The Franke Filtration of the Spaces of Automorphic Forms on the Symplectic Group of Rank Two

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To my dearest beloved wife Tiki
for her infinite love and always shining so brightly,
and to our children Mak, Vid and Lana
for bringing so much joy and happiness in our lives...

Contents

Abstract

The Franke filtration is a finite filtration of the space of automorphic forms on a connected reductive linear algebraic group defined over an algebraic number field. The main feature of the filtration is that its quotients can be described in terms of parabolically induced representations using the main values of derivatives of Eisenstein series and the residues of these. The goal of this paper is to provide a complete explicit description of the Franke filtration of the space of automorphic forms on the symplectic group of rank two. The approach is different from the original approach of Franke, and takes into account the full cuspidal support of automorphic forms, that is, the cuspidal automorphic representation from which the Eisenstein series is built and the evaluation point at which it is evaluated. This considerably simplifies the exposition and makes it possible to obtain very explicit results and reveal the phenomena present in the filtration. The considered low rank case exhibits many of the properties and phenomena present in the cases of arbitrary rank. The explicit description of the Franke filtration in this case has important implications and applications in cohomology of congruence subgroups related to the Hilbert–Siegel modular forms of degree two, the Hilbert–Siegel modular variety of degree two and the corresponding Shimura variety.

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CHAPTER 1

Introduction

The Franke filtration. The Franke filtration is a finite descending filtration of the spaces of automorphic forms on the adèlic points of a reductive linear algebraic group defined over an algebraic number field, where the notion of automorphic forms is the same as in [**BJ79**]. The filtration was defined by Jens Franke in [**Fra98**]. Its main property and advantage is that the quotients of the filtration may be described in terms of representations parabolically induced from the cuspidal and residual representations of the Levi factors of parabolic subgroups of the considered group.

The representation theoretic description is obtained using the main values of derivatives of the (degenerate) Eisenstein series associated to these representations of the Levi factors. In particular, the construction of the Franke filtration implies that any non-cuspidal automorphic form on an adèlic reductive linear algebraic group over a number field is a derivative of an Eisenstein series. This fact, which has several important consequences and applications, was previously known only in the function field case by [**MW95**, App. II].

Consequences and applications of the filtration. The applications of the Franke filtration and its construction are numerous. In particular, the fact that any non-cuspidal automorphic form is a derivative of an Eisenstein series, implies the existence of an Eisenstein spectral sequence that converges to the cohomology of arithmetic congruence subgroups. This spectral sequence, constructed already in Franke's paper [**Fra98**] using Eisenstein series and their derivatives, implies that the cohomology of an arithmetic congruence subgroup can be calculated as the relative Lie algebra cohomology of the space of automorphic forms with respect to the considered arithmetic group. This is an improvement of Borel's regularization theorem [**Bor83**], sometimes referred to as the Borel–Harder conjecture [**Bor06**], [**Har90**].

Further consequence of the existence of the filtration, obtained in [**Fra98**] and [**FS98**] in the case of the general linear group, is the rationality of the decomposition along the cuspidal support in cohomology of the space of automorphic forms. This result generalizes the rationality result of Clozel in the case of cuspidal automorphic representations [**Clo90**]. Another result in [**Fra98**] is a certain trace formula for Hecke operators in full cohomology, which is of different type than the Goresky–MacPherson trace formula [**GM92**]. For a brief summary of Franke's results see the Seminaire Bourbaki exposition of Waldspurger [**Wal97**].

Although there exists a general construction of the Franke filteration, it is highly demanding to write down the explicit description of the filtration in terms of parabolic induction. The reason is that the combinatorics of the filtration is quite involved, but also that the description depends on the analytic properties of Eisenstein series, which are not known in many cases. On the other hand, it is very desirable to have such explicit description in particular cases, because of possible applications in calculations of cohomology of the space of automorphic forms, as in [**GG13b**], [**GS21**], [**Gro13**]. In view of the results of Jun Su [**Su19**], [**Su21**], certain $(\mathfrak{p}, K)$-cohomology of spaces of automorphic forms is isomorphic to the coherent cohomology of the admissible toroidal compactification of a Shimura variety with respect to the canonical extension over the compactification of the automorphic vector bundle. Hence, the explicit descriptions of the Franke filtration may play a role in the study of such coherent cohomology of Shimura varieties in future.

Motivation behind this paper. The Franke filtration is so far explicitly described only in few special cases. In the case of the cuspidal support in a maximal proper parabolic subgroup of any reductive group, the explicit description is provided in [**Grb12**]. Interesting cases of the Franke filtration for the general linear group studied in [**GG22**] reveal certain unexpected phenomena present in the filtration. A partial result in the case of the symplectic group of rank two, which is the group also considered in this paper, is obtained in [**GG13b**]. It considers only the cases of cuspidal support which may possibly contribute to cohomology, and these are rather regular cases from the point of view of the Franke filtration.

The motivation for writing this paper is three-fold. Firstly, despite the partial result in [**GG13b**], there is no complete description of the Franke filtration of the spaces of automorphic forms on the symplectic group of rank two. Our first goal here is to provide such a description.

Secondly, the original paper [**Fra98**] of Franke is written in wide generality, technically highly demanding and the considered spaces of automorphic forms, although appropriate for theoretical considerations, are not convenient for explicit calculations and applications. Our second goal in this paper is to introduce a different approach, more convenient for applications, in which the Franke filtration is described for spaces of automorphic forms supported in a given full cuspidal support. Using this approach, the Franke filtration can be described more clearly, and in the case of the symplectic group of rank two very explicitly. The Franke filtration of higher rank groups can also be tackled using this approach, and our motivation for writing this paper is to foster such endeavours.

Thirdly, the explicit description of the Franke filtration in the case of the symplectic group of rank two reveals interesting features and phenomena present in the filtration in general. Hence, our third goal in this paper is to point out clearly these features and explain the underlying reasons for their presence in the filtration. These are related to the functional equations and the analytic properties of Eisenstein series used in the construction of the filtration.

In view of these motivating goals, the style of exposition in this paper is deliberately chosen to be highly explicit, as elementary and accessible as possible, in order to reach a wider readership. Hopefully we provide a clearer picture of the Franke filtration and pave the way towards further study of spaces of automorphic forms and applications of the filtration.

The Franke filtration revisited. As mentioned above, one of the goals of this paper is to provide the complete explicit description of the Franke filtration of the space of automorphic forms with arbitrary cuspidal support on the symplectic group of rank two. This considerably improves the partial results of [**GG13b**]. Our approach is different from the original approach of Franke in [**Fra98**], and the one taken in [**FS98**] and [**GG13b**]. The main point is that we fix the full cuspidal support, and describe the Franke filtration of the space of automorphic forms with the fixed full cuspidal support.

In the original paper [**Fra98**], an ideal of finite codimension is fixed in the center of the universal enveloping algebra of the complexification of the real Lie algebra of the Lie group of real points of the considered algebraic group. The filtration is then defined for the space of automorphic forms with cuspidal support in a given associate class of parabolic subgroups and annihilated by a power of the fixed ideal, without further notice regarding the cuspidal representation on the Levi factor. This approach is appropriate for theoretical considerations of cohomological applications, because the ideal naturally arises from the coefficient system. However, it is not so convenient for explicit calculations of cohomology and description of the Franke filtration.

The papers [**FS98**] and [**GG13b**] consider the space of automorphic forms with cuspidal support in an associate class of parabolic subgroups and the fixed unitary part of a cuspidal automorphic representation of the Levi factor. This approach is more convenient than the original one for explicit calculations, but still the description depends on the analytic properties of Eisenstein series at several points of evaluation. Therefore, it is not as clean as the approach taken here, in which the full cuspidal support is fixed. Fixing the full cuspidal support of automorphic forms makes it possible to formulate the results very explicitly and reveal otherwise hidden phenomena of the filtration.

Reasons for considering the symplectic group of rank two. The motivation for considering the symplectic group of rank two in this paper stems from several reasons. It is an important group as the ambient group for the theory of Hilbert–Siegel modular forms of degree two, related to the Hilbert–Siegel modular variety of degree two, and the corresponding Shimura variety. Thus, it would be useful to have the complete explicit description of the Franke filtration in this case. The theory of Eisenstein series on the symplectic group of rank two is already rather well understood by the work [**Kim95**] on the residual spectrum, as well as [**Wat92**], [**HM15**]. The cohomology in that case is studied and explicitly calculated in the works [**Sch86**], [**OS90**], [**GG13b**], [**MG18**].

The symplectic group of rank two is the group of low rank, in which the features and properties of the Franke filtration can be pointed out more clearly than in the case of larger groups. On the other hand, the Franke filtration of the space of automorphic forms on that group exhibits several important properties of the filtration in arbitrary rank, and is much more involved than the rank one cases. Therefore, another goal of this paper is to point out clearly and explain in details the underlying reasons for the features and phenomena present in the Franke filtration of spaces of automorphic forms on the symplectic group of rank two. We explain the general mechanisms, which are incorporated in the definition of the filtration in order to settle certain issues in its construction using the Eisenstein series. In particular, the functional equations of Eisenstein series are taken into account using colimits of a certain functor. The problems arising from the analytic properties of Eisenstein series, in particular, the problem of Eisenstein series that is not holomorphic at the relevant value of its complex parameter, are settled using certain partial order on the contributions from various Eisenstein series to the same space of automorphic forms.

And last, but not least, the author has learned from personal communication and discussions with Joachim Schwermer, that Franke himself has often used the case of the symplectic group of rank two in order to point out the underlying ideas and properties of the filtration in his talks. However, the complete explicit description of the Franke filtration of the space of automorphic forms on the symplectic group of rank two for all possible cuspidal supports is given in this paper for the first time.

Extraordinary phenomena – a model case. As a model case which may serve as a preview of the results of this paper and resembles the flavor of the phenomena that occur in the Franke filtration, we formulate below case (4j) of Theorem 7.7. It describes the Franke filtration of the space of automorphic forms on the symplectic group of rank two, with cuspidal support in the Borel parabolic subgroup and a certain twist of the trivial character of the maximal split torus. This case of the filtration is an example of the phenomenon in which two different quotients of the filtration must be described as parabolically induced representations from the parabolic subgroups of the same rank. It was apparently overlooked by Franke in Remark 2 on page 242-243 of [**Fra98**], in which he claims that the only rank two example of this phenomenon that he knows is in the case of the exceptional group G_2. My speculation would be that the cause of Franke's omission of this example is most likely arising from the following two difficulties. In his approach, Franke did not consider the full cuspidal support to distinguish spaces of automorphic forms, so that our example below is only a part of much larger space of automorphic forms considered by Franke. Besides that, the cuspidal support in our example is such that it does not contribute to cohomology of arithmetic groups with respect to any coefficient system arising from a finite dimensional algebraic representation of G. For more details regarding this and other phenomena occurring in the filtration see Chapter 9.

THEOREM. *Let F be an algebraic number field with the ring of adèles $\mathbb{A}$ and the group of idèles $\mathbb{I}$. Let $G = Sp_2$ be the symplectic group of rank two defined over F. Let B be a Borel subgroup of G, and T a maximal split torus in B. Consider the character*

$$\pi \cong |\cdot|_{\mathbb{I}} \otimes \mathbf{1}_{\mathbb{I}}$$

of $T(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$, where $|\cdot|_{\mathbb{I}}$ is the adèlic absolute value on $\mathbb{I}$, and $\mathbf{1}_{\mathbb{I}}$ is the trivial character on $\mathbb{I}$. The space of automorphic forms on $G(\mathbb{A})$, with cuspidal support in (the associate class of) B and π, is denoted $\mathcal{A}_{\{B\},\varphi(\pi)}$. Then, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \left(\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})} \left(\mathbf{1}_{\mathbb{I}} \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^{w_{121}} \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})} \left((\mathbf{1}_{\mathbb{I}} \circ \det)|\det|_{\mathbb{I}}^{1/2}\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \left(\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(|\cdot|_{\mathbb{I}} \otimes \mathbf{1}_{\mathbb{I}}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2} \end{aligned}$$

where Ind *stands for the parabolic induction,* $P_1(\mathbb{A})$*, resp.* $P_2(\mathbb{A})$*, is the standard parabolic subgroup of* G *with the Levi factor isomorphic to* $GL_2(\mathbb{A})$*, resp.* $\mathbb{I}\times SL_2(\mathbb{A})$*, and* $S(\check{\mathfrak{a}}_{R,\mathbb{C}})$ *is the symmetric algebra on the complexification* $\check{\mathfrak{a}}_{R,\mathbb{C}}$ *of the* $\mathbb{Z}$*-module of rational characters of the parabolic subgroup* R*. The exponents* w_{121} *and* w_2 *denote the space of invariant vectors for the action of certain intertwining operators acting on the induced representations.*

In this model theorem, observe that there are two quotients of the filtration isomorphic to induced representations from a parabolic subgroup of the same rank. The isomorphisms between the quotients of the filtration and the induced representations are constructed using the main values of the derivatives of the Eisenstein series. The sample theorem exhibits the situation in which the degenerate Eisenstein series associated to the residual representation $\mathbf{1}\circ\det$ of the Levi factor $L_1(\mathbb{A})\cong GL_2(\mathbb{A})$ has a pole at the relevant value $s=1/2$ of its complex parameter. The main value of the derivatives of such degenerate Eisenstein series are not well defined as elements of the full space of automorphic forms. The principal part of the Laurent series, which is in this example only the residue, of the degenerate Eisenstein series should contribute to a deeper filtration step, so that the main values of the derivatives of the degenerate Eisenstein series can be well defined as elements of the quotient of the filtration.

If the residues were square-integrable, then they would naturally form a deeper filtration step isomorphic to an induced representation from a parabolic subgroup of lower relative rank. However, this is not the case in our model theorem. The residues of the degenerate Eisenstein series are not square-integrable, and still they must contribute to a deeper filtration step. This is achieved by finding in the associate class of the cuspidal support $\pi\cong|\cdot|_{\mathbb{I}}\otimes\mathbf{1}_{\mathbb{I}}$ another character of $T(\mathbb{A})$, which is the cuspidal support of a degenerate Eisenstein series, holomorphic at the relevant value of its complex parameter, whose Taylor coefficients contain the non-square-integrable residues in question.

In our example, we take the character $\mathbf{1}_{\mathbb{I}}\otimes|\cdot|_{\mathbb{I}}$, which is associate to π, and which is the cuspidal support of the degenerate Eisenstein series associated to the residual representation $\mathbf{1}_{\mathbb{I}}\otimes\mathbf{1}_{SL_2(\mathbb{A})}$ with the relevant value of its complex parameter $s=0$. Since the Eisenstein series is holomorphic at $s=0$, the main values of its derivatives are well defined and contain the residues of the previous degenerate Eisenstein series. This is the underlying reason for the existence of two different quotients of the filtration isomorphic to the induced representations from parabolic subgroups of the same rank. See Section 9.4 for more details.

The model theorem also exhibits the way in which the Franke filtration deals with functional equations of the Eisenstein series. These functional equations relate Eisenstein series associated to residual or cuspidal representations which are associate under certain elements of the Weyl group, and the relationship is established via the standard intertwining operators attached to these Weyl group elements. Invariant vectors for these intertwining operators are taken in the first and the last quotient to avoid taking twice into account the Eisenstein series which are equal by the functional equation. For more details see Section 9.2.

Outline of the paper. At the end of the introduction, we outline the contents of the paper. Following this Introduction, the structure of the symplectic group of rank two is recalled in Chapter 2 and the basic notation is fixed. The preliminaries regarding the automorphic forms, parabolically induced representations and

Eisenstein series are collected in Chapter 3. The Franke filtration is introduced in Chapter 4. Explicit description of the spaces of automorphic forms with cuspidal support in maximal proper parabolic subgroups is given in Chapter 5. In the case of the cuspidal support in the Borel subgroup, the analytic properties of Eisenstein series required in the paper are summarized in Chapter 6, and the results regarding the explicit description of the Franke filtration are stated in Chapter 7. The proofs of these results are the subject of Chapter 8. Finally, Chapter 9 points out and explains the underlying reasons for the properties and features of the Franke filtration present in the case of the symplectic group of rank two. The calculation of colimits required in the paper is made in the Appendix A at the end of the paper.

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CHAPTER 2

The symplectic group of rank two

Let F be an algebraic number field. The set of places of F is denoted by V, the subset of all archimedean places by V_∞ and the subset of all non-archimedean places by V_f. For the place $v \in V$, let F_v denote the completion of F at v. If $v \in V_f$, let $\mathcal{O}_v$ be the ring of integers of F_v. Let $\mathbb{A}$ be the ring of adèles of F, and $\mathbb{A}_f$ the subring of finite adèles. Let $\mathbb{I}$ be the group of idèles of F. Throughout the paper, the adèlic absolute value on $\mathbb{I}$ is denoted by $|\cdot|$.

Let $G = Sp_2$ be the symplectic group of rank two defined over F. It is the group of isometries of the symplectic form on a four-dimensional vector space over F. As in [**Tad94**], we fix a basis of the vector space in which the matrix of the symplectic form is

$$J = \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix}, \quad \text{where } J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, for any F-algebra R, we have

$$G(R) = \left\{ g \in GL_4(R) \, : \, {}^t g J g = J \right\},$$

where ${}^t g$ denotes the transpose of g.

