{Q})} f_s(\gamma g).$$

It is meromorphic in s with only finitely many poles with $s > 0$ and all other poles lie in the half-plane $Re(s) < 0$. See [27, Sect. IV.1] for these facts.

Consider the Eisenstein series $E(f_s, g)$ evaluated at $s = s_0 > 0$. The possible pole at $s = s_0$ is at most simple. In any case, there exists a polynomial $F(s)$ such that $F(s)E(f_s, g)$ is holomorphic at $s = s_0$ for every section f_s. Thus, we may write the Taylor expansion of $F(s)E(f_s, g)$ around $s = s_0$. We define the space $\mathcal{A}_\pi$ as the span of all Taylor coefficients in that expansion. If the Eisenstein series $E(f_s, g)$ has a pole at $s = s_0 > 0$, then the residues span a $(\mathfrak{g}_\infty, K_\infty; U(\mathbb{A}_f))$-submodule $\mathcal{L}_\pi$ of $\mathcal{A}_\pi$ consisting of all square-integrable automorphic forms with the cuspidal support in (the associate class of) π.

The Franke filtration is a finite filtration of the spaces $\mathcal{A}_\pi$, first defined in [6, Sect. 6], and refined with respect to cuspidal support in [7, Thm. 1.4]. The successive quotients of the filtration may be described as induced representations. The explicit description of the Franke filtration in the case of cuspidal support in a maximal proper parabolic subgroup is given in [10].

The Franke filtration of $\mathcal{A}_\pi$, where $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{s_0}$, depends on the existence of the pole of the Eisenstein series $E(f_s, g)$ attached to π^u at $s = s_0 > 0$. If the Eisenstein series $E(f_s, g)$ is holomorphic at $s = s_0$, then the Franke filtration is trivial, and we have

$$\mathcal{A}_\pi \cong I(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}}),$$

where $S(\check{\mathfrak{a}}_{\infty,\mathbb{C}})$ is the symmetric algebra of the complexification $\check{\mathfrak{a}}_{\infty,\mathbb{C}}$ of $\check{\mathfrak{a}}_\infty$.

On the other hand, if the Eisenstein series $E(f_s, g)$ has a pole at $s = s_0$ for some section f_s, then the Franke filtration is a two-step filtration

$$\mathcal{L}_\pi \subset \mathcal{A}_\pi.$$

We have

$$\mathcal{L}_\pi \cong J(s_0, \pi^u),$$

where $J(s_0, \pi^u)$ is the quotient of the induced representation $I(s_0, \pi^u)$ obtained as the image of the residue of the standard intertwining operator in the constant term of Eisenstein series. According to [10], the filtration quotient is isomorphic to

$$\mathcal{A}_\pi/\mathcal{L}_\pi \cong I(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}}).$$

See Sect. 5 for more details concerning constant terms and analytic properties of Eisenstein series for rank one unitary groups.

3. Cohomological representations of $U(n,1)$ – various parameterizations

In this section we recall the classification of cohomological unitary representations of the real Lie group $U(n, 1)$, the archimedean component of the unitary group $U(\mathbb{A})$, introduced in Sect. 1. In the whole section, we are dealing only with the archimedean component, so we simplify the notation accordingly. In particular, we suppress the subscript ∞ in the whole section.

Unitary cohomological representations of a real connected semisimple Lie group (with finite center) are classified in [34], up to infinitesimal equivalence. Their classification, in terms of certain $A_\mathfrak{q}(\lambda)$, is made explicit for unitary groups in [33, Sect. 2], [1, Chap. 5], and in the special case of rank one in [4, Sect. VI.4]. We have to adjust the parameterizations given there to the needs of the actual construction of Eisenstein cohomology classes attached to cuspidal representations of type (π, w), with $w \in W^{P_0}$ a minimal coset representative.

3.1. Classification of $A_\mathfrak{q}$. Let $G = U(n, 1)$ be the unitary real Lie group of signature $(n, 1)$. We fix the basis of the underlying hermitian space in such a way that the matrix of the hermitian form is

$$\begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. We let K be the maximal compact subgroup $U(n) \times U(1)$ of $U(n, 1)$. Let $\mathfrak{g}_0$, resp. $\mathfrak{k}_0$, be the real Lie algebra of G, resp. K. Then, $\mathfrak{g}_0 = \mathfrak{u}(n, 1)$, and

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & x_2 \end{pmatrix} : X_1 \in \mathfrak{u}(n), x_2 \in \mathfrak{u}(1) \right\} \cong \mathfrak{u}(n) \oplus \mathfrak{u}(1).$$

Let $\mathfrak{g}$, resp. $\mathfrak{k}$, be the complexification of $\mathfrak{g}_0$, resp. $\mathfrak{k}_0$. We have $\mathfrak{g} = \mathfrak{gl}(n + 1, \mathbb{C})$, and $\mathfrak{k}$ is the subalgebra of block diagonal matrices with blocks of size $n \times n$ and 1×1. Let θ_K denote the Cartan involution, and

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

be the corresponding Cartan decomposition, where

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix} : Y \in M_{n,1}(\mathbb{C}) \right\},$$

with Y^* the conjugate transpose of the matrix Y. Then

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} : Y \in M_{n,1}(\mathbb{C}), Z \in M_{1,n}(\mathbb{C}) \right\}$$

is the complexification of the real Lie algebra $\mathfrak{p}_0$.

Given any $(\mathfrak{g}, K)$-module Y, we denote by $H^*(\mathfrak{g}, K; Y)$ the relative Lie algebra cohomology of Y (with respect to the trivial coefficient system), as defined in [4, Chap. I]. We will now make explicit the classification of unitary $(\mathfrak{g}, K)$-modules with non-zero relative Lie algebra cohomology.

In the case of trivial coefficients, the main result of [34] is that the unitary $(\mathfrak{g}, K)$-modules with non-zero relative Lie algebra cohomology are parameterized by the θ_K-stable parabolic subalgebras of $\mathfrak{g}$. Given such subalgebra $\mathfrak{q}$, the corresponding $(\mathfrak{g}, K)$-module is denoted by $A_{\mathfrak{q}}$.

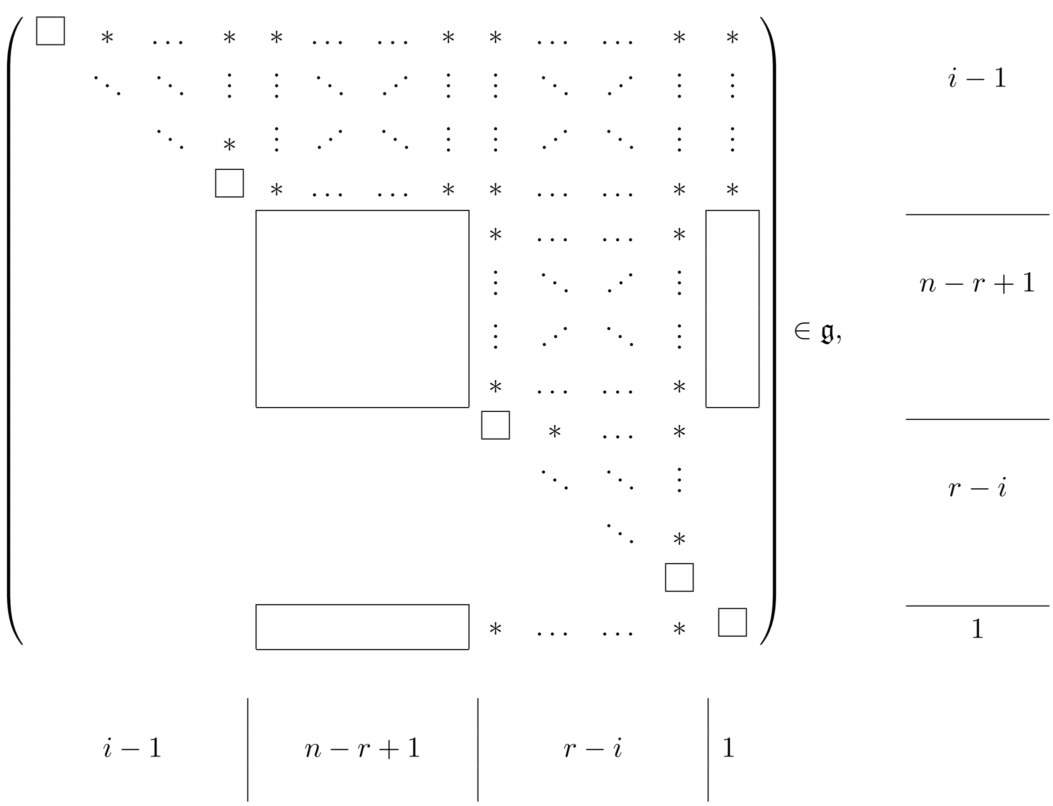
According to [33, Thm. 2.4], the θ_K-stable parabolic subalgebras $\mathfrak{q}$ are in one-to-one correspondence with ordered partitions of n into r non-negative integers of the form

$$n = \underbrace{1 + \cdots + 1}_{i-1} + (n - r + 1) + \underbrace{1 + \cdots + 1}_{r-i},$$

where $r \leq n+1$, and $n-r+1$ is at the ith place in the sum. Given such partition with $1 \leq r \leq n+1$ and $1 \leq i \leq r$, the corresponding θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is given as

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u},$$

where $\mathfrak{l}$ is the Levi subalgebra, and $\mathfrak{q}$ consists of all matrices of the form



where squares and rectangles are parts of $\mathfrak{l}$ and $*$'s are parts of $\mathfrak{u}$. Empty cells represent zeros. The numbers on the right-hand side and below the matrix indicate the sizes of blocks. Note that the Levi subalgebra $\mathfrak{l}$ is the complexification of a real Lie algebra

$$\mathfrak{l}_0 \cong \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n - r + 1, 1),$$

where $\mathfrak{u}(1)$ appears $r - 1$ times.

3.2. Cohomology of $A_{\mathfrak{q}}$. According to [34, p. 58], $A_{\mathfrak{q}}$ is discrete series, resp. tempered, if and only if $\mathfrak{l} \subset \mathfrak{k}$, resp. $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{k}$. Hence, in our case, $A_{\mathfrak{q}}$ is discrete series if and only if $\mathfrak{q}$ corresponds to an ordered partition of n into $r = n+1$ pieces, and depending on the position $1 \leq i \leq n+1$ of $n+r-1 = 0$ in that partition we obtain $n+1$ discrete series with non-zero cohomology. However, these are of no interest here, because discrete series representations are never local components of non-cuspidal square-integrable automorphic representations, hence, cannot contribute to Eisenstein cohomology. For completeness, we mention here that the cohomology

$$H^q(\mathfrak{g}, K; A_{\mathfrak{q}}) = \begin{cases} \mathbb{C}, & \text{if } q = n, \\ 0, & \text{otherwise,} \end{cases}$$

for all discrete series $A_{\mathfrak{q}}$.

The remaining $A_{\mathfrak{q}}$, that is, those corresponding to partitions of n with $1 \leq r \leq n$, are non-tempered. If $\mathfrak{q}$ corresponds to the partition of n given by $1 \leq r \leq n$ and $1 \leq i \leq r$, according to [34, Thm. 3.3], the cohomology

$$H^q(\mathfrak{g}, K; A_{\mathfrak{q}}) \cong H^{q-R(\mathfrak{q})}(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k}; \mathbb{C}),$$

where $\mathbb{C}$ is the trivial module, and, given $\mathfrak{q}$, $R(\mathfrak{q})$ is the dimension of $\mathfrak{u} \cap \mathfrak{p}$. It is clear, from the description of $\mathfrak{q}$, that $R(\mathfrak{q}) = r-1$. In fact, it is the number of $*$'s outside of the $n \times n$ block in the left upper corner. This last cohomology can be calculated, for example, using [33, Cor. 2.16]. We obtain

$$H^q(\mathfrak{g}, K; A_{\mathfrak{q}}) = \begin{cases} \mathbb{C}, & \text{if } q = r-1+2j \text{ with } 0 \leq j \leq n-r+1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the non-vanishing degrees are symmetric with respect to the middle degree $q = n$, that is, half the dimension of the unipotent radical of the only proper parabolic subgroup of $U(n,1)$.

3.3. Langlands data for $A_{\mathfrak{q}}$. We have recalled the classification of the $(\mathfrak{g}, K)$-modules $A_{\mathfrak{q}}$ with non-zero cohomology (with respect to trivial coefficients), and the calculation of their cohomology in cases that matter for describing the Eisenstein cohomology. However, in order to relate them to possible poles of Eisenstein series, it is convenient to know the standard module which $A_{\mathfrak{q}}$ is the Langlands quotient of (cf. [21]). We follow [1, Chap. 5].

The unitary group $G = U(n,1)$ is of rank one, so it has a unique conjugacy class of proper parabolic subgroups. Let P be a representative of that class. Write $P = MN$ for the Levi decomposition of P, where M is the Levi factor and N the unipotent radical. Then $M \cong \mathbb{C}^\times \times U(n-1)$, where $U(n-1)$ is the compact unitary group in $n-1$ variables.