We fix, once and for all, the choice of the Borel subgroup B of G such that $B(R)$ consists of all upper-triangular matrices in $G(R)$. Let $B = TU$ be the Levi decomposition of B, where T is the maximal F-split torus of G such that $T(R)$ consists of all diagonal matrices in $G(R)$, and U the unipotent radical. Then, T is isomorphic to the product of two copies of GL_1, that is,

$$T(R) = \left\{ t(x,y) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^{-1} & 0 \\ 0 & 0 & 0 & x^{-1} \end{pmatrix} \, : \, x, y \in R^\times \right\}.$$

The Weyl group of G with respect to T is denoted by W. It is generated by two simple reflections, which we denote by w_1 and w_2. They are given by their action on T as

$$w_1(t(x,y)) = t(y,x),$$
$$w_2(t(x,y)) = t(x,y^{-1}),$$

for $x, y \in R^\times$. The Weyl group is then

$$W = \{1, w_1, w_2, w_{12}, w_{21}, w_{121}, w_{212}, w_{1212}\},$$

where we abbreviate the product $w_{i_1} w_{i_2} \dots w_{i_k}$ by $w_{i_1 i_2 \dots i_k}$. The action of the Weyl group on the torus T is given in Table 3.1 in Chapter 3 below.

The group G has two standard maximal proper parabolic F-subgroups, which we denote by P_1 and P_2. By a standard parabolic F-subgroup we mean a parabolic subgroup, defined over F, which contains the fixed Borel subgroup B. Let $P_i = L_iN_i$, for $i = 1, 2$, be the Levi decomposition, where L_i is the Levi factor and N_i the unipotent radical of P_i. In our notation, as in [**Kim95**], the Levi factor L_1 of P_1 is isomorphic to GL_2, that is,

$$L_1(R) = \left\{ l_1(g) = \begin{pmatrix} g & 0 \\ 0 & J_2{}^t g^{-1} J_2 \end{pmatrix} : g \in GL_2(R) \right\},$$

while the Levi factor L_2 of P_2 is isomorphic to $GL_1 \times SL_2$, that is,

$$L_2(R) = \left\{ l_2(x, h) = \begin{pmatrix} x & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & x^{-1} \end{pmatrix} : x \in R^\times, h \in SL_2(R) \right\},$$

for any F-algebra R. We often use maps l_1 and l_2 to identify the Levi factors L_1 and L_2 with GL_2 and $GL_1 \times SL_2$, respectively.

For the moment, let $P = LN$ be one of the standard parabolic F-subgroups of G, i.e., $P = B$, $P = P_1$, $P = P_2$ or $P = G$. Let $X^*(P)$ be the $\mathbb{Z}$-module of F-rational characters of P. Let $\check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\check{\mathfrak{a}}_{P,\mathbb{C}}$ its complexification. The elements of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ may be identified with certain characters of $L(\mathbb{A})$. We have $\check{\mathfrak{a}}_{P,\mathbb{C}} \cong \mathbb{C}^r$, where r is the relative rank of P, i.e., $r = 2$ for $P = B$, and $r = 1$ for $P = P_1$ and $P = P_2$, and $r = 0$ for $P = G$. The isomorphism is fixed in such a way that, $\underline{s} = (s_1, s_2) \in \mathbb{C}^2$, for $P = B$, corresponds to the character

$$t(x, y) \mapsto |x|^{s_1} |y|^{s_2}$$

of $T(\mathbb{A})$, and $\underline{s} = s \in \mathbb{C}$ corresponds for $P = P_1$ to the character

$$l_1(g) \mapsto |\det g|^s$$

of $L_1(\mathbb{A})$, and for $P = P_2$ to the character

$$l_2(x, h) \mapsto |x|^s$$

of $L_2(\mathbb{A})$.

The restrictions of characters from $L(\mathbb{A})$ to $T(\mathbb{A})$ give rise to the inclusions of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ into $\check{\mathfrak{a}}_{B,\mathbb{C}}$, which we denote by ι_P. In particular, for $P = P_1$ and $s \in \check{\mathfrak{a}}_{P_1,\mathbb{C}}$, we have

$$\iota_{P_1}(s) = (s, s) \in \check{\mathfrak{a}}_{B,\mathbb{C}},$$

and for $P = P_2$ and $s \in \check{\mathfrak{a}}_{P_2,\mathbb{C}}$, we have

$$\iota_{P_2}(s) = (s, 0) \in \check{\mathfrak{a}}_{B,\mathbb{C}}.$$

For $P = B$, the inclusion ι_B is the identity map on $\check{\mathfrak{a}}_{B,\mathbb{C}}$. Although $\check{\mathfrak{a}}_{G,\mathbb{C}}$ is trivial, it is convenient to denote its inclusion into $\check{\mathfrak{a}}_{B,\mathbb{C}}$ by ι_G.

The choice of the Borel subgroup determines the positive Weyl chamber in $\check{\mathfrak{a}}_{B,\mathbb{C}}$. In coordinates, it consists of all $(s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}}$ such that $Re(s_1) > Re(s_2) > 0$. We denote by

$$\mathfrak{C}^+ = \{(s_1, s_2) \in \check{\mathfrak{a}}_B : s_1 > s_2 > 0\}$$

the real part of the positive Weyl chamber, and by

$$\overline{\mathfrak{C}^+} = \{(s_1, s_2) \in \check{\mathfrak{a}}_B : s_1 \geq s_2 \geq 0\}$$

the real part of its closure. The choice of the Borel subgroup also determines the positive Weyl chamber in $\check{\mathfrak{a}}_{P,\mathbb{C}}$ for $P = P_1$ and $P = P_2$. In both cases, the positive Weyl chamber consists of all $s \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ such that $Re(s) > 0$, the real part of the positive Weyl chamber is given by $s > 0$, and its closure by $s \geq 0$.

In the definition of the Franke filtration, the closure of the negative obtuse Weyl chamber in $\check{\mathfrak{a}}_B$ is also required. It consists of all $(s_1, s_2) \in \check{\mathfrak{a}}_B$ such that

$$s_1 \leq 0$$
$$s_1 + s_2 \leq 0$$

in coordinates as above.

We fix, once and for all, a maximal compact subgroup K of the adèlic group $G(\mathbb{A})$, which is the product $K = \prod_{v \in V} K_v$ of the fixed maximal compact subgroups K_v of $G(F_v)$ such that $K_v = G(\mathcal{O}_v)$ for every place $v \in V_f$. Then, K is in good position with respect to B in the sense of [**MW95**, Sect. I.1.4].

Let $G_\infty = \prod_{v \in V_\infty} G(F_v)$ be the archimedean part of the adèlic group $G(\mathbb{A})$. Then the product $K_\infty = \prod_{v \in V_\infty} K_v$ is a maximal compact subgroup of G_∞. The real Lie algebra of G_∞ is denoted by $\mathfrak{g}_\infty$, and its complexification by $\mathfrak{g}_{\infty,\mathbb{C}}$. The universal enveloping algebra of $\mathfrak{g}_{\infty,\mathbb{C}}$ is denoted by $\mathcal{U}$, and $\mathcal{Z}$ denotes its center.

CHAPTER 3

Spaces of automorphic forms

In this chapter we collect all the necessary preliminaries regarding automorphic forms required in the rest of the paper. In particular, we recall briefly the definition of automorphic forms following [**BJ79**], introduce parabolically induced representations and Eisenstein series, and discuss their basic properties. Finally, we define the spaces of automorphic forms with a fixed cuspidal support. These are the main objects of study in this paper.

Throughout this chapter, let $P = LN$ be one of the three standard proper parabolic F-subgroups of G, i.e., $P = B$, $P = P_1$ or $P = P_2$. We use freely the notation introduced in the previous chapter.

3.1. Automorphic forms

Let $\mathcal{A} = \mathcal{A}(G(F)\backslash G(\mathbb{A}))$ be the space of automorphic forms on $G(\mathbb{A})$, as defined in [**BJ79**]. Recall that a function on $G(\mathbb{A})$ is an automorphic form if it is

- smooth,
- left $G(F)$-invariant,
- K-finite,
- $\mathcal{Z}$-finite, and
- of uniform moderate growth,

cf. *loc. cit.* Since K-finiteness is not preserved by the action of G_∞, the space $\mathcal{A}$ of automorphic forms carries only the structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. Nevertheless, we often refer to such modules as representations of $G(\mathbb{A})$, although strictly speaking there is no action of the full adèlic group.

Given a parabolic F-subgroup Q of G, the constant term of an automorphic form $f \in \mathcal{A}$ along Q is defined as

$$f_Q(g) = \int_{N_Q(F)\backslash N_Q(\mathbb{A})} f(ng)dn,$$

where N_Q is the unipotent radical of Q, and dn is an appropriate Haar measure, cf. [**MW95**, Sect. I.1.13]. An automorphic form f is cuspidal if its constant term $f_Q = 0$ along all proper parabolic F-subgroups of G. The space of cuspidal automorphic forms on $G(\mathbb{A})$ is denoted by $\mathcal{A}_{\mathrm{cusp}} = \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$. It is a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-submodule of $\mathcal{A}$, which exhibits a direct sum decomposition into irreducible modules [**GGPS90**]. The irreducible summands in this decomposition are referred to as cuspidal automorphic representations of $G(\mathbb{A})$.

The space of automorphic forms and cuspidal automorphic forms on the Levi factors $L(\mathbb{A})$ are defined in the same way as on the full group $G(\mathbb{A})$. Hence, we may talk about cuspidal automorphic representations of $L(\mathbb{A})$.

TABLE 3.1. The action of the Weyl group on the torus and the character $\chi_1|\cdot|^{s_1}\otimes\chi_2|\cdot|^{s_2}$ of $T(\mathbb{A})$.

$w\in W$	$w(t(x,y))$	$w(s_1,s_2)$	$w(\chi_1\otimes\chi_2)$	$w\left(\chi_1\|\cdot\|^{s_1}\otimes\chi_2\|\cdot\|^{s_2}\right)$
1	$t(x,y)$	(s_1,s_2)	$\chi_1\otimes\chi_2$	$\chi_1\|\cdot\|^{s_1}\otimes\chi_2\|\cdot\|^{s_2}$
w_1	$t(y,x)$	(s_2,s_1)	$\chi_2\otimes\chi_1$	$\chi_2\|\cdot\|^{s_2}\otimes\chi_1\|\cdot\|^{s_1}$
w_2	$t(x,y^{-1})$	$(s_1,-s_2)$	$\chi_1\otimes\chi_2^{-1}$	$\chi_1\|\cdot\|^{s_1}\otimes\chi_2^{-1}\|\cdot\|^{-s_2}$
w_{12}	$t(y^{-1},x)$	$(-s_2,s_1)$	$\chi_2^{-1}\otimes\chi_1$	$\chi_2^{-1}\|\cdot\|^{-s_2}\otimes\chi_1\|\cdot\|^{s_1}$
w_{21}	$t(y,x^{-1})$	$(s_2,-s_1)$	$\chi_2\otimes\chi_1^{-1}$	$\chi_2\|\cdot\|^{s_2}\otimes\chi_1^{-1}\|\cdot\|^{-s_1}$
w_{121}	$t(x^{-1},y)$	$(-s_1,s_2)$	$\chi_1^{-1}\otimes\chi_2$	$\chi_1^{-1}\|\cdot\|^{-s_1}\otimes\chi_2\|\cdot\|^{s_2}$
w_{212}	$t(y^{-1},x^{-1})$	$(-s_2,-s_1)$	$\chi_2^{-1}\otimes\chi_1^{-1}$	$\chi_2^{-1}\|\cdot\|^{-s_2}\otimes\chi_1^{-1}\|\cdot\|^{-s_1}$
w_{1212}	$t(x^{-1},y^{-1})$	$(-s_1,-s_2)$	$\chi_1^{-1}\otimes\chi_2^{-1}$	$\chi_1^{-1}\|\cdot\|^{-s_1}\otimes\chi_2^{-1}\|\cdot\|^{-s_2}$

Let π^u be a unitary cuspidal automorphic representation of $L(\mathbb{A})$. We may write

$$\pi^u\cong\chi_1\otimes\chi_2,\quad \text{for } L=T,$$

where χ_1 and χ_2 are unitary Hecke characters of $\mathbb{I}$, and

$$\pi^u\cong\chi\otimes\sigma,\quad \text{for } L=L_2,$$

where χ is a unitary Hecke character of $\mathbb{I}$ and σ a unitary cuspidal automorphic representation of $SL_2(\mathbb{A})$. For $L=L_1\cong GL_2$, we have π^u is a unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$. We always assume that π^u is normalized in such a way that it is trivial on the connected component of the archimedean part of the center of the Levi subgroup L. This assumption is not restricting, as explained in [**Kim04**, page 121]. It is just a convenient choice of coordinates, which makes, in what follows, the relevant poles of the Eisenstein series associated to π^u real.

The action of the Weyl group W on T gives rise to the action of the Weyl group on $\check{\mathfrak{a}}_{B,\mathbb{C}}$ and on the unitary characters $\pi^u\cong\chi_1\otimes\chi_2$ of $T(\mathbb{A})$. Combining these two actions gives rise to the action of the Weyl group on characters $\pi\cong\chi_1|\cdot|^{s_1}\otimes\chi_2|\cdot|^{s_2}$. All these actions are given in Table 3.1.

3.2. Induced representations and Eisenstein series

Let $\underline{s}\in\check{\mathfrak{a}}_{P,\mathbb{C}}$. We denote by $I_P(\underline{s},\pi^u)$ the representation of $G(\mathbb{A})$ parabolically induced from the representation π^u of the Levi factor $L(\mathbb{A})$ twisted by the character corresponding to $\underline{s}$ and extended trivially on $N(\mathbb{A})$, as in [**Sha10**], [**MW95**]. More precisely,

$$I_P(\underline{s},\pi^u)=\begin{cases}\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\left(\chi_1|\cdot|^{s_1}\otimes\chi_2|\cdot|^{s_2}\right), & \text{for } \substack{P=B,\\ \pi^u\cong\chi_1\otimes\chi_2,\\ \underline{s}=(s_1,s_2),}\\ \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}\left(\pi^u|\det|^s\right), & \text{for } \substack{P=P_1,\\ \underline{s}=s,}\\ \mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}\left(\chi|\cdot|^s\otimes\sigma,\right) & \text{for } \substack{P=P_2,\\ \pi^u\cong\chi\otimes\sigma,\\ \underline{s}=s.}\end{cases}$$

The parabolic induction is always normalized by ρ_P, so that it preserves unitarizability, where ρ_P denotes the half-sum of positive roots of G with respect to T that appear in the unipotent radical N of P. It equals

$$\rho_B = (2,1), \quad \rho_{P_1} = 3/2, \quad \rho_{P_2} = 2,$$

viewed as an element of $\check{a}_{P,\mathbb{C}} \cong \mathbb{C}^r$.

For a unitary cuspidal automorphic representation π^u as above, let $\mathcal{V}_{\pi^u}$ be the submodule of the space of cuspidal automorphic forms on $L(\mathbb{A})$ isomorphic to π^u. Since the multiplicity one theorem holds for the cuspidal spectrum of the Levi factors [**Sha74**], [**Ram00**], such $\mathcal{V}_{\pi^u}$ is well defined. Let $\mathcal{W}_{\pi^u}$ be the space of K-finite smooth functions

$$f : L(F)N(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that the function given by the assignment $m \mapsto f(mg)$ belongs to $\mathcal{V}_{\pi^u}$ for all $g \in G(\mathbb{A})$.

For $f \in \mathcal{W}_{\pi^u}$, and an element $\underline{s} \in \check{a}_{P,\mathbb{C}}$, let

$$f_{\underline{s}}(g) = f(g) \cdot (\underline{s} + \rho_P)(l_g)$$

for $g \in G(\mathbb{A})$, where $\underline{s} + \rho_P$ on the right-hand side, abusing the notation, stands for the character of $L(\mathbb{A})$ corresponding to $\underline{s} + \rho_P \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ under the isomorphism introduced above, and l_g is an element in $L(\mathbb{A})$, which appears in the decomposition $g = n_g l_g k_g$ of g according to the Iwasawa decomposition of G, i.e., $n_g \in N(\mathbb{A})$, $l_g \in L(\mathbb{A})$, $k_g \in K$. Although such $l_g \in L(\mathbb{A})$ is not uniquely determined by g, the value $(\underline{s} + \rho_P)(l_g)$ of the character on l_g is independent of that choice.

The Eisenstein series associated to π^u are defined, at least formally, as

$$E(f, \underline{s})(g) = \sum_{\gamma \in P(F)\backslash G(F)} f_{\underline{s}}(\gamma g),$$

where f ranges over $\mathcal{W}_{\pi^u}$. The defining series is absolutely and locally uniformly convergent for $\underline{s}$ in a positive cone deep enough in the positive Weyl chamber defined by P. It can be analytically continued to a meromorphic funtion of $\underline{s}$ in the whole space $\check{a}_{P,\mathbb{C}}$. For these and other properties of Eisenstein series, we refer the reader to [**MW95**, Chap. IV] and [**Lan76**, Sect. 7]. The analytic properties of the Eisenstein series associated to π^u are crucial for the description of the Franke filtration of the space of automorphic forms with the cuspidal support in the associate class of π^u twisted by a character corresponding to some $\underline{s}_0 \in \check{\mathfrak{a}}_P$, which is defined in the following section.