Given $A_{\mathfrak{q}}$ with $\mathfrak{q}$ corresponding to a pair (r,i), where $1 \leq r \leq n$ and $1 \leq i \leq r$, we now specify a complex number s', with $Re(s') > 0$, a unitary character η of $\mathbb{C}^\times$, and an irreducible representation τ of $U(n-1)$, such that $A_{\mathfrak{q}}$ is the Langlands quotient of the induced representation

$$\mathrm{Ind}_P^G(\eta|\cdot|^{s'} \otimes \tau),$$

where $|z| = \sqrt{z \cdot \overline{z}}$, for $z \in \mathbb{C}$, is the non-normalized[2] absolute value on $\mathbb{C}$. The parabolic induction is normalized. The condition $Re(s') > 0$ assures that this induced representation is a standard module. According to the construction in [1, Chap. 5], the exponent s' is the positive integer

$$s' = n - r + 1,$$

[2] We always write $|z|$ for the non-normalized absolute value on $\mathbb{C}$. The normalized absolute value is denoted either by $|z|_{\mathbb{C}}$ or $|z|_\infty$. We have $|z|_{\mathbb{C}} = |z|_\infty = |z|^2$ for $z \in \mathbb{C}$.

and the unitary character η is given by the assignment

$$\eta(z) = \left(\frac{z}{|z|}\right)^{r-2i+1}, \quad \text{for } z \in \mathbb{C}.$$

The irreducible representations of the compact unitary group $U(n-1)$ are classified by their highest weights. The highest weight for $U(n-1)$ is given by a sequence of integers $(\lambda_1, \dots, \lambda_{n-1})$ such that $\lambda_1 \geq \dots \geq \lambda_{n-1}$. The representation τ is the representation of $U(n-1)$ corresponding to the highest weight

$$\mu = (\underbrace{1, \dots, 1}_{i-1}, \underbrace{0, \dots, 0}_{n-r}, \underbrace{-1, \dots, -1}_{r-i}).$$

3.4. Kostant representatives for $A_{\mathfrak{q}}$. For future reference, we determine now explicitly the minimal coset representatives, which produce the Langlands data for $A_{\mathfrak{q}}$, as in [29] and [22, Sect. 3]. Their lengths are required when computing Eisenstein cohomology in Sect. 4.

Let W be the absolute Weyl group of $U(n,1)$. It is the Weyl group of the root system of the complexified Lie algebra $\mathfrak{g} = \mathfrak{gl}(n+1, \mathbb{C})$ with respect to the diagonal Cartan subalgebra $\mathfrak{t}$. The group W is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$. Let W_P be the absolute Weyl group of the Levi factor M. Since the complexified Lie algebra $\mathfrak{m}$ of M is block diagonal, with blocks of size 1×1, $(n-1) \times (n-1)$ and 1×1 along the diagonal, we have $W_P \cong \mathfrak{S}_{n-1}$. As in Sect. 2.1, let W^P be the set of minimal coset representatives for $W_P \backslash W$. Note that this is the same as W^{P_0} of Sect. 2.1.

Let f_j, for $j = 1, \dots, n+1$, denote the projection of $\mathfrak{t}$ on the jth coordinate. Then, the f_j form the basis of the dual $\check{\mathfrak{t}}$ of $\mathfrak{t}$. The action of W on $\check{\mathfrak{t}}$ is by permutation of this basis. In a similar way as in [13, Sect. 3] and [12, Sect. 7], we may parameterize the set W^P of Kostant representatives by two integers. Namely, if we write $(s_1, \dots, s_{n+1})$ for elements of $\check{\mathfrak{t}}$ in the basis $(f_1, \dots, f_{n+1})$, the action of a Kostant representative $w \in W^P$ is determined by a pair of indices indicating which coordinates are moved to the first and the last place in the $(n+1)$-tuple. The remaining coordinates should remain in the original order, because the identity is the shortest element in W_P, and we need the minimal coset representative.

Thus, we parameterize W^P by pairs of integers (k, l) such that $1 \leq k, l \leq n+1$ and $k \neq l$. The action of the Kostant representative $w_{k,l} \in W^P$, parameterized by (k, l), on $\check{\mathfrak{t}}$ is given by the formula

$$w_{k,l}(s_1, \dots, s_{n+1}) = \begin{cases} (s_k, s_1, \dots, \hat{s}_k, \dots, \hat{s}_l, \dots, s_{n+1}, s_l), & \text{if } k < l, \\ (s_k, s_1, \dots, \hat{s}_l, \dots, \hat{s}_k, \dots, s_{n+1}, s_l), & \text{if } k > l, \end{cases}$$

where $\hat{s}_j$ indicates that s_j is removed from its natural position in the sequence. As explained in Sect. 2.1 (cf. [22, Sect. 3]), the evaluation point $s_{w_{k,l}}$ and the highest weight $\mu_{w_{k,l}}$, attached to $w_{k,l} \in W^P$, are given by the formulas

$$s_{w_{k,l}} = -w_{k,l}(\rho)\big|_{\check{\mathfrak{a}}},$$
$$\mu_{w_{k,l}} = w_{k,l}(\rho) - \rho,$$

where $\check{\mathfrak{a}}$ is the dual of the Lie algebra of a maximal split torus in the center of M, and

$$\rho = \left(\frac{n}{2}, \frac{n}{2} - 1, \dots, -\frac{n}{2}\right) \in \check{\mathfrak{t}}$$

is the half-sum of positive roots for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

By a direct calculation, using the formula for the action of $w_{k,l}$, we obtain that the Langlands data for $A_{\mathfrak{q}}$, given by $1 \leq r \leq n$ and $1 \leq i \leq r$, is obtained for the Kostant representative $w_{k,l}$ such that

$$\begin{aligned} k &= i + n - r + 1, \\ l &= i. \end{aligned}$$

In particular, since $n-r+1 > 0$, only $w_{k,l}$ with $k > l$ appear as Kostant representatives providing the Langlands data for $A_{\mathfrak{q}}$. In what follows, we need the length $\ell(w_{k,l})$ of such Kostant representatives. It is the length of the corresponding permutation. The length is

$$\ell(w_{k,l}) = n + k - l - 1 = 2n - r,$$

because we need $k-1$ transpositions to put s_k in front, and then $(n+1)-l-1 = n-l$ transpositions to put s_l to the back. Note that -1 in the second part comes from the condition $k > l$, so that s_k has been already moved, when we start moving s_l.

We remark that the Langlands data for all $A_{\mathfrak{q}}$ are obtained from the Kostant representatives $w_{k,l}$ with $k > l$. The remaining Kostant representatives, namely those with $k < l$, cannot provide additional contributions to Eisenstein cohomology because their evaluation point would be negative. The Langlands data for $A_{\mathfrak{q}}$, written in terms of the corresponding $w_{k,l} \in W^P$ with $1 \leq l < k \leq n+1$, are given as

$$s' = \ell(w_{k,l}) + 1 - n = k - l,$$

$$\eta(z) = \left(\frac{z}{|z|}\right)^{n-(k+l)+2}, \quad \text{for } z \in \mathbb{C},$$

and

$$\mu = (\underbrace{1, \ldots, 1}_{l-1}, \underbrace{0, \ldots, 0}_{k-l-1}, \underbrace{-1, \ldots, -1}_{n-k+1})$$

is the highest weight of τ.

Since we will formulate the final result in terms of $w_{k,l}$ and its length, let us rewrite the cohomology of $A_{\mathfrak{q}}$ in these terms. Let $A_{\mathfrak{q}}$ correspond to the Weyl group element $w_{k,l}$ with $1 \leq l < k \leq n+1$. Then,

$$H^*(\mathfrak{g}, K; A_{\mathfrak{q}}) = \begin{cases} \mathbb{C}, & \text{if } q = \ell(w_{k,l}) + 1 - 2j \text{ with } 0 \leq j \leq \ell(w_{k,l}) + 1 - n, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the set of degrees in which the contribution is non-trivial is symmetric around the middle degree $q = n$.

3.5. Relation between our description of $A_{\mathfrak{q}}$ and the one given in [4]**.** We finish this section with the account of the relationship between our description of $A_{\mathfrak{q}}$, which are not discrete series, in terms of $1 \leq r \leq n$ and $1 \leq i \leq r$ as in [1, Chap. 5], and that of [4, Sect. VI.4]. In the latter, these $A_{\mathfrak{q}}$ are denoted $J_{i',j'}$, parameterized by pairs of integers (i', j') such that $i', j' \geq 0$ and $i' + j' \leq n - 1$. Given a representation $A_{\mathfrak{q}}$, attached to the pair (r, i), the relation is given by

$$\begin{aligned} i' &= i - 1, \\ j' &= r - i, \end{aligned}$$

that is, $A_{\mathfrak{q}} = J_{i-1,r-i}$. The Kostant representative giving this $A_{\mathfrak{q}}$ is $w_{k,l} \in W^P$, where

$$k = n + 1 - j',$$
$$l = i' + 1.$$

Then

$$\ell(w_{k,l}) = 2n - i' - j' - 1$$

is the length of $w_{k,l}$ in terms of the parametrization in [4, Sect. VI.4].

4. Eisenstein cohomology – non-vanishing conditions and filtration quotients

We are now ready to compute explicitly the summands

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$$

in the decomposition of Eisenstein cohomology along the cuspidal support (see Sect. 2.1). The strategy is to use the Franke filtration and the long exact sequence in cohomology induced by the filtration, in a way first used in [11] for the split symplectic group of rank two over a totally real number field. The final results depend on the position of poles of Eisenstein series with the given cuspidal support. This topic is discussed in Sect. 5.

4.1. Necessary conditions for non-vanishing. We start by reducing the possible cuspidal supports that may contribute to cohomology. By the construction of Eisenstein cohomology classes in [22, Sect. 3], as already explained in Sect. 2.1, the cohomology space

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$$

is trivial, except possibly if the cuspidal support π satisfies certain compatibility conditions. These necessary conditions for non-vanishing of the cohomology of $\mathcal{A}_\pi$ are already made explicit in Sect. 3.4.

It turns out, writing $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$ as in Sect. 2.1, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a unitary cuspidal automorphic representation of $U'(\mathbb{A})$, and $s_0 > 0$, that the necessary conditions for non-vanishing imply for the archimedean components $\chi_\infty = \eta$ and $\sigma_\infty = \tau$, and the evaluation point[3] $s_0 = s'/2$, where the possible data η, τ and s' are those given in Sect. 3.3. In particular, the representation of $U(n,1)$, parabolically induced from the archimedean component π_∞, is a standard module, whose Langlands quotient is one of $A_{\mathfrak{q}}$.

Therefore, in what follows, we consider only cuspidal supports π satisfying these necessary conditions. As already mentioned in Sect. 2.1, given a cuspidal support π, there is a unique $w_{k,l} \in W^{P_0}$, with $1 \leq l < k \leq n+1$, giving these necessary conditions, that is, such that there are non-trivial cohomology classes of type $(\pi, w_{k,l})$, which form the launch pad for possible Eisenstein cohomology classes. This also determines uniquely the parameters $1 \leq r \leq n$ and $1 \leq i \leq r$ for $A_{\mathfrak{q}}$. All the results in this section are stated in terms of representatives $w_{k,l} \in W^{P_0}$, with $1 \leq l < k \leq n+1$, as well as the parameters (r, i), with $1 \leq r \leq n$ and $1 \leq i \leq r$, for $A_{\mathfrak{q}}$.

[3] The factor of $\frac{1}{2}$ appears here due to different normalizations of absolute value on $\mathbb{C}$. The local component $|\cdot|_\infty$ of the adèlic absolute value $|\cdot|_{\mathbb{I}_F}$ is not the same as the absolute value $|\cdot|$ used in Sect. 3.3. In fact, $|z|_\infty = |z|^2$, for $z \in \mathbb{C}$.

4.2. Cohomology of filtration quotients. Let π be a cuspidal automorphic representation of $M_0(\mathbb{A})$, satisfying the necessary conditions for non-vanishing with the minimal coset representative $w_{k,l} \in W^{P_0}$, where $1 \leq l < k \leq n+1$. Write $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$. According to Sect. 4.1, let $1 \leq r \leq n$ and $1 \leq i \leq r$ be the parameters of the representation $A_{\mathfrak{q}}$, which appears as the Langlands quotient of the representation parabolically induced from π_∞ at the archimedean place.

Assume first that the Eisenstein series $E(f_s, g)$ attached to $\pi^u \cong \chi \otimes \sigma$ has a pole at $s = s_0 > 0$. Then, the space $\mathcal{L}_\pi$ of square-integrable forms in $\mathcal{A}_\pi$ is non-trivial, and as already mentioned in Sect. 2.2, the Franke filtration of $\mathcal{A}_\pi$ is the two-step filtration $\mathcal{L}_\pi \subset \mathcal{A}_\pi$. In this case,

$$\mathcal{L}_\pi \cong J(s_0, \pi^u) \cong J_\infty(s_0, \pi^u) \otimes J_{\mathrm{fin}}(s_0, \pi^u),$$

where $J(s_0, \pi^u)$ is the unique irreducible quotient of the induced representation $I(s_0, \pi^u)$, and we have decomposed it into the archimedean and non-archimedean part. Note that in our setting $J_\infty(s_0, \pi^u) = A_{\mathfrak{q}}$ for $\mathfrak{q}$ parameterized by $1 \leq r \leq n$ and $1 \leq i \leq r$. Hence, after factoring out the non-archimedean part $J_{\mathrm{fin}}(s_0, \pi^u)$, the cohomology of $\mathcal{L}_\pi$ is already calculated in Sect. 3.2. The final result is the following.

Proposition 4.1. *Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$ be a cuspidal automorphic representation of the Levi factor $M_0(\mathbb{A})$ satisfying the necessary non-vanishing conditions with the minimal coset representative $w_{k,l} \in W^{P_0}$, where $1 \leq l < k \leq n+1$, and with parameters (r,i), where $1 \leq r \leq n$ and $1 \leq i \leq r$. Assume that the Eisenstein series attached to $\pi^u \cong \chi \otimes \sigma$ has a pole at $s = s_0 > 0$. Let $\mathcal{L}_\pi$ be the space spanned by the residues. Then,*

$$\begin{aligned} H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_\pi) &= \begin{cases} J_{\mathrm{fin}}(s_0, \pi^u), & \text{if } q = \ell(w_{k,l}) + 1 - 2j \text{ with } 0 \leq j \leq \ell(w_{k,l}) + 1 - n, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} J_{\mathrm{fin}}(s_0, \pi^u), & \text{if } q = r - 1 + 2j \text{ with } 0 \leq j \leq n - r + 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

as a $U(\mathbb{A}_f)$-module.

As explained in Sect. 2.2, the quotient $\mathcal{A}_\pi/\mathcal{L}_\pi$ of the Franke filtration is isomorphic to

$$\mathcal{A}_\pi/\mathcal{L}_\pi \cong I(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}}).$$

This holds even if the Eisenstein series attached to π^u is holomorphic at $s = s_0$. In that case $\mathcal{L}_\pi$ is trivial, and the full space $\mathcal{A}_\pi$ is isomorphic to that representation. Our next task is to compute the cohomology of the induced representation on the right-hand side.

Proposition 4.2. *Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$ be a cuspidal automorphic representation of the Levi factor $M_0(\mathbb{A})$ satisfying the necessary non-vanishing conditions with the minimal coset representative $w_{k,l} \in W^{P_0}$, where $1 \leq l < k \leq n+1$, and with parameters (r,i), where $1 \leq r \leq n$ and $1 \leq i \leq r$. Let $\mathcal{L}_\pi$ be the (possibly trivial) space spanned by the residues of the Eisenstein series attached to $\pi^u \cong \chi \otimes \sigma$ at $s = s_0 > 0$. Then, the cohomology space*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi/\mathcal{L}_\pi) = \begin{cases} I_{\mathrm{fin}}(s_0, \pi^u), & \text{if } q = \ell(w_{k,l}) = 2n - r, \\ 0, & \text{otherwise,} \end{cases}$$

as a $U(\mathbb{A}_f)$-module.

Proof. We follow closely the calculation in [11]. According to the description of the Franke filtration, we must calculate the cohomology of

$$I(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}}).$$

The induced representation $I(s_0, \pi^u)$ may be decomposed

$$I(s_0, \pi^u) \cong I_\infty(s_0, \pi^u) \otimes I_{\text{fin}}(s_0, \pi^u)$$

into the archimedean and the non-archimedean part. Factoring out the non-archimedean part in cohomology, reduces the calculation to finding the dimension of the cohomology space

$$H^q\big(\mathfrak{g}_\infty, K_\infty; I_\infty(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}})\big).$$

Let $w_{k,l} \in W^{P_0}$ be the Kostant representative, which determines the archimedean component π^u_∞ and the evaluation point s_0, by the necessary non-vanishing conditions, as in Sect. 3.4. Then, according to [4, Thm. III.3.3],

$$H^q\big(\mathfrak{g}_\infty, K_\infty; I_\infty(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}})\big) \cong H^{q-\ell(w_{k,l})}\big(\mathfrak{m}, K_\infty \cap M; \pi^u_\infty \otimes |\cdot|^{s_0} \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}}) \otimes F_{\mu_{w_{k,l}}}\big),$$

where $\ell(w_{k,l})$ is the length of $w_{k,l}$, and $F_{\mu_{w_{k,l}}}$ the finite-dimensional representation of ${}^0M$ of highest weight $\mu_{w_{k,l}}$ (see Sect. 3.3). Then, applying the Künneth rule to the decomposition $M = {}^0MA$, and using [6, p. 256], we obtain

$$H^q\big(\mathfrak{g}_\infty, K_\infty; I_\infty(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{\infty,\mathbb{C}})\big) \cong H^{q-\ell(w_{k,l})}\big({}^0\mathfrak{m}, K_\infty \cap {}^0M; \pi^u_\infty \otimes F_{\mu_{w_{k,l}}}\big),$$

and by compatibility of the cuspidal support π with the Kostant representative $w_{k,l}$, the last cohomology space is one-dimensional in degree zero, and vanishes in all other degrees. Since the length $\ell(w_{k,l}) = 2n - r$, in terms of parameters r and i, is given in Sect. 3.4, the proposition follows. □

4.3. Full Eisenstein cohomology. Having calculated the cohomology of the quotients of the Franke filtration, we are now in position to determine the full Eisenstein cohomology. As in Sect. 2.1, the decomposition along the cuspidal support, gives rise to

$$H^*_{\text{Eis}}(U) = \bigoplus_\pi H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi),$$

where the sum is over all cuspidal supports π compatible with the trivial coefficients. As already explained above, every such π gives rise to a minimal coset representative $w_{k,l} \in W^{P_0}$, where $1 \leq l < k \leq n+1$, and a pair (r, i) of parameters $1 \leq r \leq n$ and $1 \leq i \leq r$, which determine π_∞ uniquely. We fix one such π and calculate the corresponding summand in the Eisenstein cohomology.

Theorem 4.3. *Let U be a unitary group in $n+1$ variables defined over $\mathbb{Q}$ and of $\mathbb{Q}$-rank one. Let $P_0 = M_0N_0$ be the Levi decomposition of a representative of the unique conjugacy class of parabolic $\mathbb{Q}$-subgroups of U. Let $\pi \cong \chi|\cdot|^{s_0}_{\mathbb{I}} \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A})$, satisfying the necessary conditions for non-vanishing with the minimal coset representative $w_{k,l} \in W^{P_0}$, where $1 \leq l < k \leq n+1$, and with parameters (r, i), where $1 \leq r \leq n$ and $1 \leq i \leq r$. Then, the summand in the Eisenstein cohomology with support in π is given as follows.*

A. *If the Eisenstein series attached to $\pi^u \cong \chi \otimes \sigma$ is holomorphic at $s = s_0$, then the cohomology space*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\text{fin}}(s_0, \pi^u), & \text{if } q = \ell(w_{k,l}) = 2n - r, \\ 0, & \text{otherwise,} \end{cases}$$

as a $U(\mathbb{A}_f)$-module.

B. *If the Eisenstein series attached to* $\pi^u \cong \chi\otimes\sigma$ *has a pole at* $s = s_0$*, then the cohomology space*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong$$
$$\begin{cases} H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = \ell(w_{k,l}) - 1 - 2j \text{ with } 0 \le j \le \ell(w_{k,l}) - n, \\ \text{non-trivial submodule of } I_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = \ell(w_{k,l}), \\ H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \text{quotient of } J_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = \ell(w_{k,l}) + 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{cases} H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = r - 1 + 2j \text{ with } 0 \le j \le n - r, \\ \text{non-trivial submodule of } I_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = 2n - r, \\ H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \text{quotient of } J_{\mathrm{fin}}(s_0,\pi^u), & \text{if } q = 2n - r + 1, \\ 0, & \text{otherwise,} \end{cases}$$

as a $U(\mathbb{A}_f)$*-module, where the quotient in degree* $q = \ell(w_{k,l})+1 = 2n-r+1$ *is possibly trivial. In particular, the map in cohomology, induced by the inclusion* $\mathcal{L}_\pi \hookrightarrow \mathcal{A}_\pi$*, is injective, except possibly in degree* $q = \ell(w_{k,l}) + 1 = 2n - r + 1$*.*

Proof. In case A there is nothing to prove, as the cohomology of $\mathcal{A}_\pi$ has already been computed in Proposition 4.2. In case B, the proof just uses the long exact sequence in cohomology, obtained from the short exact sequence

$$0 \longrightarrow \mathcal{L}_\pi \longrightarrow \mathcal{A}_\pi \longrightarrow \mathcal{A}_\pi/\mathcal{L}_\pi \longrightarrow 0$$

given by the Franke filtration. Using Propositions 4.1 and 4.2, the long exact sequence can be written explicitly.

In all degrees except $q = \ell(w_{k,l})$ and $q = \ell(w_{k,l}) + 1$ we immediately get the result. For the remaining two degrees we get the exact sequence

$$0 \longrightarrow H^{\ell(w_{k,l})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \longrightarrow I_{\mathrm{fin}}(s_0,\pi^u) \longrightarrow J_{\mathrm{fin}}(s_0,\pi^u) \longrightarrow H^{\ell(w_{k,l})+1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \longrightarrow 0.$$

It is clear that $H^{\ell(w_{k,l})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ embeds into $I_{\mathrm{fin}}(s_0,\pi^u)$, and that $J_{\mathrm{fin}}(s_0,\pi^u)$ is mapped surjectively onto $H^{\ell(w_{k,l})+1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$, so that $H^{\ell(w_{k,l})+1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is a quotient of $J_{\mathrm{fin}}(s_0,\pi^u)$.

Suppose now that $H^{\ell(w_{k,l})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is trivial. This means that the map $I_{\mathrm{fin}}(s_0,\pi^u) \to J_{\mathrm{fin}}(s_0,\pi^u)$ is injective. But we know that $J_{\mathrm{fin}}(s_0,\pi^u)$ is a proper quotient of $I_{\mathrm{fin}}(s_0,\pi^u)$, as it is the non-archimedean part of $\mathcal{L}_\pi$, obtained as the image of the intertwining operator in the residue of the constant term of the Eisenstein series at the pole. This is a contradiction, showing that the cohomology space in degree $q = \ell(w_{k,l})$ is indeed non-trivial. □

5. On the analytic behavior of Eisenstein series

The explicit description of Eisenstein cohomology for the unitary group of relative rank one, obtained in Theorem 4.3, is given in terms of analytic properties of the Eisenstein series. In this section we study those properties in some relevant cases.

Having in mind cohomological applications, we work in this section with the unitary groups of $\mathbb{Q}$-rank one associated to imaginary quadratic extensions of $\mathbb{Q}$, as in Theorem 4.3. However, all the results regarding the analytic behavior of Eisenstein series hold, with the same proofs, for a more general setting of unitary groups of relative rank one obtained from any imaginary quadratic extension of algebraic number fields. The reason for our restriction is that the description of cohomology is less technical and more clear when working over $\mathbb{Q}$, and the general case may be approached by the same methods and using the Küneth rule, although the computations and the final result are more tedious.

Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of the Levi factor $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$, and σ a cuspidal automorphic representation of $U'(\mathbb{A})$. As in Sect. 2.2, for $s \in \mathbb{C}$ a complex parameter, let

$$\begin{aligned} I(s,\pi^u) &= \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})} \left(\pi^u \otimes |\cdot|_{\mathbb{I}_F}^s\right) \\ &= \mathrm{Ind}_{P_0(\mathbb{A})}^{U(\mathbb{A})} \left(\chi|\cdot|_{\mathbb{I}_F}^s \otimes \sigma\right) \end{aligned}$$

be the induced representation. Given a section f_s of induced representations, let $E(f_s,g)$ be the Eisenstein series associated to π^u, as introduced in Sect. 2.2. Our aim in this section is to determine the poles of $E(f_s,g)$ at s such that $Re(s)>0$, whose residues may possibly contribute to Eisenstein cohomology, under certain simplifying assumptions on π^u which are specified below.

5.1. Constant term of Eisenstein series. According to [25, Chapter 1], the parabolic subgroup P_0 is self-associate. From the general theory of Eisenstein series (cf. [20], [27]), the poles of the Eisenstein series $E(f_s,g)$ are the same as the poles of its constant term $E(f_s,g)_{P_0}$ along the parabolic subgroup P_0. The constant term can be expressed as

$$\begin{aligned} E(f_s,g) &= \int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} E(f_s,ng)dn \\ &= f_s(g) + M(s,\pi^u,w)f_s(g), \end{aligned}$$

where dn is the appropriate measure on the unipotent radical of P_0 and $M(s,\pi^u,w)$ is the standard intertwining operator on the induced representation $I(s,\pi^u)$, with w the unique non-trivial element of the relative Weyl group of the unitary group U. Thus, the poles of $E(f_s,g)$ coincide with the poles of the standard intertwining operator $M(s,\pi^u,w)f_s$.

Let S be the finite set of places of $\mathbb{Q}$, containing the archimedean place, and such that, for a non-archimedean place p of $\mathbb{Q}$, we have $p \notin S$ if and only if the following three assertions hold

- the extension $F/\mathbb{Q}$ is not ramified over p,
- the group U, viewed as an algebraic group over $\mathbb{Q}_p$, is quasi-split over $\mathbb{Q}_p$,
- the representation π^u is unramified at p.

Let $M(s,\pi^u_p,w)$ be the local standard intertwining operator at a place p of $\mathbb{Q}$. For $p \notin S$, let $f^\circ_{p,s}$ be the unique unramified vector in the local induced representation $I(s,\pi^u_p)$ normalized by the condition that it takes value one on the identity. The action of $M(s,\pi^u_p,w)$ on the normalized unramified vector $f^\circ_{p,s}$ is calculated in [19]. It is given by a ratio of L-functions

$$M(s,\pi^u_p,w)f^\circ_{p,s} = r(s,\pi^u_p,w)\widetilde{f}^\circ_{p,-s}\cdot,$$

where

$$r(s,\pi_p^u,w) = \begin{cases} \dfrac{L(s,\pi_p^u,r_1)L(2s,\chi_p,r_A)}{L(1+s,\pi_p^u,r_1)L(1+2s,\chi_p,r_A)}, & \text{for } n+1 \text{ even and } n>1, \\ \dfrac{L(s,\pi_p^u,r_1')L(2s,\chi_p,r_A')}{L(1+s,\pi_p^u,r_1')L(1+2s,\chi_p,r_A')}, & \text{for } n+1 \text{ odd}, \\ \dfrac{L(2s,\chi_p,r_A)}{L(1+2s,\chi_p,r_A)}, & \text{for } n=1. \end{cases}$$

In the above, $\widetilde{f}^\circ_{p,-s}$ is the unique normalized unramified vector in the induced representation $I(-s,w(\pi_p^u))$, and $w(\pi_p^u)$ is the conjugate of π_p^u by w. The automorphic L-functions appearing in the formula are the Asai (resp. twisted Asai) L-function $L(s,\chi_p,r_A)$ (resp. $L(s,\chi_p,r_A)$) attached to the character χ, and the automorphic L-functions of Rankin–Selberg type $L(s,\pi_p^u,r_1)$ and $L(s,\pi_p^u,r_1')$ attached to the pair χ and σ, where the finite-dimensional representations r_1 and r_1' are the other irreducible summands in the decomposition of the adjoint action of the L-group of the Levi factor on the Lie algebra of the L-group of the unipotent radical of P_0. See [8] for more details.