However, in the description of the Franke filtration, besides the Eisenstein series associated to the cuspidal automorphic representation π^u, we must deal with the Eisenstein series associated to certain residual representations Π of the Levi factor $L_R(\mathbb{A})$ of standard parabolic subgroups R of G. These Eisenstein series are referred to as the degenerate Eisenstein series. They are constructed in the same way as the Eisenstein series associated to π^u. The function f is chosen in the space $\mathcal{W}_\Pi$, which is the analogue of $\mathcal{W}_{\pi^u}$, and the complex parameter is taken from $\check{\mathfrak{a}}_{R,\mathbb{C}}$.

3.3. Decomposition along the cuspidal support

We now recall the definition of the space of automorphic forms with a given cuspidal support following [**FS98**, Sect. 1.3]. An alternative definition is provided in [**MW95**, Chap. III], and it is proved in [**FS98**, Sect. 1.4] that the two definitions are equivalent. In fact, the Franke filtration of the space of automorphic forms is precisely the underlying reason for the equivalence of the two definitions. It allows the definition of the space of automorphic forms with a given cuspidal support in terms of Eisenstein series.

We first fix a cuspidal support. Let P be one of the three standard proper parabolic F-subgroups of G with the Levi decomposition $P = LN$. We denote by $\{P\}$ the associate class of parabolic F-subgroups represented by P. It consists of all parabolic F-subgroups of G such that its Levi factor is conjugate to L. We also consider the associate class $\{G\}$ of the full group G, which is a singleton.

Let π^u be a unitary cuspidal automorphic representation of the Levi factor $L(\mathbb{A})$, as above. Let $\underline{s}_0 \in \check{\mathfrak{a}}_P$. The associate class of cuspidal automorphic representations represented by $\pi = \pi^u \otimes \underline{s}_0$, i.e., π^u twisted by the character of $L(\mathbb{A})$ corresponding to $\underline{s}_0$, is denoted by $\varphi(\pi)$, cf. [**FS98**, Sect. 1.2]. More precisely, $\varphi(\pi)$ is a finite family of finite sets $\varphi_Q(\pi)$, indexed by parabolic F-subgroups Q in the associate class $\{P\}$, such that $\varphi_Q(\pi)$ consists of cuspidal automorphic representations of the Levi factor $L_Q(\mathbb{A})$ of Q, constructed as follows. For $Q \in \{P\}$, if w is an element of the Weyl group W such that the Levi factors are conjugate by w, i.e., $wL_Qw^{-1} = L$, then the conjugate $w(\pi)$ of π by w is in $\varphi_Q(\pi)$. We will choose, as we may, the representative of the associate class $\varphi(\pi)$ in such a way that $\underline{s}_0 \in \check{\mathfrak{a}}_P$ belongs to the closure of the positive Weyl chamber determined by P. Hence, we always assume that π is chosen in such a way. For the special case of the associate class $\{G\}$, we take $\pi = \pi^u$ to be a unitary cuspidal automorphic representation of $G(\mathbb{A})$.

We refer to the pair $(\{P\}, \varphi(\pi))$ as the (full) cuspidal support, where $\{P\}$ is an associate class of parabolic F-subgroups and $\varphi(\pi)$ an associate class of cuspidal automorphic representations of the Levi factors of parabolic subgroups in $\{P\}$. We now define the space $\mathcal{A}_{\{P\},\varphi(\pi)}$ of automorphic forms supported in a given cuspidal support $(\{P\}, \varphi(\pi))$, where $\pi \cong \pi^u \otimes \underline{s}_0$. Stating the results on the Franke filtration with regard to the full cuspidal support π, instead of only its unitary part π^u, is an important improvement in making the statements simpler and more comprehensible.

Let $E(f, \underline{s})$ be the Eisenstein series associated to π^u, where f ranges over the space $\mathcal{W}_{\pi^u}$, as defined in Section 3.2. According to the analytic properties of the Eisenstein series, cf. [**MW95**, Chap. IV], the poles of the Eisenstein series $E(f, \underline{s})$ in the closure of the positive Weyl chamber determined by P all lie along a locally finite family of singular hyperplanes. Hence, we may find a polynomial p on $\check{\mathfrak{a}}_P$ such that

$$p(\underline{s})E(f, \underline{s})$$

is holomorphic around $\underline{s}_0$ for all $f \in \mathcal{W}_{\pi^u}$. The space $\mathcal{A}_{\{P\},\varphi(\pi)}$ is defined as the span of all the coefficients in the Taylor expansions around $\underline{s} = \underline{s}_0$ of these holomorphic functions, as f ranges over $\mathcal{W}_{\pi^u}$. It carries the structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

In the special case of the associate class $\{G\}$ and a cuspidal automorphic representation $\pi = \pi^u$ of $G(\mathbb{A})$, the above construction of the Eisenstein series is empty, and the space $\mathcal{A}_{\{G\},\varphi(\pi)}$ is the space of cuspidal automorphic forms on which the representation π acts. This special case gives rise to the space of cuspidal automorphic forms on $G(\mathbb{A})$, which decomposes into a direct sum of irreducible cuspidal automorphic representations [**GGPS90**], and thus the Franke filtration of each summand is trivial.

The goal of this paper is to provide an explicit description of the Franke filtration for the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules $\mathcal{A}_{\{P\},\varphi(\pi)}$, with $P \neq G$, for all possible cuspidal supports. Since the space $\mathcal{A}$ of automorphic forms on $G(\mathbb{A})$ exhibits a direct sum decomposition indexed by the cuspidal support, cf. [**FS98**, Sect. 1.4], [**MW95**, Sect. III.2.6], this is sufficient to completely describe the Franke filtration of $\mathcal{A}$.

CHAPTER 4

Definition of the Franke filtration

The Franke filtration is a filtration of spaces of automorphic forms, defined in [**Fra98**, Sect. 6], such that the successive quotients of the filtration may be described in terms of parabolically induced representations. In this chapter we recall the definition, but take a different approach than the original one. The main point is to fix the full cuspidal support of the considered automorphic forms.

4.1. Fixing the full cuspidal support

The Franke filtration was first defined in [**Fra98**, Sect. 6] in a slightly different setting than the one taken here. More precisely, in *loc. cit.* only the associate class of parabolic F-subgroups, but not the full cuspidal support, is fixed. Instead of the full cuspidal support, an ideal $\mathcal{J}$ of finite codimension in the center $\mathcal{Z}$ of the universal enveloping algebra is fixed. Then the space $\mathcal{A}_{\mathcal{J}}$ of all automorphic forms on $G(\mathbb{A})$ which are annihilated by a power of $\mathcal{J}$ is considered. This is not restrictive, because the condition of $\mathcal{Z}$-finiteness in the definition of an automorphic form implies that, for any automorphic form, there is an ideal of finite codimension in $\mathcal{Z}$ which annihilates it. From the cohomological point of view, the approach taken by Franke is natural, because the coefficient system for the cohomology determines the ideal $\mathcal{J}$, cf. [**BW00**]. The space $\mathcal{A}_{\mathcal{J}}$ carries the structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. Franke defines in [**Fra98**, Sect. 6] the filtration for submodules $\mathcal{A}_{\mathcal{J},\{P\}}$ of $\mathcal{A}_{\mathcal{J}}$ of automorphic forms annihilated by a power of $\mathcal{J}$ and with the cuspidal support in the associate class $\{P\}$, without further reference to a cuspidal automorphic representation of the Levi factor.

In our setting, the fixed full cuspidal support $(\{P\}, \varphi(\pi))$ provides the necessary finiteness condition for the description of the Franke filtration, as well as the possibility to make the description very explicit. In fact, given such full cuspidal support, there exists an ideal $\mathcal{J}$ of finite codimension in $\mathcal{Z}$, although not unique, such that $\mathcal{A}_{\{P\},\varphi(\pi)}$ is a submodule of $\mathcal{A}_{\mathcal{J}}$ and $\mathcal{A}_{\mathcal{J},\{P\}}$. The approach taken here was already applied in the calculation of residual Eisenstein cohomology in [**GG13b**], [**GS21**], [**Gro13**], and also implicitly in [**FS98**], [**GS11b**], [**GS11a**], [**GS10**], [**GS14**], [**GS19**], [**GG13a**], [**Grb18**], in the description of the Franke filtration for spaces of automorphic forms supported in a maximal proper parabolic subgroup in [**Grb12**], and in the study of certain phenomena in the Franke filtration of the spaces of automorphic forms on the general linear group [**GG22**]. Observe that the paper [**GG13b**] also considers the symplectic group of rank two. However, in that paper, the Franke filtration is determined only for those cuspidal supports which may possibly contribute to cohomology. This simplifies the description considerably, as the possible values of $\underline{s}_0$ are quite regular. Many of the interesting features of the filtration do not occur in such regular situations. Note that in [**GG13b**], we only fix the unitary cuspidal support π^u, and the possible values of $\underline{s}_0$ are determined by the coefficient system for the cohomology through the ideal $\mathcal{J}$. Fixing also the character corresponding to $\underline{s}_0$ makes the exposition much more comprehensive.

4.2. The category of triples

We now describe the Franke filtration of the space $\mathcal{A}_{\{P\},\varphi(\pi)}$ in terms of the cuspidal support $(\{P\},\varphi(\pi))$. Essentially, we follow [**Fra98**, Sect. 6], with the addition of keeping track of the cuspidal support, as in [**Grb12**]. Let $\mathcal{M}_{\{P\},\varphi(\pi)}$ be the set of triples $(R,\Pi,\underline{z})$, where

- R is a standard parabolic F-subgroup of G containing an element of the associate class $\{P\}$, with the Levi factor L_R,
- Π is a discrete spectrum representation of the Levi factor $L_R(\mathbb{A})$,
- $\underline{z}$ is an element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R$, such that the twisted representation $\Pi\otimes\underline{z}$, by the character of $L_R(\mathbb{A})$ corresponding to $\underline{z}$, has cuspidal support in $(\{P\},\varphi(\pi))$.

The infinitesimal character $\underline{z}_\Pi$ of Π and $\underline{z}$ may be viewed as elements of $\check{\mathfrak{a}}_B$ under inclusions arising from restrictions of characters as in [**FS98**, page 769] and Chapter 2. The third requirement in the definition imposes that the sum of $\underline{z}_\Pi$ and $\underline{z}$ belongs to the Weyl group orbit of $\iota_P(\underline{s}_0)$. Observe that the set of possible $\underline{z}$ that appear in the triples, for a given cuspidal support, is finite. This is because it is certainly a subset of the finite set of all natural projections of the elements of the Weyl group orbit of $\iota_P(\underline{s}_0)$ to $\check{\mathfrak{a}}_Q$ for all standard parabolic F-subgroups Q of G. We denote by $\mathcal{S}_{\{P\},\varphi(\pi)}$ the finite set of natural inclusions $\iota_R(\underline{z})$ into $\check{\mathfrak{a}}_B$ of all $\underline{z}$ that appear as the last entry in the triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$.

Given a non-negative integer k not greater than the rank of G, i.e., $k\in\{0,1,2\}$, let $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ be the subset of triples $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ such that the rank of R is k. Consider the set $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ as the set of objects of a groupoid, with morphisms defined as follows. Given a parabolic F-subgroup Q of G, with the Levi factor L_Q, let W_{L_Q} be the Weyl group of L_Q, and let W^Q denote the set of minimal coset representatives for right cosets in $W_{L_Q}\backslash W$. By the minimal coset representative, also known as the Kostant representative, we mean the unique element of minimal length in its right coset [**Kos61**]. Let $W(L_Q)$ denote the set of all $w\in W^Q$ such that the conjugate of L_Q by w is the Levi factor of a standard parabolic subgroup of G, as in [**MW95**, Sect. I.1.7]. If $(R,\Pi,\underline{z})$ and $(R',\Pi',\underline{z}')$ are two (possibly equal) triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$, the set of morphisms between them is the set of all $w\in W(R)$ such that $wL_Rw^{-1}=L_{R'}$, and the conjugates $w(\pi)=\pi'$ and $w(\underline{z})=\underline{z}'$.

Having organized the set $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ into a groupoid, we may define a functor M from that groupoid to the category of $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-modules. It acts on the objects as

$$M\big((R,\Pi,\underline{z})\big)=I_R(\underline{z},\Pi)\otimes S(\check{\mathfrak{a}}_{R,\mathbb{C}}),$$

where $I_R(\underline{z},\Pi)$ is the induced representation as above, and $S(\check{\mathfrak{a}}_{R,\mathbb{C}})$ is the symmetric algebra of $\check{\mathfrak{a}}_{R,\mathbb{C}}$. The action of $\mathfrak{g}_\infty$, K_∞ and $G(\mathbb{A}_f)$ is defined as in [**Fra98**, page 218]. The action of the functor M on a morphism w is defined in terms of the intertwining operator associated to w and its derivatives, as in [**Fra98**, page 234]. We omit the precise formula, as it is not required for the purposes of this paper.

In order to define the filtration, we need some way to organize the contributions of triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$. On the finite set $\mathcal{S}_{\{P\},\varphi(\pi)}$ of inclusions $\iota_R(\underline{z})$ into $\check{\mathfrak{a}}_B$, we define a partial order in which

$$\iota_R(\underline{z}) \succ \iota_{R'}(\underline{z}')$$

if and only if $\iota_R(\underline{z}) \neq \iota_{R'}(\underline{z}')$ and $\iota_R(\underline{z})-\iota_{R'}(\underline{z}')$ belongs to the closure of the negative obtuse Weyl chamber in $\check{\mathfrak{a}}_B$ introduced in Chapter 2. If we write in coordinates $\iota_R(\underline{z}) = (\zeta_1, \zeta_2)$ and $\iota_{R'}(\underline{z}') = (\zeta_1', \zeta_2')$, then $\iota_R(\underline{z}) \succ \iota_{R'}(\underline{z}')$ if and only if

$$\zeta_1 \leq \zeta_1'$$
$$\zeta_1 + \zeta_2 \leq \zeta_1' + \zeta_2'$$

and $(\zeta_1, \zeta_2) \neq (\zeta_1', \zeta_2')$.

Let $T_{\{P\},\varphi(\pi)}$ be a function on the finite set $\mathcal{S}_{\{P\},\varphi(\pi)}$, taking integer values, such that

$$T_{\{P\},\varphi(\pi)}(\iota_R(\underline{z})) > T_{\{P\},\varphi(\pi)}(\iota_{R'}(\underline{z}'))$$

whenever $\iota_R(\underline{z}) \succ \iota_{R'}(\underline{z}')$. Such function is not unique, but any choice of $T_{\{P\},\varphi(\pi)}$ defines equivalent filtrations in the following sense. If a non-trivial quotient of the filtration obtained by certain choice of the function $T_{\{P\},\varphi(\pi)}$ is a direct sum of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules arising from several triples in $\mathcal{M}^k_{\{P\},\varphi(\pi)}$, then some other choice of $T_{\{P\},\varphi(\pi)}$ may result in the filtration in which the summands appear as different quotients. Examples of this phenomenon are provided below, see also Chapter 9. Except for this minor deviation, the non-trivial quotients are the same and appear in the same order.

We now fix a function $T = T_{\{P\},\varphi(\pi)}$ satisfying the condition above. Given an integer i, let $\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}$ denote the set of triples $(R, \Pi, \underline{z})$ in $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ such that $T(\iota_R(\underline{z})) = i$. Then the set $\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}$ determines a full subcategory of the groupoid $\mathcal{M}^k_{\{P\},\varphi(\pi)}$.

4.3. Definition and construction of the filtration

We are now ready to state the main theorem of [**Fra98**] regarding the Franke filtration. Our statement slightly differs from the original one, because we take into account the full cuspidal support, as explained in Section 4.1.

THEOREM 4.1. [**Fra98**, Thm. 14] *There exists a descending filtration*

$$\cdots \supseteq \mathcal{A}^i_{\{P\},\varphi(\pi)} \supseteq \mathcal{A}^{i+1}_{\{P\},\varphi(\pi)} \supseteq \cdots$$

of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{P\},\varphi(\pi)}$*, indexed by the integers, such that*

$$\begin{aligned}\mathcal{A}^i_{\{P\},\varphi(\pi)}/\mathcal{A}^{i+1}_{\{P\},\varphi(\pi)} &\cong \bigoplus_{k=0}^{2} \operatorname*{colim}_{(R,\Pi,\underline{z})\in\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}} M\big((R,\Pi,\underline{z})\big) \\ &\cong \bigoplus_{k=0}^{2} \operatorname*{colim}_{(R,\Pi,\underline{z})\in\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}} I_R(\underline{z},\Pi)\otimes S(\check{\mathfrak{a}}_{R,\mathbb{C}}),\end{aligned}$$

where the colimit is the colimit in the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules of the functor* M*, restricted to* $\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}$*, in the sense of* [**Mac71**, Sect. III.3] *and Appendix A below. Since* T *is defined on a finite set, only finitely many quotients of the filtration are non-trivial.*

The isomorphisms in the above theorem are constructed using the main values of the derivatives of the Eisenstein series. More precisely, given a triple $(R, \Pi, \underline{z})$ in $\mathcal{M}^{k,T,i}_{\{P\},\varphi(\pi)}$, the degenerate Eisenstein series $E(f, \underline{s})$, associated to the residual representation Π of $L_R(\mathbb{A})$, are constructed in the same way as the Eisenstein series associated to π^u in Section 3.2. Here $f \in \mathcal{W}_\Pi$ and $\underline{s} \in \check{\mathfrak{a}}_{R,\mathbb{C}}$, so that $f_{\underline{s}}$ is a section of the family of induced representations $I_R(\underline{s}, \Pi)$. The elements of the symmetric algebra $S(\check{\mathfrak{a}}_{R,\mathbb{C}})$ are viewed as linear combinations of iterative derivatives. We denote by $\partial^\alpha/\partial\underline{s}^\alpha$ the iterative derivative given in coordinates by a multi-index α, and view it as an element of $S(\check{\mathfrak{a}}_{R,\mathbb{C}})$. The isomorphism is then given by the assignment

$$f_{\underline{s}} \otimes \frac{\partial^\alpha}{\partial \underline{s}^\alpha} \mapsto \mathrm{MV}_{\underline{s}=\underline{z}} \left(\frac{\partial^\alpha E(f, \underline{s})}{\partial \underline{s}^\alpha} \right).$$

The main value MV at $\underline{s} = \underline{z}$ of the derivative of the Eisenstein series is defined as the constant term in its Laurent expansion around $\underline{s} = \underline{z}$ along any line in generic position passing through $\underline{z}$. However, the main value at $\underline{s} = \underline{z}$ is well-defined only if the Eisenstein series is holomorphic at $\underline{s} = \underline{z}$. Otherwise, the main value is only well-defined as an element of the quotient

$$\mathcal{A}^i_{\{P\},\varphi(\pi)} / \mathcal{A}^{i+1}_{\{P\},\varphi(\pi)}.$$

The underlying reason is that the coefficients in the principal part of the Laurent series should have already been assigned into deeper quotients of the filtration. Hence, whenever the Eisenstein series associated to $(R, \Pi, \underline{z})$ have a pole at $\underline{s} = \underline{z}$, there should be other triples which contribute to deeper quotients of the filtration, i.e., those with larger i's, such that the coefficients in the principal part of the Laurent series already occur as main values of derivatives of Eisenstein series associated to these other triples. For more details see [**Fra98**, Sect. 6] and [**FS98**, page 775]. This interesting phenomenon occurs in the Franke filtration of $G = Sp_2$, as exhibited below, as well as in the case of the general linear group [**GG22**], and the exceptional group G_2 as observed in [**Fra98**].