The unramified calculation motivates a definition of normalized intertwining operators. For $p \notin S$, we define the local normalized intertwining operator $N(s,\pi_p^u,w)$ by the relation

$$M(s,\pi_p^u,w) = r(s,\pi_p^u,w)N(s,\pi_p^u,w).$$

Let

$$r^S(s,\pi^u,w) = \prod_{p\notin S} r(s,\pi_p^u,w).$$

The product converges in some right half-plane, and admits analytic continuation to a meromorphic function on $\mathbb{C}$. It is given in terms of partial L-functions.

For a decomposable section $f_s = \otimes f_{p,s}$ of the induced representation $I(s,\pi^u)$, let $T(f)$ be a finite set of places, containing all the places in S, and such that for all $p \notin T(f)$ we have $f_{p,s} = f^\circ_{p,s}$. The action of the standard intertwining operator on a decomposable section f decomposes over places of $\mathbb{Q}$. Hence, we have

$$M(s,\pi^u,w)f_s(g) = r^S(s,\pi^u,w)\cdot\left[\otimes_{p\in S} M(s,\pi_p^u,w)f_{p,s}(g_p) \otimes_{p\in T(f)\setminus S} N(s,\pi_p^u,w)f_{p,s}(g_p) \otimes_{p\notin T(f)} \widetilde{f}^\circ_{p,-s}\right]. \qquad (\star)$$

Having this expression for the action of the intertwining operator, we are ready to relate the poles of the Eisenstein series $E(f_s,g)$ for s such that $Re(s)>0$ to the poles of the factor $r^S(s,\pi^u,w)$ involving partial automorphic L-functions.

5.2. Poles of Eisenstein series I – relation to automorphic L-functions. We now study the poles of Eisenstein series $E(f_s,g)$ associated to π^u, under the simplifying assumption on ramified non-archimedean places. More precisely, we assume that the local component π_p^u of π^u is a tempered representation for all non-archimedean places $p \in S$. Under this assumption, the following theorem relates the relevant poles of Eisenstein series to those of the normalizing factor.

Theorem 5.1. *Let $\pi^u \cong \chi\otimes\sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$*

such that σ is cohomological at the archimedean place. Suppose that σ_p is tempered for all non-archimedean places $p \in S$. Then, the poles of the Eisenstein series $E(f_s, g)$ associated to π^u at s such that $Re(s) > 0$, coincide with the poles of the normalizing factor $r^S(s, \pi^u, w)$.

Proof. Looking at the formula ($\star$), proving the theorem amounts to proving that, for all s such that $Re(s) > 0$, the expression in the square-bracket is holomorphic for all f_s, and that there is a function f_s for which it is non-zero. This means that one should prove that

- the local intertwining operator $M(s, \pi_p^u, w)$ at every place $p \in S$ is holomorphic and not identically vanishing for $Re(s) > 0$, and
- the local normalized intertwining operator $N(s, \pi_p^u, w)$ at every place $p \notin S$ is holomorphic and not identically vanishing for $Re(s) > 0$.

Consider first a place $p \in S$. If p is the archimedean place, the group $U'(\mathbb{Q}_p) = U(n-1)$ is compact, so σ_p is a discrete series. If p is non-archimedean, then by the assumption of the theorem, σ_p is tempered. In any case, the induced representation $I(s, \pi_p^u)$ for s such that $Re(s) > 0$ is a standard module of the Langlands classification. Since $M(s, \pi_p^u, w)$ is the long intertwining operator acting on $I(s, \pi_p^u)$, it is holomorphic and non-vanishing for $Re(s) > 0$, as its image is isomorphic to the Langlands quotient.

Now let $p \notin S$ be a place of $\mathbb{Q}$ that splits in F. Then σ_p is a unitary unramified irreducible representation of $GL_{n-1}(\mathbb{Q}_p)$ and $I(s, \pi_p^u)$ is a representation of $GL_{n+1}(\mathbb{Q}_p)$. Thus, we may apply [26, Prop. I.10] to show that $N(s, \pi_p^u, w)$ is holomorphic and non-vanishing for $Re(s) > 0$.

It remains to consider the case of a place $p \notin S$ that does not split in F. To do that, we first need to understand a weak base change lift of σ to a representation of $GL_{n-1}(\mathbb{A}_F)$. The most precise result is due to Shin in the appendix to [9], which is a slight improvement of the work of Morel [28]. The result we need is also contained in Labesse [18, Cor. 5.3], because in our case the condition ($*$) in *loc. cit.* is satisfied, although he makes the assumption that the base field is not $\mathbb{Q}$. According to [9, Thm. A.1] of Shin, since σ_p is cohomological at the archimedean place, there exists an automorphic representation Σ of $GL_{n-1}(\mathbb{A}_F)$, which is a weak base change of σ to $GL_{n-1}(\mathbb{A}_F)$ given as a local base change at all $p \notin S$ and all split non-archimedean places $p \in S$. Moreover, Σ is conjugate self-dual and isomorphic to an isobaric sum $\Sigma_1 \boxplus \cdots \boxplus \Sigma_d$ of conjugate self-dual representations Σ_i in the discrete spectrum of (smaller) general linear groups.

For $p \notin S$ which does not split in F, we look at the Satake parameters of the local component $\Sigma_{\mathfrak{P}}$ of Σ, at the place $\mathfrak{P}$ of F lying above p. By the classification of the discrete spectrum of the general linear group [26], each Σ_i is the unique irreducible quotient of an induced representation of the form

$$\mathrm{Ind}_{Q_i(\mathbb{A}_F)}^{GL_{m_i}(\mathbb{A}_F)} \left(\Pi_i |\det|_{\mathbb{I}_F}^{\frac{l_i-1}{2}} \otimes \Pi_i |\det|_{\mathbb{I}_F}^{\frac{l_i-3}{2}} \otimes \cdots \otimes \Pi_i |\det|_{\mathbb{I}_F}^{-\frac{l_i-1}{2}} \right)$$

where

- $m_i = k_i l_i$,
- Q_i is the standard parabolic subgroup of GL_{m_i} with the Levi factor isomorphic to a product $GL_{k_i} \times \cdots \times GL_{k_i}$ of l_i copies of GL_{k_i},
- Π_i is a cuspidal automorphic representation of $GL_{k_i}(\mathbb{A}_F)$.

For a non-split $p \notin S$, the local component $\Pi_{i,\mathfrak{P}}$ is a unitary generic unramified representation of $GL_{m_i}(F_{\mathfrak{P}})$. Hence, by [32] and [35], it is a fully induced representation of the form

$$\Pi_{i,\mathfrak{P}} \cong \mathrm{Ind}_{B_{k_i}(F_{\mathfrak{P}})}^{GL_{k_i}(F_{\mathfrak{P}})} \left(\otimes_j (\eta_j | \, |^{\alpha_j} \otimes \eta_j | \, |^{-\alpha_j}) \otimes_{j'} \eta'_{j'} \right)$$

where B_{k_i} is a Borel subgroup of GL_{k_i}, the characters η_j and $\eta'_{j'}$ of $F_{\mathfrak{P}}^\times$ are unitary and unramified, and $0 < \alpha_j < 1/2$. Hence, the Satake parameters of $\Sigma_{\mathfrak{P}}$ are given by sequences of characters of the following form

$$\eta|\,|^{\alpha+\frac{l-1}{2}}, \dots, \eta|\,|^{\alpha-\frac{l-1}{2}},$$

where η is a unitary unramified character of $F_{\mathfrak{P}}^\times$, l a positive integer, and $|\alpha| < 1/2$.

Since $\Sigma_{\mathfrak{P}}$ is the unramified base change of σ_p, using [23, Sect. 4], we determine the Satake parameters of σ_p. They consist of sequences of characters of $F_{\mathfrak{P}}^\times$ of the following forms:

- $\eta|\,|^{\frac{1}{2}+\alpha}, \dots, \eta|\,|^{\frac{l-1}{2}+\alpha}$, with $|\alpha| < 1/2$;
- $\eta|\,|^{\alpha}, \dots, \eta|\,|^{\frac{l-1}{2}+\alpha}$, with $0 \leq \alpha < 1/2$;
- $\eta|\,|^{1-\alpha}, \dots, \eta|\,|^{\frac{l-1}{2}-\alpha}$, with $0 \leq \alpha < 1/2$;

where η is a unitary unramified character of $F_{\mathfrak{P}}^\times$.

On the other hand, as an irreducible unramified representation of $U'(\mathbb{Q}_p)$, by the Langlands classification, σ_p is the unique irreducible subrepresentation of an induced representation of the form

$$\sigma_p \hookrightarrow \mathrm{Ind}_{R(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(\mu_1|\,|^{-s_1} \otimes \cdots \otimes \mu_m|\,|^{-s_m} \otimes \tau\right),$$

where R is the appropriate standard parabolic subgroup of U', viewed as a quasi-split $\mathbb{Q}_p$-group, μ_i are unitary unramified characters of $F_{\mathfrak{P}}^\times$, τ is a tempered unramified representation of a smaller quasi-split unitary group over $\mathbb{Q}_p$, and $s_1 \geq s_2 \geq \cdots \geq s_m > 0$ are the non-zero exponents appearing in the Satake parameters of σ_p, that is, the non-zero exponents appearing in the three possible sequences listed above.

The induced representation

$$\mathrm{Ind}_{B(F_{\mathfrak{P}})}^{GL_2(F_{\mathfrak{P}})}\left(\mu|\,|^{s} \otimes \mu'|\,|^{s'}\right)$$

of $GL_2(F_{\mathfrak{P}})$, where B is a Borel subgroup of GL_2, μ and μ' unitary characters of $F_{\mathfrak{P}}^\times$ and $s, s' \in \mathbb{R}$, is reducible if and only if $\mu = \mu'$ and $s - s' \in \{\pm 1\}$. Hence, we may permute the characters in the induced representation

$$\mathrm{Ind}_{R(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(\mu_1|\,|^{-s_1} \otimes \cdots \otimes \mu_m|\,|^{-s_m} \otimes \tau\right),$$

in such a way that we put together all the characters with the same unitary part and the exponent in the same class modulo $\mathbb{Z}$. The induced representation obtained in this way is isomorphic to the original one, so it contains σ_p as the unique irreducible subrepresentation. Taking into account the three possible sequences of characters that may appear in the Satake parameter of σ_p, a typical block with the same unitary character and the exponents in the same class modulo $\mathbb{Z}$ is of the form

$$\underbrace{\eta|\,|^{-x-k} \otimes \cdots \otimes \eta|\,|^{-x-k}}_{j_k \text{ times}} \otimes \underbrace{\eta|\,|^{-x-k+1} \otimes \cdots \otimes \eta|\,|^{-x-k+1}}_{j_{k-1} \text{ times}} \otimes \cdots \otimes \underbrace{\eta|\,|^{-x} \otimes \cdots \otimes \eta|\,|^{-x}}_{j_0 \text{ times}},$$

where $k \geq 0$ is an integer, $x \in \mathbb{R}$ is such that $0 < x \leq 1$, and $j_0 \geq j_1 \geq \cdots \geq j_k$.

According to the Zelevinsky classification [35] of unramified representations of the p-adic general linear group, the induced representation

$$\mathrm{Ind}_{B(F_{\mathfrak{P}})}^{GL_{l+1}(F_{\mathfrak{P}})}\left(\eta|\,|^{-x-l} \otimes \eta|\,|^{-x-l+1} \otimes \cdots \otimes \eta|\,|^{-x}\right),$$

where B is now a Borel subgroup of GL_{l+1}, contains a unique irreducible subrepresentation. Since η is unramified, this subrepresentation is unramified. It is called the Zelevinsky representation attached to the segment $[-x-l,-x]$ and η. We denote it by

$$J(\eta,-x-l,-x),$$

following [26].

Using the irreducibility criterion for Zelevinsky representations (cf. [35]), one may show that the representation of the general linear group induced from a typical block as above contains as a subrepresentation the representation induced from

$$\underbrace{J_k(\eta,x)\otimes\cdots\otimes J_k(\eta,x)}_{j_k \text{ times}}\otimes\underbrace{J_{k-1}(\eta,x)\otimes\cdots\otimes J_{k-1}(\eta,x)}_{j_{k-1}-j_k \text{ times}}\otimes\cdots\otimes\underbrace{J_0(\eta,x)\otimes\cdots\otimes J_0(\eta,x)}_{j_0-j_1 \text{ times}},$$

where $J_l(\eta,x)$ denotes $J(\eta,-x-l,-x)$ for $l=0,\dots,k$.