CHAPTER 5

Filtration for the support in a maximal parabolic subgroup

Let P_i, $i = 1, 2$, be one of the maximal proper parabolic F-subgroups of G, as in Chapter 2. Let $P_i = L_i N_i$ be its Levi decomposition, where L_i is the Levi factor, and N_i the unipotent radical. Recall that in our notation $L_1 \cong GL_2$ and $L_2 \cong GL_1 \otimes SL_2$.

We fix the cuspidal support $(\{P_i\}, \varphi(\pi))$, where $\pi \cong \pi^u \otimes \underline{s}_0$ is a unitary cuspidal automorphic representation π^u of $L_i(\mathbb{A})$ twisted by the character of $L_i(\mathbb{A})$ corresponding to $s_0 \in \check{\mathfrak{a}}_{P_i}$ in the real part of the closure of the positive Weyl chamber, i.e., $s_0 \geq 0$, as in Section 3.3.

We form the Eisenstein series $E(f, s)$ associated to π^u, as in Section 3.2. The analytic properties of these Eisenstein series are crucial for the explicit description of the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P_i\},\varphi(\pi)}$. The Franke filtration in the case of cuspidal support in a maximal proper parabolic F-subgroup of any connected reductive linear algebraic group G is described in [**Grb12**] in terms of the analytic properties of the Eisenstein series $E(f, s)$ at the value $s = s_0$ of its complex parameter. Observe that the condition $F = \mathbb{Q}$ in *loc. cit.* is not really necessary, it only slightly simplifies the notation, and the results hold, with the same proof, for any algebraic number field F. We now recall the main theorem of *loc. cit.* in our setting.

THEOREM 5.1. [**Grb12**, Thm. 3.1] *Let $G = Sp_2$ be the symplectic group of rank two defined over F. Let $(\{P_i\}, \varphi(\pi))$ be a fixed cuspidal support, where $\pi \cong \pi^u \otimes \underline{s}_0$ is a unitary cuspidal automorphic representation π^u of $L_i(\mathbb{A})$ twisted by the character of $L_i(\mathbb{A})$ corresponding to $s_0 \in \check{\mathfrak{a}}_{P_i}$ with $s_0 \geq 0$.*

Then, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P_i\},\varphi(\pi)}$ is at most a two-step filtration

$$\mathcal{A}_{\{P_i\},\varphi(\pi)} \supsetneqq \mathcal{L}_{\{P_i\},\varphi(\pi)} \supseteq \{0\},$$

where $\mathcal{L}_{\{P_i\},\varphi(\pi)}$ is the (possibly trivial) space of automorphic forms spanned by the residues at $s = s_0$ of the Eisenstein series $E(f, s)$ associated to π^u. This space is precisely the space of square-integrable automorphic forms in $\mathcal{A}_{\{P_i\},\varphi(\pi)}$. The quotient is always non-trivial and isomorphic to

$$\mathcal{A}_{\{P_i\},\varphi(\pi)}/\mathcal{L}_{\{P_i\},\varphi(\pi)} \cong \begin{cases} \left(I_{P_i}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_i,\mathbb{C}})\right)^w, & \text{if } s_0 = 0 \text{ and } w(\pi^u) \cong \pi^u, \\ I_{P_i}(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_i,\mathbb{C}}), & \text{otherwise,} \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where $w(\pi^u)$ denotes the representation conjugate to π^u by the unique non-trivial element $w \in W(L_i)$, and $\left(I_{P_i}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_i,\mathbb{C}})\right)^w$ stands for the space of invariant vectors for the action of the operator $M(w)$ on $I_{P_i}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_i,\mathbb{C}})$, where M is the functor introduced in Section 4.2.

5.1. Analytic properties of Eisenstein series supported in a maximal parabolic subgroup

The analytic properties of the Eisenstein series $E(f,s)$, associated to a unitary cuspidal automorphic representation π^u of $L_i(\mathbb{A})$ as in Section 3.2, are determined by Kim in [**Kim95**]. He also describes the space of automorphic forms spanned by their residues at a pole. We now recall two of his results, which are required to make Theorem 5.1 explicit in the case of the symplectic group of rank two.

THEOREM 5.2. [**Kim95**, Thm. 3.3] *Let $G = Sp_2$ be the symplectic group of rank two. Let P_1 be the standard parabolic F-subgroup of G with the Levi factor $L_1 \cong GL_2$. Let π^u be a unitary cuspidal automorphic representation of $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$. Suppose that s_0 is in the real part of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_{P_1,\mathbb{C}}$, i.e., $s_0 \geq 0$.*

Then, the Eisenstein series $E(f,s)$, associated to π^u, is holomorphic at $s = s_0$ for all $f \in \mathcal{W}_{\pi^u}$, except in the case

- *$s_0 = 1/2$,*
- *the central character of π^u is trivial, and*
- *the principal automorphic L-function $L(s,\pi^u)$, attached to π^u, is non-zero at $s = 1/2$.*

In that case, for all $f \in \mathcal{W}_{\pi^u}$ the pole at $s = 1/2$ is at most of order one, and the space spanned by the residues of the Eisenstein series $E(f,s)$, associated to π^u, is the residual representation of $G(\mathbb{A})$ isomorphic to

$$\operatorname{span}\left\{\operatorname{Res}_{s=1/2} E(f,s) \,:\, f \in \mathcal{W}_{\pi^u}\right\} \cong J_{P_1}(1/2,\pi^u),$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where $J_{P_1}(1/2,\pi^u) \cong \otimes_{v\in V} J_{P_1}(1/2,\pi^u_v)$ is isomorphic to the restricted tensor product over all places v of F of irreducible representations $J_{P_1}(1/2,\pi^u_v)$ of $G(F_v)$. The representation $J_{P_1}(1/2,\pi^u_v)$ is the unique irreducible quotient of the induced representation $I_{P_1}(1/2,\pi^u_v)$.

Before stating the second result of [**Kim95**], we need a technical remark. Let σ be a cuspidal automorphic representation of $SL_2(\mathbb{A})$. According to [**Fli92**, Lemma 1.9.2], see also [**LL79**], σ is an irreducible summand in the restriction from $GL_2(\mathbb{A})$ to $SL_2(\mathbb{A})$ of a unitary cuspidal automorphic representation σ' of $GL_2(\mathbb{A})$. The representation σ is called monomial if σ' is monomial, i.e., if there exists a non-trivial character η of $\mathbb{I}$ such that $\sigma' \otimes \eta \cong \sigma'$. Then η is quadratic, and hence determines a quadratic extension E/F by class field theory. The main result of [**LL79**], see also [**GJ78**, Sect. 3.7], associates to σ' a (not necessary unitary) Hecke character Ω of the group of idèles $\mathbb{I}_E$ of E. Let Ω^c be the Hecke character of $\mathbb{I}_E$ conjugate to Ω.

If $\Omega(\Omega^c)^{-1}$ factors through the norm map $N_{E/F}$ as

$$\Omega(\Omega^c)^{-1} = \eta' \circ N_{E/F},$$

then η' is another quadratic Hecke character of the group of idèles of $\mathbb{I}$. In this case there are three quadratic characters η, η' and $\eta\eta'$ such that

$$\sigma' \cong \sigma' \otimes \eta \cong \sigma' \otimes \eta' \cong \sigma' \otimes \eta\eta',$$

that is, σ' twisted by one of these characters is again isomorphic to σ'. Otherwise, that is, if $\Omega(\Omega^c)^{-1}$ does not factor through the norm map $N_{E/F}$, then η is the unique such character determined by σ.

THEOREM 5.3. [**Kim95**, Thm. 4.1] *Let $G = Sp_2$ be the symplectic group of rank two. Let P_2 be the standard parabolic F-subgroup of G with the Levi factor $L_2 \cong GL_1 \times SL_2$. Let $\pi^u \cong \chi \otimes \sigma$ be a unitary cuspidal automorphic representation of $L_2(\mathbb{A}) \cong \mathbb{I} \times SL_2(\mathbb{A})$, where χ is a unitary Hecke character of the group of idèles $\mathbb{I}$, and σ a cuspidal automorphic representation of $SL_2(\mathbb{A})$. Suppose that s_0 is in the real part of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_{P_2,\mathbb{C}}$, i.e., $s_0 \geq 0$.*

Then, the Eisenstein series $E(f,s)$, associated to π^u, is holomorphic at $s = s_0$ for all $f \in \mathcal{W}_{\pi^u}$, except in the case

- $s_0 = 1$,
- *the representation σ is monomial, and*
- *the character χ is determined by σ, more precisely, in the notation introduced before the statement of the theorem, χ is one of the quadratic characters η, η' and $\eta\eta'$ if the character $\Omega(\Omega^c)^{-1}$ factors through the norm map $N_{E/F}$, and $\chi = \eta$ otherwise.*

In that case, for all $f \in \mathcal{W}_{\pi^u}$ the pole at $s = 1$ is at most of order one, and the space spanned by the residues of the Eisenstein series $E(f,s)$, associated to π^u, is the residual representation of $G(\mathbb{A})$ isomorphic to

$$\operatorname{span}\{\operatorname{Res}_{s=1}E(f,s) \,:\, f \in \mathcal{W}_{\pi^u}\} \cong J_{P_2}(1,\pi^u),$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where $J_{P_2}(1,\pi^u) \cong \otimes_{v\in V} J_{P_2}(1,\pi^u_v)$ is the restricted tensor product over all places v of F of irreducible representations $J_{P_2}(1,\pi^u_v)$ of $G(F_v)$. The representation $J_{P_2}(1,\pi^u_v)$ is the unique irreducible quotient of the induced representation $I_{P_2}(1,\pi^u_v)$.

5.2. Results for the support in a maximal parabolic subgroup

As a consequence of these two theorems of Kim, we can make Theorem 5.1 explicit. We omit the proofs, as they follow directly from the general description of the Franke filtration, once the analytic properties of the Eisenstein series are known. Note that the condition $\widetilde{\pi}^u \cong \pi^u$ in Theorem 5.4 below, where $\widetilde{\pi}^u$ is the representation contragredient to π^u, and the condition that χ^2 is trivial in Theorem 5.5 below, arise from the fact that the conjugate of π^u by w is

$$w(\pi^u) \cong \begin{cases} \widetilde{\pi}^u, & \text{for } i = 1 \\ \chi^{-1} \otimes \sigma, & \text{for } i = 2, \end{cases}$$

where w is the unique non-trivial element in $W(L_i)$ with $i = 1, 2$. Thus, the condition $w(\pi^u) \cong \pi^u$ in Theorem 5.1, becomes $\widetilde{\pi}^u \cong \pi^u$ in Theorem 5.4, and $\chi^{-1} = \chi$, i.e., $\chi^2 = \mathbf{1}$ in Theorem 5.5.

THEOREM 5.4. *Let $G = Sp_2$ be the symplectic group of rank two. Let P_1 be the standard parabolic F-subgroup of G with the Levi factor $L_1 \cong GL_2$. Let $(\{P_1\}, \varphi(\pi))$ be a fixed cuspidal support, where $\pi \cong \pi^u \otimes |\det|^{s_0}$ is a unitary cuspidal automorphic representation π^u of $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ twisted by the character of $L_1(\mathbb{A})$ corresponding to $s_0 \in \check{\mathfrak{a}}_{P_1}$ with $s_0 \geq 0$.*

(1) *If the following assertions*
   - $s_0 = 1/2$,
   - *the central character of π^u is trivial, and*
   - *the principal automorphic L-function $L(s,\pi^u)$ is non-zero at $s = 1/2$,*

are all satisfied, then the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{P_1\},\varphi(\pi)}$ *is a two-step filtration*

$$\mathcal{A}_{\{P_1\},\varphi(\pi)} \supsetneqq \mathcal{L}_{\{P_1\},\varphi(\pi)} \supsetneqq \{0\},$$

where, in the notation of Theorem 5.2,

$$\mathcal{L}_{\{P_1\},\varphi(\pi)} \cong J_{P_1}(1/2, \pi^u),$$

and

$$\mathcal{A}_{\{P_1\},\varphi(\pi)}/\mathcal{L}_{\{P_1\},\varphi(\pi)} \cong I_{P_1}(1/2, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}),$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(2) *Otherwise, that is, if one of the three assertions above is not satisfied, the Franke filtration is only one-step filtration, and the full space* $\mathcal{A}_{\{P_1\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{P_1\},\varphi(\pi)} \cong \begin{cases} \left(I_{P_1}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\right)^w, & \text{if } s_0 = 0 \text{ and } \widetilde{\pi}^u \cong \pi^u, \\ I_{P_1}(s_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}), & \text{otherwise,} \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where* $\widetilde{\pi}^u$ *is the representation contragredient to* π^u*, and* $\left(I_{P_1}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\right)^w$ *stands for the space of invariant vectors for the action of the operator* $M(w)$ *on* $I_{P_1}(0, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$*, where* $w = w_{212}$ *is the unique non-trivial element in* $W(L_1)$ *and* M *is the functor introduced in Section 4.2.*

THEOREM 5.5. *Let* $G = Sp_2$ *be the symplectic group of rank two. Let* P_2 *be the standard parabolic* F*-subgroup of* G *with the Levi factor* $L_2 \cong GL_1 \times SL_2$*. Let* $(\{P_2\}, \varphi(\pi))$ *be a fixed cuspidal support, where* $\pi \cong \chi|\cdot|^{s_0} \otimes \sigma$ *is a unitary cuspidal automorphic representation* $\pi^u \cong \chi \otimes \sigma$ *of* $L_2(\mathbb{A}) \cong \mathbb{I} \times SL_2(\mathbb{A})$ *twisted by the character of* $L_2(\mathbb{A})$ *corresponding to* $s_0 \in \check{\mathfrak{a}}_{P_2}$ *with* $s_0 \geq 0$*, where* χ *is a unitary Hecke character of the group of idèles* $\mathbb{I}$*, and* σ *is a cuspidal automorphic representation of* $SL_2(\mathbb{A})$*.*

(1) *If the following assertions*
- $s_0 = 1$,
- σ *is a monomial representation, and*
- χ *is one of the characters determined by* σ *as in Theorem 5.3,*

are all satisfied, then the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{P_2\},\varphi(\pi)}$ *is a two-step filtration*

$$\mathcal{A}_{\{P_2\},\varphi(\pi)} \supsetneqq \mathcal{L}_{\{P_2\},\varphi(\pi)} \supsetneqq \{0\},$$

where, in the notation of Theorem 5.3,

$$\mathcal{L}_{\{P_2\},\varphi(\pi)} \cong J_{P_2}(1, \pi^u),$$

and

$$\mathcal{A}_{\{P_2\},\varphi(\pi)}/\mathcal{L}_{\{P_2\},\varphi(\pi)} \cong I_{P_2}(1, \pi^u) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}}),$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(2) *Otherwise, that is, if one of the three assertions above is not satisfied, the Franke filtration is only one-step filtration, and the full space* $\mathcal{A}_{\{P_2\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{P_2\},\varphi(\pi)} \cong \begin{cases} \left(I_{P_2}(0,\pi^u) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^w, & \text{if } s_0 = 0 \text{ and } \chi^2 \text{ is trivial,} \\ I_{P_2}(s_0,\pi^u) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}}), & \text{otherwise,} \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where* $\left(I_{P_2}(0,\pi^u) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^w$ *stands for the space of invariant vectors for the action of the operator* $M(w)$ *on* $I_{P_2}(0,\pi^u) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$*, where* $w = w_{121}$ *is the unique non-trivial element in* $W(L_2)$ *and* M *is the functor introduced in Section 4.2.*

CHAPTER 6

Analytic properties of Eisenstein series supported in the Borel subgroup

In this chapter, the Eisenstein series supported in the Borel subgroup B are studied. The Borel subgroup B of G, with the Levi decomposition $B = TU$, is fixed as in Chapter 2. We retain the notation of previous chapters.