Combining such subrepresentations of all typical blocks we obtain a subrepresentation of the form

$$\mathrm{Ind}_{R'(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(J_1\otimes\cdots\otimes J_{m'}\otimes\tau\right)\quad\hookrightarrow\quad\mathrm{Ind}_{R(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(\mu_1|\,|^{-s_1}\otimes\cdots\otimes\mu_m|\,|^{-s_m}\otimes\tau\right),$$

where R' is a parabolic subgroup of U' viewed as a quasi-split $\mathbb{Q}_p$-group, and J_j are Zelevinsky representations attached to segments given by sequences of characters of the three possible forms as above appearing in the Satake parameters of σ_p. Note that for the second sequence and $\alpha=0$ the first character $\eta|\,|^\alpha$ is unitary, so it is a part of cuspidal support of τ, and does not appear in the segments for J_j's.

Since σ_p is the unique irreducible subrepresentation of the induced representation

$$\mathrm{Ind}_{R(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(\mu_1|\,|^{-s_1}\otimes\cdots\otimes\mu_m|\,|^{-s_m}\otimes\tau\right),$$

it is also a subrepresentation of

$$\sigma_p\hookrightarrow\mathrm{Ind}_{R'(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)}\left(J_1\otimes\cdots\otimes J_{m'}\otimes\tau\right).$$

Thus, by induction in stages,

$$I(s,\pi_p^u)=\mathrm{Ind}_{P(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)}\left(\chi_{\mathfrak{P}}|\,|^s\otimes\sigma_p\right)\quad\hookrightarrow\quad\mathrm{Ind}_{P'(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)}\left(\chi_{\mathfrak{P}}|\,|^s\otimes J_1\otimes\cdots\otimes J_{m'}\otimes\tau\right),$$

where P' is the standard parabolic subgroup of U, viewed as a quasi-split $\mathbb{Q}_p$-group, with the Levi factor isomorphic to $F_{\mathfrak{P}}^\times\times M_{R'}(\mathbb{Q}_p)$, where $M_{R'}$ is the Levi factor of R'.

Finally, we are ready to prove the holomorphy and non-vanishing for s such that $Re(s)>0$ of the normalized intertwining operator $N(s,\pi_p^u,w)$ acting on $I(s,\pi_p^u)$ for non-split $p\notin S$. Since $N(s,\pi_p^u,w)$ is the restriction of the corresponding normalized operator acting on

$$\mathrm{Ind}_{P'(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)}\left(\chi_{\mathfrak{P}}|\,|^s\otimes J_1\otimes\cdots\otimes J_{m'}\otimes\tau\right),$$

it is sufficient to show holomorphy on this larger induced representation. By Zhang's lemma [36], the non-vanishing follows from holomorphy. Thus, we may decompose the normalized operator into a composition of intertwining operators, and prove the holomorphy of each of them. The composition of holomorphic operators would again be holomorphic.

We decompose the operator on the larger induced representation as follows.

$$\begin{array}{cccc} & I(s,\pi_p^u) & \hookrightarrow & \mathrm{Ind}\,(\chi_{\mathfrak{P}}|\,|^s\otimes J_1\otimes\cdots\otimes J_{m'}\otimes\tau) \\ & & & \downarrow (1) \\ & & & \mathrm{Ind}\,(J_1\otimes\cdots\otimes J_{m'}\otimes\chi_{\mathfrak{P}}|\,|^s\otimes\tau) \\ {\scriptstyle N(s,\pi_p^u,w)} & \downarrow & & \downarrow (2) \\ & & & \mathrm{Ind}\left(J_1\otimes\cdots\otimes J_{m'}\otimes\overline{\chi}_{\mathfrak{P}}^c|\,|^{-s}\otimes\tau\right) \\ & & & \downarrow (3) \\ & I(-s,w(\pi_p^u)) & \hookrightarrow & \mathrm{Ind}\left(\overline{\chi}_{\mathfrak{P}}^c|\,|^{-s}\otimes J_1\otimes\cdots\otimes J_{m'}\otimes\tau\right) \end{array}$$

Here $\overline{\chi}_{\mathfrak{P}}^c$ is the conjugate by the non-trivial Galois automorphism c of the complex conjugate $\overline{\chi}_{\mathfrak{P}}$ of the character $\chi_{\mathfrak{P}}$. We must show that the three normalized operators, drawn as vertical arrows denoted (1), (2) and (3), are holomorphic.

For (1) the holomorphy for $Re(s)>0$ follows from the fact that $J_j = J(\eta,-x-l,-x)$ for some unitary character η, integer $l\geq 0$ and $0<x\leq 1$, so that

$$J_j = J(\eta,-l/2,l/2)|\det|^{-x-l/2},$$

which is a twist of a unitary Zelevinsky representation $J(\eta,-l/2,l/2)$ by a negative exponent $-x-l/2<0$. Since we are interested in s with $Re(s)>0$, we have $Re(s-(-x-l/2))>0$ and it follows from [26, Prop. I.10] that the operator (1) is holomorphic.

The operator (2) is holomorphic because τ is tempered, $\chi_{\mathfrak{P}}$ unitary, and $Re(s)>0$, so it acts on the standard module of the Langlands classification.

For (3), we use [26, Lemme I.8]. First of all, if the segment of J_j is not linked (in the sense of Zelevinsky [35]) to the segment of $\chi_{\mathfrak{P}}|\,|^{-s}$, which is just a singleton consisting of $-s$, then the induction is irreducible and the normalized operator is holomorphic. Since the segments $[-x-l,-x]$ of all J_j end with $-x\in[-1,0\rangle$, and $Re(-s)<0$, the linking cannot happen on that side of the segment. But if the linking happens on the other side of the segment, that is, $-s=-x-l-1$, then the segment $[-x-l,-x]$ dominates $-s$, so that condition (**P**) of [26, Sect. I.8] is satisfied. Thus, by [26, Lemme I.8], the operator (3) is also holomorphic. □

The same conclusion as in Theorem 5.1 regarding the poles of Eisenstein series may be deduced under a certain assumption on a weak base change of σ. Although this assumption is of different nature, in fact it implies that σ_p is tempered for all non-archimedean places $p\in S$. This is explained in the following corollary.

Corollary 5.2. *Let $\pi^u\cong\chi\otimes\sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A})\cong\mathbb{I}_F\times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ_p is cohomological at the archimedean place p. Suppose that U', viewed as an algebraic $\mathbb{Q}_p$-group, is quasi-split over $\mathbb{Q}_p$ for all non-archimedean $p\in S$, and that a weak base change of σ is cuspidal. Then, the poles of the Eisenstein series $E(f_s,g)$ associated to π^u at s such that $Re(s)>0$, coincide with the poles of the normalizing factor $r^S(s,\pi^u,w)$.*

Proof. As in the proof of Theorem 5.1, a weak base change Σ of σ to $GL_{n-1}(\mathbb{A}_F)$ exists according to Shin [9, Thm. A.1]. The additional assumption that Σ is cuspidal, and that U' is quasi-split at all $p\in S$, allows us to apply [18, Th. 5.9] of Labesse. The conclusion is that the weak base change Σ is in fact the strong base change of σ, that is, it is compatible with local base change at all $p\in S$.

Moreover, Σ is cohomological at the archimedean place, and conjugate self-dual, i.e., $\widetilde{\Sigma}^c = \Sigma$, where $\widetilde{\Sigma}$ denotes the contragredient representation and c stands for conjugation by the non-trivial Galois automorphism c. Recall that in our setting the condition $(*)$ of Labesse is satisfied because $U'(\mathbb{R})$ is compact (cf. [18, Remarque 5.2]).

By the result of Caraiani [5, Thm. 1.2], generalizing the earlier work of Shin [31], the local components of a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A}_F)$, which is conjugate self-dual and cohomological at the archimedean place, are tempered at all non-archimedean places of F. As Σ satisfies these requirements, we have that $\Sigma_{\mathfrak{P}}$ is tempered for all non-archimedean places $\mathfrak{P}$ of F.

For p a non-archimedean place of $\mathbb{Q}$ which splits in F, this means that σ_p is tempered. Since Σ is a strong base change of σ, if p is a non-archimedean place of $\mathbb{Q}$ which does not split in F, then $\Sigma_{\mathfrak{P}}$ is the local base change of σ_p, where $\mathfrak{P}$ is the place of F lying above p. By the description of the local base change, due to Mœglin [24], if the base change $\Sigma_{\mathfrak{P}}$ is tempered, then σ_p is necessarily tempered as well. Thus, we have that σ_p is tempered for all non-archimedean places p of $\mathbb{Q}$, so that the assumptions of Theorem 5.1 are satisfied. □

Remark 5.3. In the corollary we may have assumed that there exists a strong base change of σ to $GL_{n-1}(\mathbb{A}_F)$, which is cuspidal, conjugate self-dual and cohomological at the archimedean place. This would have been sufficient to apply the result of Caraiani, and prove the corollary. In fact, our assumption implies these requirements by the work of Labesse.

5.3. Poles of Eisenstein series II – unitarity argument. In this subsection, we determine certain regions of holomorphy for the Eisenstein series $E(f_s, g)$, using a unitarity argument, that goes back to Shahidi and Kim, as in [14, Sect. 4.3], where the case of the Siegel parabolic subgroup of a quasi-split unitary group was considered. The assumptions of the theorem below are slightly more general than those in Corollary 5.2, as it is not assumed that the group U is quasi-split at non-archimedean places $p \in S$.

Theorem 5.4. *Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ is cohomological at the archimedean place. Suppose that a weak base change of σ, constructed in* [5, Thm. A.1]*, is a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. Then the Eisenstein series $E(f_s, g)$, associated to π^u, is holomorphic at s with $Re(s) \geq 3/2$.*

Proof. The strategy of the proof is the following. If the Eisenstein series $E(f_s, g)$, associated to π^u, had a pole at $s = s_0$ with $Re(s_0) > 0$, then the residues would generate an automorphic representation in the residual spectrum of $U(\mathbb{A})$. In particular, this automorphic representation would be unitary, and thus, its local components would be unitary at every place.

On the other hand, the space of residues is isomorphic to the image of some intertwining operator acting on the induced representation $I(s_0, \pi^u)$. Therefore, if the Eisenstein series had a pole at $s = s_0$, then the induced representation $I(s_0, \pi_p^u)$ would have a unitary subquotient for every place p of $\mathbb{Q}$. Thus, proving that $I(s_0, \pi_p^u)$ has no unitary subquotients for any place p of $\mathbb{Q}$, implies the holomorphy of $E(f_s, g)$ at $s = s_0$.

Let p be a place of $\mathbb{Q}$ that splits in F and such that $p \notin S$. As in the proof of Theorem 5.1, according to [5, Thm. A.1] of Shin, there exists a weak base change Σ of σ to $GL_{n-1}(\mathbb{A}_F)$. By the assumption of the theorem, the base change Σ is cuspidal.

Hence, at the place p, the representation σ_p is an unramified local component of a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. In particular, it is generic and unitary. By the classification of the generic unitary dual of the p-adic general linear group (cf. [32]), σ_p is a fully induced representation of the form

$$\sigma_p \cong \mathrm{Ind}_{B_{n-1}(\mathbb{Q}_p)}^{GL_{n-1}(\mathbb{Q}_p)} \left(\eta_1|\,|^{\beta_1} \otimes \cdots \otimes \eta_{n-1}|\,|^{\beta_{n-1}}\right)$$

where, for $j = 1, \ldots, n-1$, η_j are unitary characters of $\mathbb{Q}_p^\times$, β_j are real numbers such that $|\beta_j| < 1/2$, and B_k is a Borel subgroup of GL_k, for k a positive integer. Since p splits in F, the local component of χ at p is a product of two characters $\chi_{\mathfrak{P}_1}$ and $\chi_{\mathfrak{P}_2}$, where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are the two places of F lying above p.

Hence, $I(s, \pi_p^u)$ is of the form

$$I(s, \pi_p^u) \cong \mathrm{Ind}_{B_{n+1}(\mathbb{Q}_p)}^{GL_{n+1}(\mathbb{Q}_p)} \left(\chi_{\mathfrak{P}_1}|\,|^{s} \otimes \eta_1|\,|^{\beta_1} \otimes \cdots \otimes \eta_{n-1}|\,|^{\beta_{n-1}} \otimes \chi_{\mathfrak{P}_2}|\,|^{-s}\right).$$

This induced representation is irreducible for $Re(s) \geq 3/2$, because the exponents satisfy inequalities $s \pm \beta_j > 1$ and $|\beta_j - \beta_{j'}| < 1$, so that no pair of characters can produce reducibility, by the representation theory of $GL_2(\mathbb{Q}_p)$. By the classification of the unitary dual of the general linear group over a non-archimedean field (cf. [32]), this induced representation is not unitary, since the exponents s and $-s$ satisfy $|Re(s)| \geq 1/2$. This shows that the induced representation $I(s, \pi_p^u)$ has no unitary subquotients for $Re(s) \geq 3/2$, and thus, the Eisenstein series $E(f_s, g)$ is indeed holomorphic for $Re(s) \geq 3/2$ as claimed. □

5.4. Poles of Eisenstein series III – non-self-conjugate case. We begin now with describing the poles of Eisenstein series $E(f_s, g)$ in the half-plane $Re(s) > 0$. First we consider the case in which χ is *not* conjugate self-dual, that is, χ is *non-trivial* on the norm subgroup $N_{F/\mathbb{Q}}(\mathbb{I}_F)$. In that case the Eisenstein series $E(f_s, g)$ is holomorphic for $Re(s) > 0$ by a general result which we now recall. Note that in this case there is no assumption on σ.