Let $\pi^u \cong \chi_1 \otimes \chi_2$ be a unitary character of $T(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$, where χ_1 and χ_2 are unitary Hecke characters of the group of idèles $\mathbb{I}$. We fix an element $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \overline{\mathfrak{C}^+}$, i.e., $s_{0,1} \geq s_{0,2} \geq 0$. Let

$$\pi = \chi_1 |\cdot|^{s_{0,1}} \otimes \chi_2 |\cdot|^{s_{0,2}}$$

be the unitary character π^u of $T(\mathbb{A})$ twisted by the character of $T(\mathbb{A})$ corresponding to $\underline{s}_0$. Table 3.1 gives the action of the Weyl group W on the character π, which also determines the action on $(s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}}$ and on the unitary character $\pi^u = \chi_1 \otimes \chi_2$ of $T(\mathbb{A})$.

6.1. Eisenstein series on $GL_2(\mathbb{A})$ and $SL_2(\mathbb{A})$

Given a cuspidal support $(\{B\}, \varphi(\pi))$, where $\varphi(\pi)$ is the associate class of π, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $(\{B\}, \varphi(\pi))$ is closely related to the analytic properties at $\underline{s} = \underline{s}_0$ of the Eisenstein series $E(f, \underline{s})$ associated to π^u, where $f \in \mathcal{W}_{\pi^u}$, as in Section 3.2. Moreover, the analytic properties of the Eisenstein series on the Levi factors $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ and $L_2(\mathbb{A}) \cong GL_1(\mathbb{A}) \times SL_2(\mathbb{A})$ with the support in $\varphi(\pi)$ are also required for the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$. Therefore, we first recall the well known analytic properties of the Eisenstein series on $GL_2(\mathbb{A})$ and $SL_2(\mathbb{A})$.

THEOREM 6.1. *Let GL_2 be the general linear group of rank one defined over F. Let B_{GL_2} be a Borel subgroup and $T_{GL_2} \cong GL_1 \times GL_1$ the maximal split torus of GL_2 contained in B_{GL_2}. Let $\mu_1 \otimes \mu_2$ be a unitary character of $T_{GL_2}(\mathbb{A}) \cong \mathbb{I} \otimes \mathbb{I}$, where μ_1 and μ_2 are unitary Hecke characters of the group of idèles $\mathbb{I}$ of F, normalized as in Section 3.1. Let $E^{GL_2}(f, \underline{s})$ be the Eisenstein series constructed from $f \in \mathcal{W}^{GL_2}_{\mu_1 \otimes \mu_2}$, where $\mathcal{W}^{GL_2}_{\mu_1 \otimes \mu_2}$ is the analogue in the case of GL_2 of the space $\mathcal{W}_{\pi^u}$ of Section 3.2, and $\underline{s} = (s_1, s_2) \in \check{a}_{B_{GL_2},\mathbb{C}}$ is the complex parameter. Suppose that $\underline{s}_0 = (s_{0,1}, s_{0,2})$ is in the real part of the closure of the positive Weyl chamber determined by B_{GL_2}, i.e., $s_{0,1} \geq s_{0,2}$ are real numbers.*

Then, the Eisenstein series $E^{GL_2}(f, \underline{s})$ is holomorphic at $\underline{s} = \underline{s}_0$ for all $f \in \mathcal{W}^{GL_2}_{\mu_1 \otimes \mu_2}$, except in the case

- *$\mu_1 = \mu_2$, and*
- *$s_{0,1} - s_{0,2} = 1$.*

TABLE 6.1. Singular hyperplanes for the Eisenstein series $E(f, \underline{s})$ associated to π^u, where $\pi^u = \chi_1 \otimes \chi_2$ is a unitary character of $T(\mathbb{A})$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}}$ is the complex parameter, with equations in $(s_1, s_2) \in \check{\mathfrak{a}}_B$ and conditions on χ_1 and χ_2 for the pole along singular hyperplanes.

Hyperplane	Equation	Condition on $\pi^u = \chi_1 \otimes \chi_2$
$\mathfrak{S}_1$	$s_1 - s_2 = 1$	$\chi_1 = \chi_2$
$\mathfrak{S}_2$	$s_2 = 1$	$\chi_2 = \mathbf{1}$
$\mathfrak{S}_3$	$s_1 + s_2 = 1$	$\chi_1 = \chi_2^{-1}$
$\mathfrak{S}_4$	$s_1 = 1$	$\chi_1 = \mathbf{1}$

In that case, for all $f \in \mathcal{W}^{GL_2}_{\mu_1 \otimes \mu_2}$ the pole of $E^{GL_2}(f, \underline{s})$ is at most of order one, and the residues span the residual representation isomorphic to

$$(\mu \circ \det) \, |\det|^{\frac{s_{0,1}+s_{0,2}}{2}},$$

where we write $\mu = \mu_1 = \mu_2$.

THEOREM 6.2. *Let SL_2 be the special linear group of rank one defined over F. Let B_{SL_2} be a Borel subgroup and $T_{SL_2} \cong GL_1$ the maximal split torus of SL_2 contained in B_{SL_2}. Let μ be a unitary Hecke character of $T_{SL_2}(\mathbb{A}) \cong \mathbb{I}$, normalized as in Section 3.1. Let $E^{SL_2}(f, s)$ be the Eisenstein series constructed from $f \in \mathcal{W}^{SL_2}_\mu$, where $\mathcal{W}^{SL_2}_\mu$ is the analogue in the case of SL_2 of the space $\mathcal{W}_{\pi^u}$ of Section 3.2, and $s \in \check{a}_{B_{SL_2},\mathbb{C}}$ is the complex parameter. Suppose that s_0 is in the real part of the closure of the positive Weyl chamber determined by B_{SL_2}, i.e., $s_0 \geq 0$ is a real number.*

Then, the Eisenstein series $E^{SL_2}(f, s)$ is holomorphic at $s = s_0$ for all $f \in \mathcal{W}^{SL_2}_\mu$, except in the case

- *$\mu = \mathbf{1}$ is the trivial character of $\mathbb{I}$, and*
- *$s_0 = 1$.*

In that case, for all $f \in \mathcal{W}^{SL_2}_\mu$ the pole of $E^{SL_2}(f, s)$ is at most of order one, and the residues span the residual representation isomorphic to the trivial representation $\mathbf{1}_{SL_2(\mathbb{A})}$ of $SL_2(\mathbb{A})$.

6.2. Eisenstein series on $G(\mathbb{A})$

We now turn our attention back to the Eisenstein series $E(f, \underline{s})$ on $G(\mathbb{A})$, associated to π^u. The singular hyperplanes of these Eisenstein series, which intersect the closure of the positive Weyl chamber, are determined in [**Kim95**, page 141]. They are listed in Table 6.1 and shown in Figure 6.1. For each singular hyperplane, Table 6.1 contains its equation and the condition on the characters χ_1 and χ_2 for the pole along the hyperplane.

The singular hyperplanes intersect at certain points, which may result with a pole of higher order of the Eisenstein series $E(f, \underline{s})$. These points are $\underline{s}_0 = T_1(2, 1)$, $\underline{s}_0 = T_2(1, 0)$ and $\underline{s}_0 = T_3(1, 1)$ as shown in Figure 6.1. They are studied by Kim [**Kim95**] as part of his complete description of the residual spectrum of $G(\mathbb{A})$. For the convenience of the reader we recall the results below.

THEOREM 6.3 (H. H. Kim [**Kim95**, Thm. 5.4, Prop. 5.1.2, Prop. 5.2.1]). *Let $G = Sp_2$ be the symplectic group of rank two. Let B be the Borel subgroup of G with the Levi factor $T \cong GL_1 \times GL_1$. Let $\pi^u = \chi_1 \otimes \chi_2$ be a unitary character of $T(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$, where χ_1 and χ_2 are unitary Hecke characters of the group of idèles $\mathbb{I}$. Let $E(f, \underline{s})$ be the Eisenstein series associated to π^u, where $f \in \mathcal{W}_{\pi^u}$ as in Section 3.2, and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}}$ is the complex parameter.*

Suppose that:

(i) $\underline{s}_0 = T_1(2, 1)$.
Then, there exists $f \in \mathcal{W}_{\pi^u}$ such that the Eisenstein series $E(f, \underline{s})$ has a pole of order two at $\underline{s} = \underline{s}_0 = (2, 1)$ if and only if $\chi_1 = \chi_2 = \mathbf{1}$ is the trivial character of $\mathbb{I}$. In that case, the iterated residues of $E(f, \underline{s})$ at $\underline{s} = \underline{s}_0$ span a residual representation which is isomorphic to the trivial representation $\mathbf{1}_{G(\mathbb{A})}$ of $G(\mathbb{A})$.

(ii) $\underline{s}_0 = T_2(1, 0)$.
Then, there exists $f \in \mathcal{W}_{\pi^u}$ such that the Eisenstein series $E(f, \underline{s})$ has a pole of order two at $\underline{s} = \underline{s}_0 = (1, 0)$ if and only if χ_1 and χ_2 are equal and $\chi^2 = \mathbf{1}$, where we denote $\chi = \chi_1 = \chi_2$. In that case,
- *if $\chi = \mathbf{1}$ is the trivial character, the iterated residues of $E(f, \underline{s})$ at $\underline{s} = \underline{s}_0$ are not square-integrable;*
- *if $\chi \neq \mathbf{1}$ is a non-trivial quadratic character, the iterated residues of $E(f, \underline{s})$ at $\underline{s} = \underline{s}_0$ are square-integrable and they span a residual representation isomorphic to the representation $J(\chi)$, which is defined as a direct sum of irreducible summands described explicitly in* [**Kim95**, page 148] *as a restricted tensor product with certain parity condition imposed on the local factors.*

(iii) $\underline{s}_0 = T_3(1, 1)$.
Then, for every $f \in \mathcal{W}_{\pi^u}$ the Eisenstein series $E(f, \underline{s})$ has a pole at most of order one at $\underline{s} = \underline{s}_0 = (1, 1)$.

Besides singular hyperplanes, the Franke filtration depends on the existence of non-trivial morphisms in the groupoids $\mathcal{M}^k_{\{B\},\varphi(\pi)}$ introduced in Section 4.2. These non-trivial morphisms are given by the Weyl group elements which stabilize $\underline{s}_0 \in \overline{\mathfrak{C}^+}$. The stabilizer of $\underline{s}_0$ is non-trivial along certain hyperplanes. We refer to these hyperplanes as stabilizing hyperplanes. They are listed in Table 6.2 and shown in Figure 6.1. For each stabilizing hyperplane, Table 6.2 contains its equation, the stabilizer in W of points on the hyperplane, and the condition on χ_1 and χ_2 assuring that the character π^u is also stabilized along the hyperplane.

Observe that the stabilizing hyperplanes intersect in a single point in $\check{\mathfrak{a}}_B$, and that is the origin $\underline{s}_0 = (0, 0)$. The stabilizer in W of this point is generated by stabilizers along all stabilizing hyperplanes, i.e., the stabilizer is W. Because of such large stabilizer, the cases of cuspidal support with $\underline{s}_0 = (0, 0)$ are treated separately in the statement of the results in Chapter 7.

The singular and stabilizing hyperplanes are shown in Figure 6.1. The figure shows only the real part $\check{\mathfrak{a}}_B$ of the space $\check{\mathfrak{a}}_{B,\mathbb{C}}$, because, as explained in Section 3.1, our normalization of the cuspidal support makes the poles of the Eisenstein series real. The hyperplanes are seen as lines of their intersection with $\check{\mathfrak{a}}_B$. The figure is similar to the one in [**Kim95**, page 141], except that we use a different coordinate system.

TABLE 6.2. Stabilizing hyperplanes for the cuspidal support $(B, \varphi(\pi))$, with $\pi \cong \pi^u \otimes \underline{s}$, where $\pi^u \cong \chi_1 \otimes \chi_2$ is a unitary character of $T(\mathbb{A})$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_B$. The columns contain equations in (s_1, s_2), stabilizers of $\underline{s}$ in W, and conditions on χ_1 and χ_2 for stabilization of π^u along the stabilizing hyperplanes.

Hyperplane	Equation	Stabilizer	Stabilizing condition for $\pi^u = \chi_1 \otimes \chi_2$
$\mathfrak{S}'_1$	$s_1 - s_2 = 0$	$1, w_1$	$\chi_1 = \chi_2$
$\mathfrak{S}'_2$	$s_2 = 0$	$1, w_2$	$\chi_2^2 = \mathbf{1}$
$\mathfrak{S}'_3$	$s_1 + s_2 = 0$	$1, w_{212}$	$\chi_1 = \chi_2^{-1}$
$\mathfrak{S}'_4$	$s_1 = 0$	$1, w_{121}$	$\chi_1^2 = \mathbf{1}$

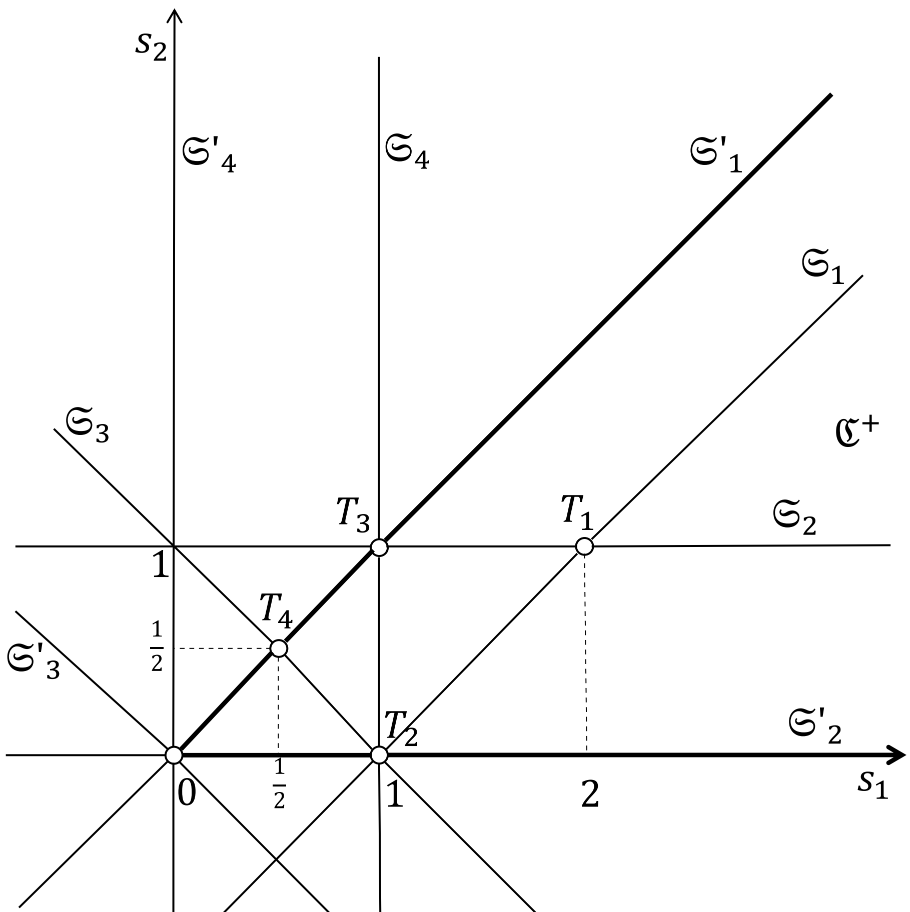

FIGURE 6.1. Singular and stabilizing hyperplanes for the Franke filtration of $\mathcal{A}_{B,\varphi(\pi)}$.

In the statement of results, we refer to different regions in $\overline{\mathfrak{C}^+}$ defined below. The distinguished points in Figure 6.1 are intersections of singular and stabilizing hyperplanes in the closure of the positive Weyl chamber. They are given in coordinates as $T_1(2,1)$, $T_2(1,0)$, $T_3(1,1)$, $T_4(1/2,1/2)$ and $O(0,0)$. The regions are disjoint and defined as follows

$$\begin{aligned}
\widehat{\mathfrak{C}}^+ &= \mathfrak{C}^+ \setminus (\mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 \cup \mathfrak{S}_4),\\
\widehat{\mathfrak{S}}_1 &= \left(\mathfrak{S}_1 \cap \mathfrak{C}^+\right) \setminus \{T_1\},\\
\widehat{\mathfrak{S}}_2 &= \left(\mathfrak{S}_2 \cap \mathfrak{C}^+\right) \setminus \{T_1\},\\
\widehat{\mathfrak{S}}_3 &= \mathfrak{S}_3 \cap \mathfrak{C}^+,\\
\widehat{\mathfrak{S}}_4 &= \mathfrak{S}_4 \cap \mathfrak{C}^+,\\
\widehat{\mathfrak{S}}'_1 &= \left(\mathfrak{S}'_1 \cap \overline{\mathfrak{C}^+}\right) \setminus \{T_3, T_4, O\},\\
\widehat{\mathfrak{S}}'_2 &= \left(\mathfrak{S}'_2 \cap \overline{\mathfrak{C}^+}\right) \setminus \{T_2, O\}.
\end{aligned}$$

Observe that the stabilizing hyperplanes $\mathfrak{S}'_3$ and $\mathfrak{S}'_4$ intersect the closure $\overline{\mathfrak{C}^+}$ of the positive Weyl chamber in point O, and that point is treated in a separate theorem. Hence, they do not generate additional regions above.