Theorem 5.5. *Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$. Suppose that χ is non-trivial on the norm subgroup $N_{F/\mathbb{Q}}(\mathbb{I}_F)$. Then the Eisenstein series $E(f_s, g)$, constructed from π^u, is holomorphic in the half-plane $Re(s) > 0$.*

Proof. By [27, Sect. IV.3.12], a necessary condition for the Eisenstein series $E(f_s, g)$ to have a pole at s with $Re(s) > 0$ is that $\pi^w \cong \pi$. In our case, $\pi^w = (\chi^c)^{-1} \otimes \sigma$, so that if $(\chi^c)^{-1} \neq \chi$, this necessary condition is not satisfied. But the latter condition in fact says that χ is non-trivial on the norm group $N_{F/\mathbb{Q}}(\mathbb{I}_F)$. Thus, for such χ the Eisenstein series $E(f_s, g)$ is holomorphic at s such that $Re(s) > 0$, as claimed. □

5.5. Poles of Eisenstein series IV – conclusions. In this section, we provide an explicit description of poles at s such that $Re(s) > 0$ of the Eisenstein series $E(f_s, g)$, associated to $\pi^u \cong \chi \otimes \sigma$ where χ is a unitary Hecke character of $\mathbb{I}_F$ and σ a cuspidal automorphic representation of $U'(\mathbb{A})$, in terms of the analytic properties of the automorphic L-functions appearing in the normalizing factor $r^S(s, \pi^u, w)$.

Since in the case when χ is not conjugate self-dual, the Eisenstein series is holomorphic in the region $Re(s) > 0$ by Theorem 5.5, we will assume now that χ *is conjugate self-dual*, that is, trivial on the norm subgroup $N_{F/\mathbb{Q}}(\mathbb{I}_F)$. Thus, χ restricted to $\mathbb{I}$ is either trivial or the quadratic character $\delta_{F/\mathbb{Q}}$ of $\mathbb{I}$ attached to the extension $F/\mathbb{Q}$ by class field theory.

We treat first the case $n \geq 3$, because in the low rank cases $n = 1$ and $n = 2$ we provide a more precise description below.

Theorem 5.6. *Let $n \geq 3$. Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$ which is conjugate self-dual, and σ a cuspidal automorphic representation of $U'(\mathbb{A})$ such that σ is cohomological at the archimedean place. Suppose that σ_p is tempered for all non-archimedean places $p \in S$. Then, the Eisenstein series $E(f_s, g)$, associated to π^u, is holomorphic at s such that $Re(s) > 0$, except for possible simple poles at $s \in \{1/2, 1, 3/2, \ldots, n/2\}$.*

The pole at $s = 1/2$ occurs if and only if

- *either condition $\mathcal{C}_{\mathrm{even}}$, given by*

$$\mathcal{C}_{\mathrm{even}} \equiv \begin{cases} n+1 \text{ is even}, \\ \text{the restriction of } \chi \text{ to } \mathbb{I} \text{ is trivial}, \\ L^S(1/2, \chi \otimes \sigma, r_1) \neq 0, \end{cases}$$

- *or condition $\mathcal{C}_{\mathrm{odd}}$, given by*

$$\mathcal{C}_{\mathrm{odd}} \equiv \begin{cases} n+1 \text{ is odd}, \\ \text{the restriction of } \chi \text{ to } \mathbb{I} \text{ is the quadratic character } \delta_{F/\mathbb{Q}} \text{ of } \mathbb{I} \\ \qquad \text{attached to the extension } F/\mathbb{Q} \text{ by class field theory}, \\ L^S(1/2, \chi \otimes \sigma, r_1') \neq 0, \end{cases}$$

is satisfied. The pole at $s = \frac{m+1}{2}$ with $1 \leq m \leq n-1$ an integer occurs if and only if the weak local lift Σ of σ to $GL_{n-1}(\mathbb{A}_F)$ contains as a summand in the isobaric sum the discrete spectrum representation of $GL_m(\mathbb{A}_F)$ isomorphic to the unique irreducible quotient $J(m, \chi^c)$ of the induced representation

$$\mathrm{Ind}_{B_m(\mathbb{A}_F)}^{GL_m(\mathbb{A}_F)} \left(\chi^c |\,|_{\mathbb{I}_F}^{\frac{m-1}{2}} \otimes \chi^c |\,|_{\mathbb{I}_F}^{\frac{m-3}{2}} \otimes \cdots \otimes \chi^c |\,|_{\mathbb{I}_F}^{-\frac{m-1}{2}} \right),$$

where B_m is a Borel subgroup of GL_m, and χ^c is the conjugate of χ by the non-trivial Galois automorphism c.

Proof. According to Theorem 5.1, the poles of $E(f_s, g)$ in the region $Re(s) > 0$ coincide with the poles of $r^S(s, \pi^u, w)$. Recall that

$$r^S(s, \pi^u, w) = \begin{cases} \dfrac{L^S(s, \pi^u, r_1) L^S(2s, \chi, r_A)}{L^S(1+s, \pi^u, r_1) L^S(1+2s, \chi, r_A)}, & \text{for } n+1 \text{ even}, \\ \dfrac{L^S(s, \pi^u, r_1') L^S(2s, \chi, r_A')}{L^S(1+s, \pi^u, r_1') L^S(1+2s, \chi, r_A')}, & \text{for } n+1 \text{ odd}. \end{cases}$$

As already explained in the proof of Theorem 5.1, the assumptions on σ imply, according to [5, Thm. A.1], that there is a weak lift Σ of σ to $GL_{n-1}(\mathbb{A}_F)$, which is an isobaric sum $\Sigma = \Sigma_1 \boxplus \cdots \boxplus \Sigma_d$ of conjugate self-dual discrete spectrum representations Σ_i of general linear groups. Since the Satake parameters of σ and Σ match, the first L-function in the formula for $r^S(s, \pi^u, w)$ may be written as the Rankin–Selberg L-function of pairs

$$L^S(s, \pi^u, r_1) = L^S(s, \chi \times \Sigma),$$

and similarly for $L^S(s,\pi^u,r_1')$. On the other hand, the Asai L-function of a Hecke character is just the Hecke L-function of the restriction

$$L^S(s,\chi,r_A) = L^S\left(s,\chi\big|_{\mathbb{I}}\right),$$

and similarly for the twisted Asai L-function

$$L^S(s,\chi,r_A') = L^S(s,\chi\otimes\widehat{\delta},r_A) = L^S(s,\delta_{F/\mathbb{Q}}\chi\big|_{\mathbb{I}}),$$

where $\widehat{\delta}$ is any extension of $\delta_{F/\mathbb{Q}}$ to a quadratic character of $\mathbb{I}_F$ (see [8]).

By the classification of the discrete spectrum of the general linear group [26], Σ_i is isomorphic to the unique irreducible quotient of

$$\mathrm{Ind}_{Q_i(\mathbb{A}_F)}^{GL_{m_i}(\mathbb{A}_F)}\left(\Pi_i|\det|_{\mathbb{I}_F}^{\frac{l_i-1}{2}}\otimes\Pi_i|\det|_{\mathbb{I}_F}^{\frac{l_i-3}{2}}\otimes\cdots\otimes\Pi_i|\det|_{\mathbb{I}_F}^{-\frac{l_i-1}{2}}\right),$$

where $m_i = k_i l_i$, Π_i is a cuspidal automorphic representation of $GL_{k_i}(\mathbb{A}_F)$, and Q_i is a parabolic subgroup of GL_{m_i} with the Levi factor isomorphic to the product of l_i copies of GL_{k_i}. By the well-known formulas for Rankin–Selberg L-functions of pairs (cf. [16]), we have

$$\begin{aligned}\frac{L^S(s,\chi\times\Sigma)}{L^S(1+s,\chi\times\Sigma)} &= \prod_{i=1}^{d}\frac{L^S(s,\chi\times\Sigma_i)}{L^S(1+s,\chi\times\Sigma_i)}\\ &= \prod_{i=1}^{d}\frac{L^S(s-\frac{l_i-1}{2},\chi\times\Pi_i)}{L^S(1+s+\frac{l_i-1}{2},\chi\times\Pi_i)}.\end{aligned}$$

In the second line of this equation, all terms in the formula for the Rankin–Selberg L-functions cancel, except the first in the numerator and the last in the denominator. Since the Rankin–Selberg and Hecke L-functions have no zeroes in $Re(s) > 1$, the denominator of $r^S(s,\pi^u,w)$ cannot produce a pole in the half-plane $Re(s) > 0$. It remains to describe the pole of the numerator.

Poles of the partial L-functions are always among the poles of the complete L-function, since local L-factors have no zeroes. In our case, the complete Hecke L-function $L(s,\mu)$, where μ is a unitary Hecke character of either $\mathbb{I}$ or $\mathbb{I}_F$, is holomorphic, except for possible simple poles at $s = 0$ and $s = 1$. The poles occur if and only if μ is trivial. Since S is not empty (contains at least the archimedean place), and the local L-function of the trivial character has a pole at $s = 0$, it follows that the partial L-function $L^S(s,\mu)$ has a simple pole at $s = 1$ if and only if μ is trivial, and it is holomorphic elsewhere. Similarly, the complete Rankin–Selberg L-function $L(s,\chi\times\Pi)$, where χ is a unitary Hecke character of $\mathbb{I}_F$ and Π a cuspidal automorphic representation of $GL_k(\mathbb{A}_F)$, is holomorphic except for possible simple poles at $s = 0$ and $s = 1$. The poles occur if and only if Π is in fact a unitary Hecke character of $GL_1(\mathbb{A}_F)$, that is, $k = 1$, and $\Pi = \chi^{-1}$. But in that case, this Rankin–Selberg L-function is just a Hecke L-function of the trivial character. Thus, as before, the partial L-function $L^S(s,\chi\times\Pi)$ has a pole at $s = 1$ if and only $\Pi = \chi^{-1}$, and it is holomorphic elsewhere.

From these properties of partial L-functions we deduce the theorem. The pole at $s = 1/2$ occurs if and only if the Asai (resp. twisted Asai) L-function has a pole at $2s = 1$ and the Rankin–Selberg L-function does not cancel that pole, that is, it is non-zero at $s = 1/2$. The pole of the Asai (resp. twisted Asai) L-function is given in terms of the restriction of χ to $\mathbb{I}$ due to the formula above relating it to the Hecke L-function of the restriction.

The pole at $s = \frac{m+1}{2}$ arises from the Rankin–Selberg L-functions. It occurs if and only if there is a Σ_i such that the associated cuspidal automorphic representation Π_i introduced above is equal to χ^{-1} and the corresponding $l_i = m$, because then $s - \frac{m-1}{2} = 1$ gives the pole, that is, $s = \frac{m+1}{2}$. Since χ is conjugate self-dual, $\chi^{-1} = \chi^c$. This gives the last claim of the theorem. □

Consider now the low rank cases $n = 1$ and $n = 2$. In these cases the relative rank one unitary group U is quasi-split as an algebraic $\mathbb{Q}$-group. Hence, we may be more precise in describing the poles of Eisenstein series.

Theorem 5.7. *Let U be the quasi-split unitary group in two variables, i.e., $n = 1$. Let χ be a unitary Hecke character of the Levi factor $M_0(\mathbb{A}) \cong \mathbb{I}_F$. Then, the Eisenstein series $E(f_s, g)$, associated to χ, is holomorphic for s such that $Re(s) > 0$, except for a possible simple pole at $s = 1/2$. The pole at $s = 1/2$ occurs if and only if the restriction of χ to $\mathbb{I}$ is trivial.*

Proof. In the case $n = 1$, the poles of the Eisenstein series $E(f_s, g)$ for s such that $Re(s) > 0$ coincide with the poles of the normalizing factor $r^S(s, \chi, w)$, without any assumptions because there is no σ in the representation of $M_0(\mathbb{A})$. In the case $n = 1$, recall that

$$r^S(s, \chi, w) = \frac{L^S(2s, \chi, r_A)}{L^S(1 + 2s, \chi, r_A)}.$$

As in the proof of Theorem 5.6, the properties of the Asai L-functions imply that the normalizing factor is holomorphic for s such that $Re(s) > 0$, except possibly for $s = 1/2$, and that the pole at $s = 1/2$ occurs if and only if the restriction of χ to $\mathbb{I}$ is trivial. □

Theorem 5.8. *Let U be the quasi-split unitary group in three variables, i.e., $n = 2$. Let $\pi^u \cong \chi \otimes \sigma$ be a cuspidal automorphic representation of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}_F$, and σ is a character of the unitary group $U'(\mathbb{A})$ in one variable, that is, a norm-one subgroup of $\mathbb{I}_F$. Then, the Eisenstein series $E(f_s, g)$, associated to π^u, is holomorphic for s such that $Re(s) > 0$, except for possible simple poles at $s = 1/2$ and $s = 1$. The pole at $s = 1/2$ occurs if and only if the restriction of χ to $\mathbb{I}$ is the quadratic character $\delta_{F/\mathbb{Q}}$ of $\mathbb{I}$ attached to the extension $F/\mathbb{Q}$ by class field theory, and $L^S(1/2, \pi^u, r_1') \neq 0$. The pole at $s = 1$ occurs if and only if the character χ is equal to the conjugate Σ^c of a base change Σ of σ.*

Proof. The proof of this low rank example is the same as the proof of Theorem 5.6 in the case of $n+1$ odd. The reason for stating the result separately is that the assumption on σ in Theorem 5.6 is always satisfied, and that the necessary and sufficient conditions for the poles are a bit simplified, because the Rankin–Selberg L-function in the constant term becomes the Hecke L-function. □

5.6. Poles of Eisenstein series V – the case of the trivial representation. We consider here the case of the trivial representation of $U(\mathbb{A})$, and show how it is realized in the residual spectrum of $U(\mathbb{A})$.