CHAPTER 7

Filtration for the support in the Borel subgroup – statement of results

In this chapter, we describe the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms with cuspidal support in the associate class $(\{B\},\varphi(\pi))$, represented by the character

$$\pi \cong \pi^u \otimes \underline{s}_0 \cong \chi_1|\cdot|^{s_{0,1}} \otimes \chi_2|\cdot|^{s_{0,2}},$$

where $\pi^u = \chi_1 \otimes \chi_2$ is a unitary character of $T(\mathbb{A}) = \mathbb{I} \times \mathbb{I}$, with χ_1 and χ_2 unitary Hecke characters of $\mathbb{I}$, and

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \overline{\mathfrak{C}^+}, \quad \text{i.e.,} \quad s_{0,1} \geq s_{0,2} \geq 0.$$

However, before stating the results, we explain the plan how to systematically list all the possible cases.

7.1. Plan for the statement of results

Since the singular hyperplanes for the Eisenstein series $E(f, \underline{s})$ are determined by the conditions on the unitary character π^u, given in Table 6.1, we first distinguish several cases based on the type of π^u.

We number the cases according to the number of singular hyperplanes for the Eisenstein series associated to π^u. Such numeration of cases is aligned with the growing complexity of the Franke filtration. Thus, Case 0 is the case with no singular hyperplanes and Case 4 is the case with all four possible singular hyperplanes. There are four different cases with one singular hyperplane, so we add in our numeration of cases the index of the hyperplane. For instance, Case 1–1 is the case in which $\mathfrak{S}_1$ is the only singular hyperplane. As explained below, there is only one case with two singular hyperplanes, which is numerated as Case 2, and there is no Case 3, because there is no π^u such that there are three singular hyperplanes.

We now list these cases with explicit conditions on $\pi^u = \chi_1 \otimes \chi_2$, i.e., on Hecke characters χ_1 and χ_2. See Table 6.1.

Case 0: $\pi^u \cong \chi_1 \otimes \chi_2$ with $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_2^{-1}$, $\chi_1 \neq \mathbf{1}$, $\chi_2 \neq \mathbf{1}$,

Case 1–1: $\pi^u \cong \chi \otimes \chi$ with $\chi^2 \neq \mathbf{1}$, i.e., χ_1 and χ_2 are equal non-trivial non-quadratic characters,

Case 1–2: $\pi^u \cong \chi \otimes \mathbf{1}$ with $\chi \neq \mathbf{1}$, i.e., χ_1 is a non-trivial and χ_2 is the trivial character,

Case 1–3: $\pi^u \cong \chi \otimes \chi^{-1}$ with $\chi^2 \neq \mathbf{1}$, i.e., χ_1 and χ_2 are different non-trivial non-quadratic characters such that $\chi_1\chi_2$ is trivial,

Case 1–4: $\pi^u \cong \mathbf{1} \otimes \chi$ with $\chi \neq \mathbf{1}$, i.e., χ_1 is the trivial and χ_2 a non-trivial character,

Case 2: $\pi^u \cong \chi \otimes \chi$ with $\chi^2 = \mathbf{1} \neq \chi$, i.e., χ_1 and χ_2 are equal non-trivial quadratic characters,

Case 4: $\pi^u = \mathbf{1} \otimes \mathbf{1}$, i.e., χ_1 and χ_2 are both trivial characters,

where $\mathbf{1}$ denotes the trivial character of $\mathbb{I}$. From the conditions for singular hyperplanes in Table 6.1, observe that in the cases with more than one singular hyperplane, if one of the characters χ_1 and χ_2 is trivial, then the other is trivial as well, and we are in Case 4. If both χ_1 and χ_2 are non-trivial, then the only possibility for more than one singular hyperplane is Case 2 above. This explains why there is only one Case 2, and no Case 3.

Once we fixed the seven cases depending on π^u, we consider each of them separately. In each case, we formulate the result depending on the region in $\overline{\mathfrak{C}^+}$, introduced in Section 6.2, in which $\underline{s}_0$ lies. The only exception of this rule is that we consider separately the point

$$\underline{s}_0 = (0, 0),$$

which is denoted by O in Figure 6.1. There are no singular hyperplanes passing through this point, so that the seven cases above are irrelevant. On the other hand, all stabilizing hyperplanes intersect at $\underline{s}_0 = (0, 0)$, and we distinguish cases depending on the conditions for stabilization in Table 6.2.

7.2. Fascicule de résultats

In this section we formulate the explicit description of the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$ in all possible cases. The numeration of cases is as in Section 7.1.

To reduce repetition, in all theorems below, we have the following assumptions. Let $G = Sp_2$ be the symplectic group of rank two. Let B be the fixed Borel subgroup of G with the maximal split torus $T \cong GL_1 \times GL_1$ as the Levi factor. Let $(\{B\}, \varphi(\pi))$ be a fixed cuspidal support, where $\pi \cong \chi_1|\cdot|^{s_{0,1}} \otimes \chi_2|\cdot|^{s_{0,2}}$ is a character of $T(\mathbb{A})$ with $\pi^u \cong \chi_1 \otimes \chi_2$ a unitary character of $T(\mathbb{A})$, and $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \overline{\mathfrak{C}^+}$, i.e., $s_{0,1} \geq s_{0,2} \geq 0$. In all theorems except the last one, we also assume that $\underline{s}_0 \neq (0, 0)$, and state the results depending on the region of Section 6.2 and Figure 6.1 to which $\underline{s}_0$ belongs. The last theorem deals with the point $\underline{s}_0 = (0, 0)$, which is denoted by O in Figure 6.1.

THEOREM 7.1 (**Case 0**). *Suppose that $\pi^u = \chi_1 \otimes \chi_2$ is in Case 0, that is, the unitary Hecke characters χ_1 and χ_2 satisfy $\chi_1 \neq \chi_2$, $\chi_1 \neq \chi_2^{-1}$, $\chi_1 \neq \mathbf{1}$, and $\chi_2 \neq \mathbf{1}$. In this case, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ is always one-step filtration.*

(0a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_1 \cup \widehat{\mathfrak{S}}_2 \cup \widehat{\mathfrak{S}}_3 \cup \widehat{\mathfrak{S}}_4 \cup \widehat{\mathfrak{S}}'_1 \cup \{T_1, T_3, T_4\} = \overline{\mathfrak{C}^+} \setminus \mathfrak{S}'_2,$$

that is, $s_{0,1} \geq s_{0,2} > 0$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

(0b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_2 \cup \{T_2\},$$

that is, $s_{0,1} > s_{0,2} = 0$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \begin{cases} I_B((s_{0,1},0),\pi^u) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}), & \text{if } \chi_2^2 \neq \mathbf{1}, \\ \big(I_B((s_{0,1},0),\pi^u) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\big)^{w_2}, & \text{if } \chi_2^2 = \mathbf{1}, \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_2 *in the last line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((s_{0,1},0),\pi^u)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

THEOREM 7.2 (**Case 1–1**). *Suppose that* $\pi^u = \chi_1 \otimes \chi_2$ *is in Case 1–1, that is, the unitary Hecke characters* χ_1 *and* χ_2 *are equal non-trivial non-quadratic characters, and denote* $\chi = \chi_1 = \chi_2$*. In this case, the Franke filtration of the* $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is at most two-step filtration.*

(1–1a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_2 \cup \widehat{\mathfrak{S}}_3 \cup \widehat{\mathfrak{S}}_4 \cup \widehat{\mathfrak{S}}'_2,$$

that is, $s_{0,1} > s_{0,2} \geq 0$ *and* $s_{0,1} - s_{0,2} \neq 1$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module.*

(1–1b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_1 \cup \{T_3, T_4\},$$

that is, $s_{0,1} = s_{0,2} > 0$*, let* $t_0 = s_{0,1} = s_{0,2}$*. Then the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \big(I_B((t_0,t_0),\chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\big)^{w_1}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_1 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((t_0,t_0),\chi\otimes\chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(1–1c) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_1 \cup \{T_1, T_2\},$$

that is, $s_{0,1} - s_{0,2} = 1$ *and* $s_{0,2} \geq 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1}+s_{0,2}}{2}, \chi \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$

$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

THEOREM 7.3 (**Case 1–2**). *Suppose that $\pi^u = \chi_1 \otimes \chi_2$ is in Case 1–2, that is, the unitary Hecke character χ_1 is non-trivial and $\chi_2 = \mathbf{1}$ is trivial. In this case, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ is at most two-step filtration.*

(1–2a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_1 \cup \widehat{\mathfrak{S}}_3 \cup \widehat{\mathfrak{S}}_4 \cup \widehat{\mathfrak{S}}_1' \cup \{T_4\},$$

that is, $s_{0,1} \geq s_{0,2} > 0$ and $s_{0,2} \neq 1$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

(1–2b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_2' \cup \{T_2\},$$

that is $s_{0,1} > s_{0,2} = 0$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((s_{0,1}, 0), \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where the exponent w_2 denotes the space of invariant vectors for the action of the intertwining operator $M(w_2)$ on $I_B((s_{0,1}, 0), \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$, and M is the functor introduced in Section 4.2.

(1–2c) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_2 \cup \{T_1, T_3\},$$

that is, $s_{0,1} \geq s_{0,2} = 1$, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,1}, \chi_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((s_{0,1}, 1), \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

THEOREM 7.4 (**Case 1–3**). *Suppose that $\pi^u = \chi_1 \otimes \chi_2$ is in Case 1–3, that is, the unitary Hecke characters χ_1 and χ_2 are different non-trivial non-quadratic characters such that $\chi_1\chi_2 = \mathbf{1}$ is trivial, and denote $\chi = \chi_1 = \chi_2^{-1}$. In this case, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ is at most two-step filtration.*

(1–3a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_1 \cup \widehat{\mathfrak{S}}_2 \cup \widehat{\mathfrak{S}}_4 \cup \widehat{\mathfrak{S}}_1' \cup \widehat{\mathfrak{S}}_2' \cup \{T_1, T_3\},$$

that is, $s_{0,1} \geq s_{0,2} \geq 0$, with $(s_{0,1}, s_{0,2}) \neq (0,0)$, and $s_{0,1} + s_{0,2} \neq 1$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi^{-1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

(1–3b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_3 \cup \{T_2, T_4\},$$

that is, $s_{0,1} + s_{0,2} = 1$ and $s_{0,1} \geq s_{0,2} \geq 0$, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1} - s_{0,2}}{2}, \chi \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi^{-1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

THEOREM 7.5 (**Case 1–4**). *Suppose that $\pi^u = \chi_1 \otimes \chi_2$ is in Case 1–4, that is, the unitary Hecke character $\chi_1 = \mathbf{1}$ is trivial and χ_2 is non-trivial. In this case, the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ is at most two-step filtration.*

(1–4a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_1 \cup \widehat{\mathfrak{S}}_2 \cup \widehat{\mathfrak{S}}_3 \cup \widehat{\mathfrak{S}}'_1 \cup \{T_1, T_4\},$$

that is, $s_{0,1} \geq s_{0,2} > 0$ and $s_{0,1} \neq 1$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

(1–4b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_2,$$

that is, $s_{0,1} > s_{0,2} = 0$ and $s_{0,1} \neq 1$, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \begin{cases} I_B((s_{0,1}, 0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}), & \text{if } \chi_2^2 \neq \mathbf{1}, \\ \left(I_B((s_{0,1}, 0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}, & \text{if } \chi_2^2 = \mathbf{1}, \end{cases}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where the exponent w_2 denotes the space of invariant vectors for the action of the intertwining operator $M(w_2)$ on $I_B((s_{0,1}, 0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$, and M is the functor introduced in Section 4.2.

(1–4c) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_4 \cup \{T_3\},$$

that is, $s_{0,1} = 1$ and $1 \geq s_{0,2} > 0$, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,2}, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1, s_{0,2}), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

(1–4d) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_2,$$

that is, $s_{0,1} = 1$ *and* $s_{0,2} = 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \begin{cases} I_{P_2}\left(0, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}}), & \text{if } \chi_2^2 \neq \mathbf{1}, \\ \left(I_{P_2}\left(0, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^{w_{121}}, & \text{if } \chi_2^2 = \mathbf{1}, \end{cases}$$

$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \begin{cases} I_B((1,0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}), & \text{if } \chi_2^2 \neq \mathbf{1}, \\ \left(I_B((1,0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}, & \text{if } \chi_2^2 = \mathbf{1}, \end{cases}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where the exponent* w_{121} *in the first line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_{121})$ *on the induced representation* $I_{P_2}\left(0, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$*, and the exponent* w_2 *in the second line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((1,0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

THEOREM 7.6 (**Case 2**). *Suppose that* $\pi^u = \chi_1 \otimes \chi_2$ *is in Case 2, that is, the unitary Hecke characters* χ_1 *and* χ_2 *are equal non-trivial quadratic characters, and denote* $\chi = \chi_1 = \chi_2$*. In this case, the Franke filtration of the* $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is at most three-step filtration.*

(2a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+ \cup \widehat{\mathfrak{S}}_2 \cup \widehat{\mathfrak{S}}_4,$$

that is, $s_{0,1} > s_{0,2} > 0$ *and* $s_{0,1} - s_{0,2} \neq 1$ *and* $s_{0,1} + s_{0,2} \neq 1$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module.*

(2b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_1 \cup \{T_3\},$$

that is, $s_{0,1} = s_{0,2} > 0$*, and* $t_0 = s_{0,1} = s_{0,2} \neq 1/2$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((t_0, t_0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_1 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((t_0, t_0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(2c) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_2,$$

that is, $s_{0,1} > s_{0,2} = 0$ *and* $s_{0,1} \neq 1$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((s_{0,1}, 0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2},$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_2 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((s_{0,1}, 0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(2d) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_1 \cup \{T_1\},$$

that is, $s_{0,1} - s_{0,2} = 1$ *and* $s_{0,2} > 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1} + s_{0,2}}{2}, \chi \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)} / \mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(2e) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_3,$$

that is, $s_{0,1} + s_{0,2} = 1$ *and* $s_{0,1} > s_{0,2} > 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1} - s_{0,2}}{2}, \chi \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)} / \mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(2f) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_4,$$

that is, $\underline{s}_{0,1} = \underline{s}_{0,2} = 1/2$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_{P_1}(0, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\right)^{w_{212}}$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)} / \mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_B((1/2, 1/2), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where the exponent* w_{212} *in the first line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_{212})$ *on* $I_{P_1}(0, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$*, and the exponent* w_1 *in the second line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((1/2, 1/2), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(2g) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_2,$$

that is, $\underline{s}_{0,1} = 1$ *and* $\underline{s}_{0,2} = 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the three-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong J(\chi) \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong I_{P_1}(1/2, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \left(I_B((1,0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2} \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where* $J(\chi)$ *is the residual representation of* $G(\mathbb{A})$ *mentioned in Theorem 6.3 and explicitly described on* [**Kim95**, page 148]*, and the exponent* w_2 *in the last line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((1,0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

Theorem 7.7 (**Case 4**). *Suppose that* $\pi^u = \chi_1 \otimes \chi_2$ *is in Case 4, that is, the unitary Hecke characters* χ_1 *and* χ_2 *are both trivial. In this case, the Franke filtration of the* $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is at most three-step filtration.*

(4a) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+,$$

that is, $s_{0,1} > s_{0,2} > 0$ *and* $s_{0,1} - s_{0,2} \neq 1$ *and* $s_{0,2} \neq 1$ *and* $s_{0,1} + s_{0,2} \neq 1$ *and* $s_{0,1} \neq 1$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module.*

(4b) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_1,$$

that is, $s_{0,1} = s_{0,2} > 0$ *and* $t_0 = s_{0,1} = s_{0,2} \neq 1$ *and* $t_0 = s_{0,1} = s_{0,2} \neq 1/2$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((t_0, t_0), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_1 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((t_0, t_0), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(4c) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_2,$$

that is, $s_{0,1} > s_{0,2} = 0$ *and* $\underline{s}_{0,1} \neq 1$*, the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((s_{0,1}, 0), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2},$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_2 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((s_{0,1}, 0), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(4d) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_1,$$

that is, $s_{0,1} - s_{0,2} = 1$ *and* $\underline{s}_0 \neq (2,1)$ *and* $\underline{s}_0 \neq (1,0)$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1}+s_{0,2}}{2}, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(4e) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_2,$$

that is, $s_{0,1} > s_{0,2} = 1$ *and* $s_{0,1} \neq 2$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,1}, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((s_{0,1},1), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(4f) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_3,$$

that is, $s_{0,1} + s_{0,2} = 1$ *and* $s_{0,1} > s_{0,2} > 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1}-s_{0,2}}{2}, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(4g) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_4,$$

that is, $1 = s_{0,1} > s_{0,2} > 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,2}, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1,s_{0,2}), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules.*

(4h) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_4,$$

that is, $\underline{s}_{0,1} = \underline{s}_{0,2} = 1/2$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_{P_1}\left(0, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\right)^{w_{212}}$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_B((1/2,1/2), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where the exponent* w_{212} *in the first line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_{212})$ *on* $I_{P_1}(0, \mathbf{1}\circ\det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$*, and the exponent* w_1 *in the second line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((1/2,1/2), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(4i) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_3,$$

that is, $\underline{s}_{0,1} = \underline{s}_{0,2} = 1$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the two-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(1, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_B((1,1), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where the exponent* w_1 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_1)$ *on* $I_B((1,1), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(4j) *For*

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_2,$$

that is, $\underline{s}_{0,1} = 1$ *and* $\underline{s}_{0,2} = 0$*, the Franke filtration of the space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is the three-step filtration*

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\mathcal{A}^2_{\{B\},\varphi(\pi)} \cong \left(I_{P_2}\left(0, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^{w_{121}}$$
$$\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(1/2, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_B((1,0), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, where the exponent* w_{121} *in the first line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_{121})$ *on the induced representation* $\left(I_{P_2}\left(0, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)$*, and the exponent* w_2 *in the last line denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on the* *induced representation $I_B((1,0),\mathbf{1}\otimes\mathbf{1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$, and M is the functor introduced in Section 4.2.*

(4k) *For*

$$\underline{s}_0=(s_{0,1},s_{0,2})=T_1,$$

that is, $\underline{s}_{0,1}=2$ and $\underline{s}_{0,2}=1$, it can be arranged that the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)}=\mathcal{A}^0_{\{B\},\varphi(\pi)}\supsetneqq\mathcal{A}^1_{\{B\},\varphi(\pi)}\supsetneqq\mathcal{A}^2_{\{B\},\varphi(\pi)}\supsetneqq\{0\},$$

and the quotients of the filtration are isomorphic to

$$\begin{aligned}\mathcal{A}^2_{\{B\},\varphi(\pi)}&\cong\mathbf{1}_{G(\mathbb{A})}\\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)}&\cong\Big(I_{P_1}\left(3/2,\mathbf{1}\circ\det\right)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\Big)\\ &\qquad\bigoplus\Big(I_{P_2}\left(2,\mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right)\otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\Big)\\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)}&\cong I_B((2,1),\mathbf{1}\otimes\mathbf{1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\end{aligned}$$

as $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-modules, where $\mathbf{1}_{G(\mathbb{A})}$ denotes the trivial representation of $G(\mathbb{A})$.