Theorem 5.9. *Let $\pi^u \cong \mathbf{1}_{\mathbb{I}_F} \otimes \mathbf{1}_{U'(\mathbb{A})}$ be the trivial representation of the Levi factor $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, where $\mathbf{1}_{\mathbb{I}_F}$ is the trivial character of $\mathbb{I}_F$ and $\mathbf{1}_{U'(\mathbb{A})}$ is the trivial representation of $U'(\mathbb{A})$. Then, the Eisenstein series $E(f_s, g)$, associated to the trivial representation π^u, has a simple pole at $s = n/2$, and the space $\mathcal{L}_\pi$ spanned by the residues, where $\pi \cong \pi^u \otimes |\cdot|_{\mathbb{I}_F}^{n/2}$, is isomorphic to the trivial representation $\mathbf{1}_{U(\mathbb{A})}$ of $U(\mathbb{A})$.*

Proof. The proof is similar to the proof of Theorem 5.1. The idea is again to show that the pole of the Eisenstein series at $s = n/2$ coincides with the pole of the normalizing factor $r^S(s, \pi^u, w)$ at $s = n/2$. In view of formula $(\star)$, as in the proof of Theorem 5.1, one must prove that

- the local intertwining operator $M(s, \pi_p^u, w)$ at every place $p \in S$ is holomorphic and not identically vanishing at $s = n/2$, and
- the local normalized intertwining operator $N(s, \pi_p^u, w)$ at every place $p \notin S$ is holomorphic and not identically vanishing at $s = n/2$.

Observe that the trivial representation of any quasi-split group is unramified. Hence, in the case of the trivial representation, the finite set S consists of the archimedean place and all non-archimedean places p of $\mathbb{Q}$ such that p does not split in F and U' as a $\mathbb{Q}_p$-group is not quasi-split over $\mathbb{Q}_p$.

The same proof as in Theorem 5.1 shows that the local intertwining operator $M(s, \pi_p^u, w)$ at the archimedean place, as well as the local normalized intertwining operator $N(s, \pi_p^u, w)$ at every place $p \notin S$, is holomorphic and not identically vanishing for $Re(s) > 0$, which includes the point $s = n/2$. The image in both cases is isomorphic to the trivial representation of the local group, i.e., $U(\mathbb{R})$ at the archimedean place and $U(\mathbb{Q}_p)$ at the non-archimedean place $p \notin S$.

It remains to show that the local intertwining operator $M(s, \pi_p^u, w)$ at every non-archimedean place $p \in S$ is holomorphic at $s = n/2$ and to determine its image. As already mentioned above, for a non-archimedean place $p \in S$, we have that p does not split in F and that U' is not quasi-split as a $\mathbb{Q}_p$-group. By the classification of unitary groups over a p-adic field, recalled in Section 1, this means that $n-1$ is even, and that the minimal parabolic $\mathbb{Q}_p$-subgroup $P'_{\min}(\mathbb{Q}_p)$ of $U'(\mathbb{Q}_p)$ has the Levi factor

$$M'_{\min}(\mathbb{Q}_p) \cong \underbrace{F_{\mathfrak{P}}^\times \times \cdots \times F_{\mathfrak{P}}^\times}_{n-3} \times U^\circ,$$

where $\mathfrak{P}$ is the place of F lying above p, and U° is the unitary group of the unique (up to isomorphism) anisotropic two-dimensional hermitian space over $\mathbb{Q}_p$. In this case, the trivial representation of $U'(\mathbb{Q}_p)$ is the Langlands quotient of the induced representation

$$\mathrm{Ind}_{P'_{\min}(\mathbb{Q}_p)}^{U'(\mathbb{Q}_p)} \left(|\cdot|_{\mathfrak{P}}^{\frac{n}{2}-1} \otimes \cdots \otimes |\cdot|_{\mathfrak{P}}^{\frac{3}{2}} \otimes \mathbf{1}_{U^\circ} \right).$$

Hence, the local intertwining operator $M(s, \pi_p^u, w)$ for $s = n/2$ fits into the following diagram

$$\begin{array}{ccc}
\mathrm{Ind}_{P_{\min}(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)} \Big(|\cdot|_{\mathfrak{P}}^{\frac{n}{2}} \otimes & |\cdot|_{\mathfrak{P}}^{\frac{n}{2}-1} \otimes \cdots \otimes |\cdot|_{\mathfrak{P}}^{\frac{3}{2}} \otimes \mathbf{1}_{U^\circ} \Big) & \\
& \downarrow \; M_{\mathrm{long},U'} & \\
\mathrm{Ind}_{P_0(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)} \Big(|\cdot|_{\mathfrak{P}}^{\frac{n}{2}} \otimes & \mathbf{1}_{U'(\mathbb{Q}_p)} \Big) \hookrightarrow & \mathrm{Ind}_{P_{\min}(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)} \Big(|\cdot|_{\mathfrak{P}}^{\frac{n}{2}} \otimes |\cdot|_{\mathfrak{P}}^{-\frac{n}{2}+1} \otimes \cdots \otimes |\cdot|_{\mathfrak{P}}^{-\frac{3}{2}} \otimes \mathbf{1}_{U^\circ} \Big) \\
& \downarrow \; M(n/2, \pi_p^u, w) & \\
\mathrm{Ind}_{P_0(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)} \Big(|\cdot|_{\mathfrak{P}}^{-\frac{n}{2}} \otimes & \mathbf{1}_{U'(\mathbb{Q}_p)} \Big) \hookrightarrow & \mathrm{Ind}_{P_{\min}(\mathbb{Q}_p)}^{U(\mathbb{Q}_p)} \Big(|\cdot|_{\mathfrak{P}}^{-\frac{n}{2}} \otimes |\cdot|_{\mathfrak{P}}^{-\frac{n}{2}+1} \otimes \cdots \otimes |\cdot|_{\mathfrak{P}}^{-\frac{3}{2}} \otimes \mathbf{1}_{U^\circ} \Big),
\end{array}$$

where $P_{\min}$ is a minimal parabolic $\mathbb{Q}_p$-subgroup of U, and $M_{\mathrm{long},U'}$ is the longest intertwining operator of the Langlands classification for the trivial representation of $U'(\mathbb{Q}_p)$, viewed as an intertwining operator on $U(\mathbb{Q}_p)$. Clearly, the composition of two vertical arrows,

$$M(n/2, \pi_p^u, w) \circ M_{\mathrm{long},U'} = M_{\mathrm{long},U}$$

is the longest intertwining operator of the Langlands classification for the trivial representation of $U(\mathbb{Q}_p)$. Hence, $M(s,\pi_p^u,w)$ is holomorphic at $s=n/2$ and its image is the trivial representation of $U(\mathbb{Q}_p)$.

Thus, we have proved that the possible pole at $s=n/2$ of the Eisenstein series $E(f_s,g)$, associated to the trivial representation π^u of $M_0(\mathbb{A})$, is determined by the analytic behavior of the normalizing factor $r^S(s,\pi^u,w)$ at $s=n/2$ for the trivial representation π^u. Recall that

$$r^S(n/2,\pi^u,w)=\begin{cases}\dfrac{L^S(n/2,\pi^u,r_1)L^S(n,\mathbf{1}_{\mathbb{I}_F},r_A)}{L^S(1+n/2,\pi^u,r_1)L^S(1+n,\mathbf{1}_{\mathbb{I}_F},r_A)}, & \text{for } n+1 \text{ even},\\[2ex] \dfrac{L^S(n/2,\pi^u,r_1')L^S(n,\mathbf{1}_{\mathbb{I}_F},r_A')}{L^S(1+n/2,\pi^u,r_1')L^S(1+n,\mathbf{1}_{\mathbb{I}_F},r_A')}, & \text{for } n+1 \text{ odd},\end{cases}$$

where $\mathbf{1}_{\mathbb{I}_F}$ is the trivial character of $\mathbb{I}_F$. Since the Asai L-function $L^S(s,\mu,r_A)$, and the twisted Asai L-function $L^S(s,\mu,r_A')$, are holomorphic and non-zero for s such that $Re(s)>1$ (cf. [8] and Section 5.5), the pole does not arise from these L-functions, as long as $n>1$. The case $n=1$ is treated in Theorem 5.7. For the Rankin–Selberg type L-function, we have

$$L^S(s,\pi^u,r_1)=L^S(s,\mathbf{1}_{\mathbb{I}_F}\times\mathbf{1}_{GL_{n-1}(\mathbb{A}_F)}),$$

and similarly for $L(s,\pi^u,r_1')$, because the base change of the trivial representation of $U'(\mathbb{A})$ is the trivial representation of $GL_{n-1}(\mathbb{A}_F)$. According to [16], the Rankin–Selberg L-function equals

$$L^S(s,\mathbf{1}_{\mathbb{I}_F}\times\mathbf{1}_{GL_{n-1}(\mathbb{A}_F)})=\prod_{j=1}^{n-1}L^S(s+\frac{n}{2}-j,\mathbf{1}_{\mathbb{I}_F}),$$

where the L-functions on the right-hand side are the Hecke L-functions of the trivial character of $\mathbb{I}_F$, so that the quotient, after cancellations, becomes

$$\frac{L^S(s,\mathbf{1}_{\mathbb{I}_F}\times\mathbf{1}_{GL_{n-1}(\mathbb{A}_F)})}{L^S(1+s,\mathbf{1}_{\mathbb{I}_F}\times\mathbf{1}_{GL_{n-1}(\mathbb{A}_F)})}=\frac{L^S(s-\frac{n}{2}+1,\mathbf{1}_{\mathbb{I}_F})}{L^S(s+\frac{n}{2},\mathbf{1}_{\mathbb{I}_F})}.$$

By the analytic properties of Rankin–Selberg L-functions (cf. Section 5.5), the denominator is holomorphic and non-zero at $s=n/2$, while the numerator has a simple pole at s such that $s-n/2+1=1$, i.e., at $s=n/2$. This means that the Eisenstein series $E(f_s,g)$, associated to the trivial representation of $M_0(\mathbb{A})$, has a simple pole at $s=n/2$. The residual representation $\mathcal{L}_\pi$, spanned by the residues at $s=n/2$, is isomorphic to the restricted tensor product of the images of local intertwining operators. But we have seen above that these images are isomorphic to the trivial representation for every place p of $\mathbb{Q}$. Thus, the representation $\mathcal{L}_\pi$ is isomorphic to the trivial representation of $U(\mathbb{A})$. □

6. Eisenstein cohomology – final results

In this section we use the analytic properties of Eisenstein series, determined in Section 5, to make the general Theorem 4.3 more precise in certain cases. All the results follow directly from Theorem 4.3 using the analytic properties of Eisenstein series determined in Section 5.

We state all the results in this section for the evaluation points s_0 such that the necessary non-vanishing conditions are satisfied. Observe that for all other s_0, i.e., those for which the non-vanishing conditions are not satisfied, the corresponding summand in cohomology is trivial, and thus there is nothing to describe.

6.1. Contributions in the regions of holomorphy. We describe now the contribution to Eisenstein cohomology in the cases for which the Eisenstein series is holomorphic at the relevant point of evaluation.

Theorem 6.1. *Let* $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{\frac{k-l}{2}} \otimes \sigma$ *be a cuspidal automorphic representation of* $M_0(\mathbb{A})$*, where* χ *is a unitary Hecke character of* $\mathbb{I}_F$ *and* σ *a unitary cuspidal automorphic representation of* $U'(\mathbb{A})$*, such that the necessary conditions for non-vanishing are satisfied with the minimal coset representative* $w_{k,l} \in W^{P_0}$*, where* $1 \leq l < k \leq n+1$*, and with parameters* (r, i)*, where* $1 \leq r \leq n$ *and* $1 \leq i \leq r$*. Suppose that*

- *either* χ *is non-trivial on the norm subgroup* $N_{F/\mathbb{Q}}(\mathbb{I}_F)$*,*
- *or a weak base change of* σ*, constructed in* [9, Thm. A.1]*, is cuspidal, and* $k - l \geq 3$*.*

Then, the summand in the Eisenstein cohomology supported in π *is isomorphic as a* $U(\mathbb{A}_f)$*-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\mathrm{fin}}\left(\frac{k-l}{2}, \pi^u\right), & \text{if } q = \ell(w_{k,l}) = n+k-l-1 = 2n-r, \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi^u \cong \chi \otimes \sigma$ *is the unitary part of* π*.*

Proof. According to Theorem 5.5, the Eisenstein series associated to $\pi^u \cong \chi \otimes \sigma$ such that χ is non-trivial on the norm subgroup $N_{F/\mathbb{Q}}(\mathbb{I}_F)$ is holomorphic in the half-plane $Re(s) > 0$. Hence, the theorem in this case follows directly from part A of Theorem 4.3.

In the second case, according to Theorem 5.4, the Eisentein series associated to $\pi^u \cong \chi \otimes \sigma$ such that a weak base change of σ is cuspidal is holomorphic in the half-plane $Re(s) \geq 3/2$. Hence, if the evaluation point $s_0 = \frac{k-l}{2} \geq 3/2$, i.e., $k - l \geq 3$, the Eisenstein series is holomorphic at s_0, and thus the claim again follows directly from part A of Theorem 4.3. □

6.2. Contributions related to arithmetic conditions. The most interesting part of Eisenstein cohomology is the part in which residual Eisenstein cohomology classes may appear. This is governed by the arithmetic conditions given in terms of the analytic properties of certain automorphic L-functions and their non-vanishing at the center of symmetry for the functional equation. The precise description of this phenomenon is the subject of this section.