THEOREM 7.8 (**Case of the point $\underline{s}_0=(0,0)$**). *Let $(\{B\},\varphi(\pi))$ be a fixed cuspidal support with $\pi=\pi^u\cong\chi_1\otimes\chi_2$ a unitary character of $T(\mathbb{A})$, where χ_1 and χ_2 are unitary characters of $\mathbb{I}$, that is, $\underline{s}_0=(0,0)$. In this case, the Franke filtration of the $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ is always one-step filtration.*

(a) *If $\chi_1\neq\chi_2$ and $\chi_1\neq\chi_2^{-1}$ and $\chi_1^2\neq\mathbf{1}$ and $\chi_2^2\neq\mathbf{1}$, then the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)}\cong I_B((0,0),\chi_1\otimes\chi_2)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module.

(b) *If $\chi_1=\chi_2$ and $\chi_1\neq\chi_2^{-1}$, which implies that $\chi_1^2=\chi_2^2\neq\mathbf{1}$, then the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)}\cong\big(I_B((0,0),\chi\otimes\chi)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\big)^{w_1}$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module, where we write $\chi=\chi_1=\chi_2$, and the exponent w_1 denotes the space of invariant vectors for the action of the intertwining operator $M(w_1)$ on $I_B((0,0),\chi\otimes\chi)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$, and M is the functor introduced in Section 4.2.

(c) *If $\chi_1\neq\chi_2$ and $\chi_1=\chi_2^{-1}$, which implies that $\chi_1^2\neq\mathbf{1}$ and $\chi_2^2\neq\mathbf{1}$, then the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)}\cong\big(I_B((0,0),\chi\otimes\chi^{-1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\big)^{w_{212}}$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module, where we write $\chi=\chi_1=\chi_2^{-1}$, and the exponent w_{212} denotes the space of invariant vectors for the action of the intertwining operator $M(w_{212})$ on $I_B((0,0),\chi\otimes\chi^{-1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$, and M is the functor introduced in Section 4.2.

(d) *If $\chi_1^2=\mathbf{1}$ and $\chi_2^2\neq\mathbf{1}$, which implies that $\chi_1\neq\chi_2$ and $\chi_1\neq\chi_2^{-1}$, then the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)}\cong\big(I_B((0,0),\chi_1\otimes\chi_2)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\big)^{w_{121}}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_{121} *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_{121})$ *on* $I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(e) *If* $\chi_1^2 \neq \mathbf{1}$ *and* $\chi_2^2 = \mathbf{1}$*, which implies that* $\chi_1 \neq \chi_2$ *and* $\chi_1 \neq \chi_2^{-1}$*, then the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_2 *denotes the space of invariant vectors for the action of the intertwining operator* $M(w_2)$ *on* $I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(f) *If* $\chi_1^2 = \chi_2^2 = \mathbf{1}$ *and* $\chi_1 \neq \chi_2$*, then the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2, w_{121}}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where the exponent* w_2, w_{121} *denotes the space of invariant vectors for the action of both intertwining operators* $M(w_2)$ *and* $M(w_{121})$ *on* $I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

(g) *If* $\chi_1 = \chi_2$ *and* $\chi_1^2 = \chi_2^2 = \mathbf{1}$*, then the full space* $\mathcal{A}_{\{B\},\varphi(\pi)}$ *is isomorphic to*

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong \left(I_B((0,0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_1, w_2}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module, where we write* $\chi = \chi_1 = \chi_2$*, and the exponent* w_1, w_2 *denotes the space of invariant vectors for the action of both intertwining operators* $M(w_1)$ *and* $M(w_2)$ *on* $I_B((0,0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$*, and* M *is the functor introduced in Section 4.2.*

CHAPTER 8

Filtration for the support in the Borel subgroup – proofs

In this chapter the theorems stated in Section 7.2 are proved. The plan of the proof is different than the plan for the statement of the results in Section 7.1. Therefore, we begin by explaining the strategy and the plan of the proof.

8.1. Strategy and plan of the proof

The strategy of the proof is to prove simultaneously in Section 8.2 all the Theorems 7.1–7.7, because some cases of different theorems admit the same proof. The only exception is the last theorem, namely, Theorem 7.8, which is proved separately in Section 8.3 at the end of this chapter.

The proof is divided into steps. Each step covers certain region defined in Section 6.2 referring to Figure 6.1. The steps are then divided into substeps depending on the cuspidal support. Several substeps of different steps may admit the same proof, because of the same properties of the Eisenstein series and/or the same conditions for stabilization of the cuspidal support.

This plan of proof is different than the statement of results. The reason is that in the statement of theorems, the cuspidal support is used to distinguish cases. This is natural from the point of view of applications in which the cuspidal support is given. On the other hand, in the proof it is more practical to distinguish the steps according to the region in which $\underline{s}_0$ belongs, because cases with different cuspidal support have the same proof.

For convenience of the reader, we provide in Tables 8.1 and 8.2 the list of all theorems and their cases, together with the reference to the step of the proof in which they are proved. The table simplifies the navigation between the statement of theorems and the steps in which they are proved.

The proofs are based on

- the description of the Franke filtration in Chapter 4,
- the analytic properties of the Eisenstein series constructed from π^u recalled in Chapter 6 and summarized in Table 6.1,
- the conditions for stabilization given in Table 6.2,
- and Theorem A.2, in which the colimits required for the proof are computed.

See also Figure 6.1, in which the singular and stabilizing hyperplanes intersecting the closure of the positive Weyl chamber are shown.

According to the description of the Franke filtration in Chapter 4, the key information for the proof is the structure of the groupoid $\mathcal{M}_{\{B\},\varphi(\pi)}$. This is studied using the analytic properties of Eisenstein series and the stabilization conditions in each step of the proof separately. The partial oreder $\succ$ on the set $\mathcal{S}_{\{P\},\varphi(\pi)}$, which

TABLE 8.1. The list of Theorems 7.1–7.5 and their cases, with reference to the steps of the proof in which they are proved.

Theorem	Case	Steps
Thm. 7.1	(0a)	1, 2.1, 3.1, 4.1, 5.1, 6.1, 7.1, 10.1, 11.1
	(0b)	8.1, 8.2, 9.1, 9.2
Thm. 7.2	(1-1a)	1, 3.1, 4.1, 5.1, 8.1
	(1-1b)	7.2, 10.2, 11.2
	(1-1c)	2.2, 6.2, 9.3
Thm. 7.3	(1-2a)	1, 2.1, 4.1, 5.1, 7.1, 11.1
	(1-2b)	8.2, 9.2
	(1-2c)	3.2, 6.3, 10.3
Thm. 7.4	(1-3a)	1, 2.1, 3.1, 5.1, 6.1, 7.1, 8.1, 10.1
	(1-3b)	4.2, 9.4, 11.3
Thm. 7.5	(1-4a)	1, 2.1, 3.1, 4.1, 6.1, 7.1, 11.1
	(1-4b)	8.1, 8.2
	(1-4c)	5.2, 10.4
	(1-4d)	9.5, 9.6

is required for the definition of the Franke filtration, is made explicit for the group G in Section 4.2. It is repeatedly used in the proof to order the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ into appropriate quotients of the filtrations.

Observe that, except for Theorem 7.8, the only two non-trivial groupoids for the computation of the colimit of Theorem A.2 which appear in the proof are either the groupoid with one object X_0 and a non-trivial automorphism w_0 of X_0, or the groupoid with two objects X_0 and X_1 and the isomorphisms $w_{0,1} = w_{1,0}^{-1}$ between them as the only non-trivial morphisms. The notation is as in Theorem A.2. In the former case, we look at Theorem A.2 with $m = 0$ and $W_0 = \{1, w_0\}$, so that the colimit is isomorphic to the space of $M(w_0)$-invariant vectors in $M(X_0)$. In the latter case, we look at Theorem A.2 with $m = 1$ and $W_0 = \{1\}$, so that the colimit is isomorphic to $M(X_0)$ itself. We use this observation repeatedly in the proof below.

8.2. Proof of Theorems 7.1–7.7

As already explained above, the proof is divided into steps according to the region in which $\underline{s}_0$ belongs. The steps are then divided into substeps depending on the conditions on the cuspidal support $\pi^u = \chi_1 \otimes \chi_2$.

Step 1: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{C}}^+$.

In this step, according to Table 6.1 and Table 6.2, the point $\underline{s}_0$ is away from all singular hyperplanes and all stabilizing hyperplanes. Therefore, the

TABLE 8.2. The list of Theorems 7.6–7.7 and their cases, with reference to the steps of the proof in which they are proved.

Theorem	Case	Steps
Thm. 7.6	(2a)	1, 3.1, 5.1
	(2b)	7.2, 10.2
	(2c)	8.2
	(2d)	2.2, 6.2
	(2e)	4.2
	(2f)	11.4
	(2g)	9.7
Thm. 7.7	(4a)	1
	(4b)	7.2
	(4c)	8.2
	(4d)	2.2
	(4e)	3.2
	(4f)	4.2
	(4g)	5.2
	(4h)	11.4
	(4i)	10.5
	(4j)	9.8
	(4k)	6.4

Eisenstein series with any cuspidal support π^u are holomorphic and there is no stabilization. Thus, the set of triples $\mathcal{M}_{\{B\},\varphi(\pi)}$ in the description of the filtration is a singleton. The only possible triple is

$$(B, \pi^u, \underline{s}_0),$$

as there are no poles of the Eisenstein series which could produce residual representations with that support. The only morphism of this triple is the identity, as there is no stabilization. Hence, according to the description of the Franke filtration in Section 4.3, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is isomorphic to

$$I_B(\underline{s}_0, \pi^u) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

Step 2: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_1$, i.e., $s_{0,1} - s_{0,2} = 1$ with $s_{0,2} > 0$, and $\underline{s}_0 \neq (2, 1)$.

This step is divided into substeps depending on the condition for the pole of Eisenstein series along $\mathfrak{S}_1$ given in Table 6.1.

2.1 $\chi_1 \neq \chi_2$
In this substep, there is no pole of the Eisenstein series, and no stabilization. Hence, the proof and the result are the same as in Step 1.

2.2 $\chi_1 = \chi_2$
We denote $\chi = \chi_1 = \chi_2$. The condition for the pole of the Eisenstein series along $\mathfrak{S}_1$ is satisfied, so that the residues span the residual representation

$$(\chi \circ \det) |\cdot|^{\frac{s_{0,1}+s_{0,2}}{2}}$$

of the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ as in Theorem 6.1. Thus, there are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, and these are

$$(B, \chi \otimes \chi, \underline{s}_0) \quad \text{and} \quad \left(P_1, \chi \circ \det, \frac{s_{0,1}+s_{0,2}}{2}\right).$$

Since there is no stabilization, there are no non-trivial morphisms in $\mathcal{M}_{\{B\},\varphi(\pi)}$. In the partial order required for the description of the quotients of the filtration, which is made explicit in Section 4.2, we have

$$\iota_{P_1}\left(\frac{s_{0,1}+s_{0,2}}{2}\right) = \left(\frac{s_{0,1}+s_{0,2}}{2}, \frac{s_{0,1}+s_{0,2}}{2}\right) \succ (s_{0,1}, s_{0,2}) = \iota_B(\underline{s}_0),$$

because $s_{0,2} < \frac{s_{0,1}+s_{0,2}}{2} < s_{0,1}$. Hence, we may choose the function $T_{\{B\},\varphi(\pi)}$ to take values 0 and 1, and the two triples contribute to different quotients of the filtration. The Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1}+s_{0,2}}{2}, \chi \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

Step 3: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_2$, i.e., $s_{0,2} = 1$ with $s_{0,1} > 1$ and $s_{0,1} \neq 2$.
This step is divided into substeps depending on the condition for the pole of Eisenstein series along $\mathfrak{S}_2$ given in Table 6.1.

3.1 $\chi_2 \neq \mathbf{1}$
In this substep, there is no pole of the Eisenstein series, and no stabilization. Hence, the proof and the result are the same as in Step 1.

3.2 $\chi_2 = \mathbf{1}$

This substep is very similar to Step 2.2. The condition for the pole of the Eisenstein series along $\mathfrak{S}_2$ is satisfied, and there is no stabilization. The residues of the Eisenstein series span a residual representation as in Theorem 6.2, which produces a triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with the parabolic subgroup P_2. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi_1 \otimes \mathbf{1}, (s_{0,1}, 1)) \quad \text{and} \quad \left(P_2, \chi_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}, s_{0,1}\right),$$

and there are no non-trivial morphisms. In the partial order defined in Section 4.2, we have

$$\iota_{P_2}(s_{0,1}) = (s_{0,1}, 0) \succ (s_{0,1}, 1) = \iota_B((s_{0,1}, 1)).$$

Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,1}, \chi_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((s_{0,1}, 1), \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

Step 4: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_3$, i.e., $s_{0,1} + s_{0,2} = 1$ with $s_{0,1} > s_{0,2} > 0$. This step is divided into substeps depending on the condition for the pole of Eisenstein series along $\mathfrak{S}_3$ given in Table 6.1.

4.1 $\chi_1\chi_2 \neq \mathbf{1}$

In this substep, there is no pole of the Eisenstein series, and no stabilization. Hence, the proof and the result are the same as in Step 1.

4.2 $\chi_1\chi_2 = \mathbf{1}$

As in Step 2.2 and Step 3.2, the Eisenstein series have a pole only along $\mathfrak{S}_3$, so that there are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, and there are no non-trivial morphisms. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi \otimes \chi^{-1}, (s_{0,1}, s_{0,2})) \quad \text{and} \quad \left(P_1, \chi \circ \det, \frac{s_{0,1} - s_{0,2}}{2}\right),$$

where we set $\chi = \chi_1 = \chi_2^{-1}$, and there are no non-trivial morphisms, because there is no stabilization. In the partial order defined in Section 4.2, we have

$$\iota_{P_1}\left(\frac{s_{0,1} - s_{0,2}}{2}\right) = \left(\frac{s_{0,1} - s_{0,2}}{2}, \frac{s_{0,1} - s_{0,2}}{2}\right) \succ (s_{0,1}, s_{0,2}) = \iota_B(\underline{s}_0),$$

because $\frac{s_{0,1}-s_{0,2}}{2} < s_{0,1} - s_{0,2} < s_{0,1}$ and $s_{0,1} - s_{0,2} < s_{0,1} + s_{0,2}$. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}\left(\frac{s_{0,1}-s_{0,2}}{2}, \chi\circ\det\right)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B(\underline{s}_0, \chi\otimes\chi^{-1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

Step 5: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}_4$, i.e., $s_{0,1} = 1$ with $1 > s_{0,2} > 0$.
This step is divided into substeps depending on the condition for the pole of Eisenstein series along $\mathfrak{S}_4$ given in Table 6.1.

5.1 $\chi_1 \neq \mathbf{1}$
In this substep, there is no pole of the Eisenstein series, and no stabilization. Hence, the proof and the result are the same as in Step 1.