We first consider the contributions with the evaluation point $s_0 = 1/2$. In that case, the analytic properties of the Asai automorphic L-functions and the non-vanishing of the central value of certain Rankin–Selberg automorphic L-function determine the contribution to Eisenstein cohomology.

Theorem 6.2. *Let* $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{1/2} \otimes \sigma$ *be a cuspidal automorphic representation of* $M_0(\mathbb{A})$*, where* χ *is a unitary Hecke character of* $\mathbb{I}_F$ *which is conjugate self-dual and* σ *a unitary cuspidal automorphic representation of* $U'(\mathbb{A})$*, such that the necessary conditions for non-vanishing are satisfied with the minimal coset representative* $w_{k,l} \in W^{P_0}$*, where* $1 \leq l < k \leq n+1$*, and with parameters* (r, i)*, where* $1 \leq r \leq n$ *and* $1 \leq i \leq r$*. Then,* $k = l + 1$ *and* $r = n$*. Suppose that* σ_p *is tempered for all non-archimedean places* $p \in S$*.*

If

- *either condition* $\mathcal{C}_{\mathrm{even}}$*, given by*

$$\mathcal{C}_{\mathrm{even}} \equiv \begin{cases} n+1 \text{ is even}, \\ \text{the restriction of } \chi \text{ to } \mathbb{I} \text{ is trivial}, \\ L^S(1/2, \chi \otimes \sigma, r_1) \neq 0, \end{cases}$$

- *or condition* $\mathcal{C}_{\rm odd}$*, given by*

$$\mathcal{C}_{\rm odd} \equiv \begin{cases} n+1 \text{ is odd,} \\ \text{the restriction of } \chi \text{ to } \mathbb{I} \text{ is the quadratic character } \delta_{F/\mathbb{Q}} \text{ of } \mathbb{I} \\ \qquad\quad \text{attached to the extension } F/\mathbb{Q} \text{ by class field theory,} \\ L^S(1/2, \chi\otimes\sigma, r_1') \neq 0, \end{cases}$$

is satisfied, then the summand in the Eisenstein cohomology supported in π *is isomorphic as a* $U(\mathbb{A}_f)$*-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} H^{n-1}_{(\rm sq)}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\rm fin}\,(1/2, \pi^u)\,, & \text{if } q = n-1, \\ \text{non-trivial submodule of } I_{\rm fin}\,(1/2, \pi^u)\,, & \text{if } q = n, \\ H^{n+1}_{(\rm sq)}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \text{quotient of } J_{\rm fin}\,(1/2, \pi^u)\,, & \text{if } q = n+1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi^u \cong \chi\otimes\sigma$ *is the unitary part of* π*, and the quotient in degree* $q = n+1$ *may possibly be trivial.*

Otherwise, that is, if neither of the two sets of conditions above is satisfied, then the summand in the Eisenstein cohomology supported in π *is isomorphic as a* $U(\mathbb{A}_f)$*-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\rm fin}\,(1/2, \pi^u)\,, & \text{if } q = n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi^u \cong \chi\otimes\sigma$ *is the unitary part of* π*.*

Proof. Observe that the evaluation point $s_0 = 1/2$ is obtained if and only if the minimal coset representative $w_{k,l}$ is such that $k = l+1$. According to Theorem 5.6, the Eisenstein series associated to π^u has a pole at $1/2$ if and only if one of the two sets of conditions in the theorem is satisfied. In that case, the summand in Eisenstein cohomology is obtained directly from part B of Theorem 4.3, using the fact that the length $\ell(w_{l+1,l}) = n$. Otherwise, the Eisenstein series is holomorphic at $1/2$ and thus part A of Theorem 4.3 gives the claim. ☐

Theorem 6.3. *Let* $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{\frac{m+1}{2}} \otimes\sigma$ *be a cuspidal automorphic representation of* $M_0(\mathbb{A})$*, where* $1 \leq m \leq n-1$ *is an integer,* χ *is a unitary Hecke character of* $\mathbb{I}_F$ *which is conjugate self-dual and* σ *a unitary cuspidal automorphic representation of* $U'(\mathbb{A})$*, such that the necessary conditions for non-vanishing are satisfied with the minimal coset representative* $w_{k,l} \in W^{P_0}$*, where* $1 \leq l < k \leq n+1$*, and with parameters* (r, i)*, where* $1 \leq r \leq n$ *and* $1 \leq i \leq r$*. Then,* $k - l = m+1$ *and* $r = n - m$*. Suppose that* σ_p *is tempered for all non-archimedean places* $p \in S$*.*

If a weak base change of σ*, constructed in* [9, Thm. A.1]*, contains as a summand in the isobaric sum a representation isomorphic to* $J(m, \chi^c)$*, where* χ^c *is the conjugate of* χ *by the non-trivial Galois automorphism* c*, then the summand in the Eisenstein cohomology supported in* π *is isomorphic as a* $U(\mathbb{A}_f)$*-module to*

$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong$

$$\begin{cases} H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}\left(\frac{m+1}{2}, \pi^u\right), & \textit{if } q = n-m-1+2j \textit{ with } 0 \le j \le m, \\ \textit{non-trivial submodule of } I_{\mathrm{fin}}\left(\frac{m+1}{2}, \pi^u\right), & \textit{if } q = n+m, \\ H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \textit{quotient of } J_{\mathrm{fin}}\left(\frac{m+1}{2}, \pi^u\right), & \textit{if } q = n+m+1, \\ 0, & \textit{otherwise,} \end{cases}$$

where $\pi^u \cong \chi \otimes \sigma$ is the unitary part of π, and the quotient in degree $q = n+m+1$ may possibly be trivial.

Otherwise, that is, if a weak base change of σ, constructed in [9, Thm. A.1]*, does not contain as a summand in the isobaric sum a representation isomorphic to $J(m, \chi^c)$, then the summand in the Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\mathrm{fin}}\left(\frac{m+1}{2}, \pi^u\right), & \textit{if } q = n+m, \\ 0, & \textit{otherwise,} \end{cases}$$

where $\pi^u \cong \chi \otimes \sigma$ is the unitary part of π.

Proof. As in the proof of Theorem 6.2, the result is obtained from Theorem 4.3, using the description of poles of Eisenstein series in Theorem 5.6. □

6.3. Contribution of the trivial representation. We now state the special case in which $\mathcal{L}_\pi$ is the trivial representation of $U(\mathbb{A})$. The required analytic properties of Eisenstein series in this case are determined in Theorem 5.9.

Theorem 6.4. *Let $\pi \cong |\cdot|^{n/2}_{\mathbb{I}_F} \otimes \mathbf{1}_{U'(\mathbb{A})}$ be the trivial representation of the Levi factor $M_0(\mathbb{A})$ twisted by the character $|\cdot|^{n/2}_{\mathbb{I}_F}$. Then, the summand in the Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} \mathbf{1}_{U(\mathbb{A}_f)}, & \textit{if } q = 0, 2, \ldots, 2n-2, \\ \text{non-trivial submodule of } I_{\mathrm{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})}), & \textit{if } q = 2n-1, \\ \text{either } \mathbf{1}_{U(\mathbb{A}_f)} \text{ or } 0, & \textit{if } q = 2n, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, $H^{2n}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \mathbf{1}_{U(\mathbb{A}_f)}$ if and only if $H^{2n-1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong I_{\mathrm{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})})$, and if $H^{2n}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is trivial, then $H^{2n-1}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ is the submodule of $I_{\mathrm{fin}}(n/2, \mathbf{1}_{M_0(\mathbb{A})})$ for which the quotient is the trivial representation.

Proof. Since the highest weight of the trivial representation corresponds to the zero highest weight, it follows that the minimal coset representative $w_{k,l} \in W^{P_0}$ corresponds to $k = n+1$ and $l = 1$, and the pair (r, i) of parameters is $r = 1$ and $i = 1$. Inserting these in Theorem 4.3, and taking into account that, according to Theorem 5.9, the Eisenstein series associated to the trivial representation $\mathbf{1}_{M_0(\mathbb{A})}$ of $M_0(\mathbb{A})$ has a simple pole at $s = n/2$, and that $J(n/2, \mathbf{1}_{M_0(\mathbb{A})})$ is the trivial representation, give all the claims of the theorem. □

6.4. Cohomology of relative rank one unitary groups in two and three variables. In this section we explicitly describe the cohomology in the case of unitary groups of relative rank one in two and three variables, that is, the cases $n = 1$ and $n = 2$. These unitary groups are quasi-split as algebraic $\mathbb{Q}$-groups. The cohomology of these groups is already known from the work of Harder [15], but our approach provides a different proof. We omit the proofs, as they follow directly from Theorem 4.3, using the properties of Eisenstein series, obtained in Theorem 5.7 in the case $n = 1$, and in Theorem 5.8 in the case $n = 2$.

For the relative rank one unitary group in two variables, i.e., for $n = 1$, the Levi factor $M_0 \cong Res_{F/\mathbb{Q}} GL_1$. Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0}$ be a unitary character χ of $M_0(\mathbb{A}) \cong \mathbb{I}_F$ twisted by $|\cdot|_{\mathbb{I}_F}^{s_0}$, where $s_0 \geq 0$. From the necessary conditions for non-vanishing it follows that the summand $H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ in Eisenstein cohomology, supported in the associate class of π, is trivial, except in the case $s_0 = 1/2$. The case of the evaluation point $s_0 = 1/2$ is explicitly described in the following theorem.

Theorem 6.5. *Let U be the quasi-split unitary group in two variables, i.e., $n = 1$. Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{1/2}$ be a unitary character χ of $M_0(\mathbb{A}) \cong \mathbb{I}_F$ twisted by $|\cdot|_{\mathbb{I}_F}^{1/2}$.*

If the restriction of χ to $\mathbb{I}$ is trivial, then the summand in Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} H^0_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}(1/2, \chi), & \textit{if } q = 0, \\ \textit{non-trivial submodule of } I_{\mathrm{fin}}(1/2, \chi), & \textit{if } q = 1, \\ H^2_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \textit{quotient of } J_{\mathrm{fin}}(1/2, \chi), & \textit{if } q = 2, \\ 0, & \textit{otherwise,} \end{cases}$$

Otherwise, if the restriction of χ to $\mathbb{I}$ is not trivial, then the summand in Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\mathrm{fin}}(1/2, \chi), & \textit{if } q = 1, \\ 0, & \textit{otherwise,} \end{cases}$$

For the relative rank one unitary group in three variables, i.e., for $n = 2$, the Levi factor $M_0 \cong Res_{F/\mathbb{Q}} GL_1 \times U'$, where U' is the unitary group in one variable. Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$ be a unitary character $\chi \otimes \sigma$ of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, twisted by $|\cdot|_{\mathbb{I}_F}^{s_0}$, where $U'(\mathbb{A})$ is the norm one subgroup of $\mathbb{I}_F$. As in the case $n = 1$ above, from the necessary conditions for non-vanishing it follows that the summand $H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi)$ in Eisenstein cohomology, supported in the associate class of π, is trivial, except in the cases $s_0 = 1/2$ and $s_0 = 1$. The case of the evaluation points $s_0 = 1/2$ and $s_0 = 1$ are explicitly described in the following theorem.

Theorem 6.6. *Let U be the quasi-split unitary group in three variables, i.e., $n = 2$. Let $\pi \cong \chi|\cdot|_{\mathbb{I}_F}^{s_0} \otimes \sigma$ be a unitary cuspidal automorphic representation $\pi^u \cong \chi \otimes \sigma$ of $M_0(\mathbb{A}) \cong \mathbb{I}_F \times U'(\mathbb{A})$, twisted by $|\cdot|_{\mathbb{I}_F}^{s_0}$, where χ is a unitary Hecke character of $\mathbb{I}_F$, and σ is a character of the unitary group $U'(\mathbb{A})$ in one variable, that is, a norm-one subgroup of $\mathbb{I}_F$.*

(1) *Let $s_0 = 1/2$. If the restriction of χ to $\mathbb{I}$ is the quadratic character $\delta_{F/\mathbb{Q}}$ of $\mathbb{I}$ attached to the extension $F/\mathbb{Q}$ by class field theory, and $L^S(1/2, \pi^u, r_1') \neq 0$, then the summand in*

Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} H^1_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}\left(1/2, \pi^u\right), & \text{if } q = 1, \\ \text{non-trivial submodule of } I_{\mathrm{fin}}\left(1/2, \pi^u\right), & \text{if } q = 2, \\ H^3_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \text{quotient of } J_{\mathrm{fin}}\left(1/2, \pi^u\right), & \text{if } q = 3, \\ 0, & \text{otherwise,} \end{cases}$$

Otherwise, if any of the above two conditions regarding π^u is not satisfied, then the summand in Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\mathrm{fin}}\left(1/2, \pi^u\right), & \text{if } q = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let $s_0 = 1$. If the character χ is equal to the conjugate Σ^c of a base change Σ of σ, then the summand in Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} H^q_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong J_{\mathrm{fin}}\left(1, \pi^u\right), & \text{if } q = 0 \text{ and } q = 2, \\ \text{non-trivial submodule of } I_{\mathrm{fin}}\left(1, \pi^u\right), & \text{if } q = 3, \\ H^4_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \text{quotient of } J_{\mathrm{fin}}\left(1, \pi^u\right), & \text{if } q = 4, \\ 0, & \text{otherwise,} \end{cases}$$

Otherwise, if the character χ is not equal to the conjugate Σ^c of a base change Σ of σ, then the summand in Eisenstein cohomology supported in π is isomorphic as a $U(\mathbb{A}_f)$-module to

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_\pi) \cong \begin{cases} I_{\mathrm{fin}}\left(1, \pi^u\right), & \text{if } q = 3, \\ 0, & \text{otherwise.} \end{cases}$$

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