5.2 $\chi_1 = \mathbf{1}$
The Eisenstein series have the pole along $\mathfrak{S}_4$ and there is no stabilization. Therefore, as in Step 2.2, Step 3.2 and Step 4.2, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1}\otimes\chi_2, (1, s_{0,2})) \quad \text{and} \quad \left(P_2, \chi_2\otimes\mathbf{1}_{SL_2(\mathbb{A})}, s_{0,2}\right),$$

and there are no non-trivial morphisms. In the partial order defined in Section 4.2, we have

$$\iota_{P_2}(s_{0,2}) = (s_{0,2}, 0) \succ (1, s_{0,2}) = \iota_B((1, s_{0,2})),$$

because $1 > s_{0,2} > 0$ in this step. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(s_{0,2}, \chi_2\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right)\otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1, s_{0,2}), \mathbf{1}\otimes\chi_2)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

Step 6: $\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_1$, i.e., $s_{0,1} = 2$ and $s_{0,2} = 1$.
Since T_1 is the intersection point of hyperplanes $\mathfrak{S}_1$ and $\mathfrak{S}_2$, this step is divided into substeps depending on the condition for the pole of Eisenstein series along $\mathfrak{S}_1$ and $\mathfrak{S}_2$, as given in Table 6.1.

6.1 $\chi_1 \neq \chi_2$ and $\chi_2 \neq \mathbf{1}$
In this substep, there is no pole of the Eisenstein series, and no stabilization. Hence, the proof and the result are the same as in Step 1.

6.2 $\chi_1 = \chi_2 \neq \mathbf{1}$
In this substep, only the condition for the pole of the Eisenstein series along $\mathfrak{S}_1$ is satisfied, and there is no stabilization. Hence, the proof and the result are the same as in Step 2.2 with $s_{0,1} = 2$ and $s_{0,2} = 1$.

6.3 $\chi_1 \neq \chi_2 = \mathbf{1}$

This substep is the same as Step 3.2 with $s_{0,1} = 2$, because only the condition for the pole along the hyperplane $\mathfrak{S}_2$ is satisfied.

6.4 $\chi_1 = \chi_2 = \mathbf{1}$

In this substep, the pole of Eisenstein series is of order two and the iterated residues span the trivial representation of $G(\mathbb{A})$. Since the residues of order one along $\mathfrak{S}_1$ and $\mathfrak{S}_2$ span residual representations of the Levi factors of the maximal proper parabolic subgroups, there are four triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1} \otimes \mathbf{1}, (2,1)), \quad (P_1, \mathbf{1} \circ \det, 3/2), \quad (P_2, \mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 2) \quad \text{and} \quad (G, \mathbf{1}_{G(\mathbb{A})}, 0),$$

and there are no non-trivial morphisms, because there is no stabilization. In the partial order required for the Franke filtration, we have

$$\iota_G(0) = (0,0) \succ \iota_{P_1}(3/2) = (3/2, 3/2) \succ \iota_B(2,1) = (2,1)$$
$$\iota_G(0) = (0,0) \succ \iota_{P_2}(2) = (2,0) \succ \iota_B(2,1) = (2,1),$$

and $\iota_{P_1}(3/2) = (3/2, 3/2)$ is not comparable to $\iota_{P_2}(2) = (2,0)$. Hence, we may choose the function $T = T_{\{B\},\varphi(\pi)}$, in such a way that

$$T(\iota_G(0)) = 2, \quad T(\iota_{P_1}(3/2)) = T(\iota_{P_2}(2)) = 1, \quad \text{and} \quad T(\iota_B(2,1)) = 0.$$

Thus, the triple with G as the parabolic subgroup contributes to the deepest filtration step. The two triples with P_1 and P_2 as parabolic subgroups are incomparable, so that we may arrange that they contribute to the same quotient of the filtration, which is deeper in the filtration than the triple with the parabolic subgroup B. In this way, we obtain that the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \mathbf{1}_{G(\mathbb{A})} \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \Big(I_{P_1}\left(3/2, \mathbf{1} \circ \det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\Big) \\ &\qquad \bigoplus \Big(I_{P_2}\left(2, \mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\Big) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong I_B((2,1), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}) \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. This is the case in which a different choice of the function T may result in a slightly different filtration, as explained in Section 9.1 below.

Step 7: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_1$, i.e., $s_{0,1} = s_{0,2} > 0$ and $s_{0,1} = s_{0,2} \neq 1/2$ and $s_{0,1} = s_{0,2} \neq 1$.

In this step, we set $t_0 = s_{0,1} = s_{0,2}$. According to Table 6.1, there are no singular hyperplanes passing through $\underline{s}_0$. The type of stabilization depends on the condition on π^u given in Table 6.2. We distinguish the following substeps.

7.1 $\chi_1 \neq \chi_2$
As there are no singular hyperplanes, all triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ have B as the parabolic subgroup. The stabilization along $\mathfrak{S}'_1$ is given by the Weyl group element w_1. In this substep, w_1 does not stabilize π^u. Hence, there are two different triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, with a non-trivial isomorphism between them given by w_1. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi_1 \otimes \chi_2, (t_0, t_0)) \quad \text{and} \quad (B, \chi_2 \otimes \chi_1, (t_0, t_0)),$$

and the only non-trivial morphisms are the isomorphisms between these two triples given by w_1. The colimit required in the definition of the Franke filtration is the case of $m = 1$, $W_0 = \{1\}$ and $w_{0,1} = w_1$ in Theorem A.2. Hence, according to the description of the Franke filtration in Section 4.3, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is isomorphic to

$$I_B((t_0, t_0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module.

7.2 $\chi_1 = \chi_2$
Set $\chi = \chi_1 = \chi_2$. This substep is similar to Step 7.1, except that w_1 stabilizes π^u, so that there is only one triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with a non-trivial automorphism w_1. More precisely, $\mathcal{M}_{\{B\},\varphi(\pi)}$ contains only the triple

$$(B, \chi \otimes \chi, (t_0, t_0)),$$

and the only non-trivial morphism is the automorphism w_1 of that triple. The colimit required in the definition of the Franke filtration is the case of $m = 0$ and $W_0 = \{1, w_1\}$ in Theorem A.2. Hence, according to the description of the Franke filtration in Section 4.3, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is isomorphic to

$$\Big(I_B((t_0, t_0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_1}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where the exponent w_1 stands for the space of invariant vectors for the action of the intertwining operator $M(w_1)$.

Step 8: $\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \widehat{\mathfrak{S}}'_2$, i.e., $s_{0,1} > s_{0,2} = 0$ and $s_{0,1} \neq 1$.
This step is very similar to Step 7. There are no singular hyperplanes, and we distinguish the substeps depending on the condition on stabilization of π^u along $\mathfrak{S}'_2$ given in Table 6.2.

8.1 $\chi_2^2 \neq \mathbf{1}$
This substep is parallel to Step 7.1, as the condition for stabilization of π^u along $\mathfrak{S}'_2$ is not satisfied. Thus, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi_1 \otimes \chi_2, (s_{0,1}, 0)) \quad \text{and} \quad (B, \chi_1 \otimes \chi_2^{-1}, (s_{0,1}, 0)),$$

and the only non-trivial morphisms are the isomorphisms between these two triples given by w_2. Hence, using again Theorem A.2 and the description of the Franke filtration in Section 4.3, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is isomorphic to

$$I_B((s_{0,1},0),\chi_1\otimes\chi_2)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module.

8.2 $\chi_2^2=\mathbf{1}$
As in Step 7.2, the only triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ is now

$$(B,\chi_1\otimes\chi_2,(s_{0,1},0)),$$

and its automorphism w_2 is the only non-trivial morphism. Hence, according to Theorem A.2 and the description of the Franke filtration, the full space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is isomorphic to

$$\Big(I_B((s_{0,1},0),\chi_1\otimes\chi_2)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_2}$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module, where the exponent w_2 stands for the space of invariant vectors for the action of the intertwining operator $M(w_2)$.

Step 9: $\underline{s}_0=(s_{0,1},s_{0,2})=T_2$, i.e., $s_{0,1}=1$ and $s_{0,2}=0$.
In this step, the singular hyperplanes passing through T_2 are $\mathfrak{S}_1$, $\mathfrak{S}_3$ and $\mathfrak{S}_4$, and the stabilizing hyperplane is $\mathfrak{S}_2'$. The substeps are distinguished with respect to conditions for poles and stabilization along these hyperplanes given in Table 6.1 and Table 6.2. The possible substeps follow a similar pattern as the statement of the theorems. More precisely, the substeps are ordered with respect to growing number of singular hyperplanes.

9.1 $\chi_1\neq\chi_2$ and $\chi_1\neq\chi_2^{-1}$ and $\chi_1\neq\mathbf{1}$ and $\chi_2^2\neq\mathbf{1}$
In this substep, the Eisenstein series is holomorphic and the stabilization is the same as in Step 8.1. Hence, the proof and the result are the same as in that step with $s_{0,1}=1$.

9.2 $\chi_1\neq\chi_2$ and $\chi_1\neq\mathbf{1}$ and $\chi_2^2=\mathbf{1}$
As in the previous substep, the Eisenstein series is holomorphic, but the stabilization is the same as in Step 8.2, so that the proof and the result are the same as in that step with $s_{0,1}=1$.

9.3 $\chi_1=\chi_2$ and $\chi_2^2\neq\mathbf{1}$
In this substep, the Eisenstein series has a pole only along the singular hyperplane $\mathfrak{S}_1$. It is of order one, and the residues span a residual representation of the Levi factor of P_1, as in Theorem 6.1. This yields a triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with the parabolic subgroup P_1. The stabilizer along $\mathfrak{S}_2'$ does not stabilize π^u, so that there are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with the parabolic subgroup B. Observe that the other type of stabilization along $\mathfrak{S}_2'$ is treated in Step 9.7 below, because it implies that the Eisenstein series have two singular hyperplanes passing through T_2. To summarize, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ in this substep are

$$(B,\chi\otimes\chi,(1,0)),\quad (B,\chi\otimes\chi^{-1},(1,0))\quad\text{and}\quad (P_1,\chi\circ\det,1/2)\,,$$

where we set $\chi = \chi_1 = \chi_2$, and the only non-trivial morphisms are the isomorphisms between the first two triples given by w_2. In the partial order required for the definition of the filtration, we have

$$\iota_{P_1}(1/2) = (1/2, 1/2) \succ \iota_B(1,0) = (1,0),$$

which implies that the triple with the parabolic subgroup P_1 contributes to a deeper filtration step. The colimit is calculated using Theorem A.2 in the case of $m = 1$, $W_0 = \{1\}$ and $w_{0,1} = w_2$. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}(1/2, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1,0), \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

9.4 $\chi_1 = \chi_2^{-1}$ and $\chi_2^2 \neq \mathbf{1}$
The cuspidal support in this substep is conjugate by w_2 to the cuspidal support in Step 9.3. Hence, the Franke filtration is already obtained in that step.

9.5 $\chi_1 = \mathbf{1}$ and $\chi_2^2 \neq \mathbf{1}$
The only singular hyperplane in this substep is $\mathfrak{S}_4$. The stabilizer along $\mathfrak{S}'_2$ does not stabilize π^u. Hence, there are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with the parabolic subgroup B. The residual representation spanned by the residues of the Eisenstein series along $\mathfrak{S}_4$ produce a triple with the parabolic subgroup P_2, which admits another conjugate triple. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1} \otimes \chi_2, (1,0)), \quad (B, \mathbf{1} \otimes \chi_2^{-1}, (1,0)),$$

$$(P_2, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 0) \quad \text{and} \quad (P_2, \chi_2^{-1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 0),$$

and the only non-trivial morphisms are the non-trivial isomorphisms between the first pair of triples given by w_2 and the non-trivial isomorphisms between the second pair of triples given by w_{121}. The partial order again implies that the triples with the parabolic subgroup P_2 contribute to a deeper filtration step. The colimits are determined by Theorem A.2. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(0, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1,0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

9.6 $\chi_1 = \mathbf{1}$ and $\chi_2 \neq \mathbf{1}$ and $\chi_2^2 = \mathbf{1}$

The only difference between this substep and the previous one is that the stabilization along $\mathfrak{S}_2'$ is such that π^u is stabilized. Thus, the two pairs of triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are replaced with two triples which have non-trivial automorphisms. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1} \otimes \chi_2, (1,0)) \quad \text{and} \quad (P_2, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 0),$$

and the only non-trivial morphisms are the automorphism of the first triple given by w_2 and the automorphism of the second triple given by w_{121}. The partial order is the same as in the previous substep. The colimits are calculated using Theorem A.2. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_{P_2}\left(0, \chi_2 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^{w_{121}}$$

$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \left(I_B((1,0), \mathbf{1} \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where the exponent w_{121} in the first line denotes the space of invariant vectors for the intertwining operator $M(w_{121})$, and the exponent w_2 in the second line denotes the space of invariant vectors for the intertwining operator $M(w_2)$.

9.7 $\chi_1 = \chi_2 \neq \mathbf{1}$ and $\chi_2^2 = \mathbf{1}$

This is the substep in which the Eisenstein series has a pole along two singular hyperplanes $\mathfrak{S}_1$ and $\mathfrak{S}_3$ passing through T_2. According to Theorem 6.3, the iterated residues of the Eisenstein series along these two hyperplanes span a residual representation of $G(\mathbb{A})$ denoted by $J(\chi)$, where $\chi = \chi_1 = \chi_2$. There are also residues along each of the singular hyperplanes, which produce triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$. Hence, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi \otimes \chi, (1,0)), \quad (P_1, \chi \circ \det, 1/2) \quad \text{and} \quad (G, J(\chi), 0),$$

and the only non-trivial morphism is the automorphism of the first triple given by w_2. In the partial order required for the definition of the filtration, we have

$$\iota_G(0) = (0,0) \succ \iota_{P_1}(1/2) = (1/2, 1/2) \succ \iota_B(1,0) = (1,0),$$

which indicates that the triple with G as the parabolic subgroup contributes to the deepest filtration step, and the triple with P_1 to the step deeper than the one with B. The colimit is determined by Theorem A.2. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong J(\chi) \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong I_{P_1}(1/2, \chi\circ\det)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \Big(I_B((1,0),\chi\otimes\chi)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_2} \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where the exponent w_2 in the last line denotes the space of invariant vectors for the action of the intertwining operator $M(w_2)$.

9.8 $\chi_1 = \chi_2 = \mathbf{1}$

In this case all three hyperplanes $\mathfrak{S}_1$, $\mathfrak{S}_3$ and $\mathfrak{S}_4$ passing through T_2 are singular for the Eisenstein series. According to Theorem 6.3, the iterated residue of the Eisenstein series at T_2 is not square-integrable. Hence, there is no triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with G as the parabolic subgroup. The residues along each of the singular hyperplanes yield triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with P_1 and P_2 as the parabolic subgroups. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1}\otimes\mathbf{1}, (1,0)), \quad (P_1, \mathbf{1}\circ\det, 1/2) \quad \text{and} \quad (P_2, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}, 0),$$

and the only non-trivial morphisms are the automorphism of the first triple given by w_2 and the automorphism of the last triple given by w_{121}. In the partial order required for the definition of the filtration, we have

$$\iota_{P_2}(0) = (0,0) \succ \iota_{P_1}(1/2) = (1/2,1/2) \succ \iota_B(1,0) = (1,0),$$

which implies that the triple with the parabolic subgroup P_2 contributes to the deepest filtration step, while the triple with the parabolic subgroup P_1 contributes to the step deeper than the one with B. The colimits are calculated using Theorem A.2. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

and the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \Big(I_{P_2}(0, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})})\otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\Big)^{w_{121}} \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong I_{P_1}(1/2, \chi\circ\det)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \Big(I_B((1,0),\mathbf{1}\otimes\mathbf{1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_2} \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where the exponent w_{121} in the first line denotes the space of invariant vectors for the action of the intertwining operator $M(w_{121})$, and the exponent w_2 in the last line denotes the space of invariant vectors for the action of the intertwining operator $M(w_2)$.

Step 10: $\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_3$, i.e., $s_{0,1} = 1$ and $s_{0,2} = 1$.

In this step, the singular hyperplanes $\mathfrak{S}_2$ and $\mathfrak{S}_4$ and the stabilizing hyperplane $\mathfrak{S}'_1$ pass through point T_3. Hence, the substeps are distinguished according to the conditions for these hyperplanes given in Table 6.1 and Table 6.2.

10.1 $\chi_1 \neq \mathbf{1}$ and $\chi_2 \neq \mathbf{1}$ and $\chi_1 \neq \chi_2$
In this substep, the Eisenstein series is holomorphic, because the conditions for the singular hyperplanes $\mathfrak{S}_2$ and $\mathfrak{S}_4$ are not satisfied. Thus, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ have the parabolic subgroup B. The stabilization along $\mathfrak{S}'_1$ does not stabilize π^u. Hence, we are in the same setting as in Step 7.1 with $t_0 = 1$. The proof and the result are the same as in that step.

10.2 $\chi_1 = \chi_2 \neq \mathbf{1}$
The proof and the result in this substep are the same as in Step 7.2 with $t_0 = 1$, because the Eisenstein series is holomorphic and the stabilization along $\mathfrak{S}'_1$ stabilizes π^u.

10.3 $\chi_1 \neq \mathbf{1}$ and $\chi_2 = \mathbf{1}$
In this substep, the Eisenstein series have a pole only along $\mathfrak{S}_2$, and the residues span a residual representation of the Levi factor of P_2. The stabilization along $\mathfrak{S}'_1$ does not stabilize π^u, so that there are two triples with the parabolic subgroup B, which are conjugate by w_1. Hence, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi_1 \otimes \mathbf{1}, (1,1)), \quad (B, \mathbf{1} \otimes \chi_1, (1,1)) \quad \text{and} \quad (P_2, \chi_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 1),$$

and the only non-trivial morphisms are the isomorphisms between the first two triples given by w_1. The partial order used for the definition of the filtration implies that the triple with the parabolic subgroup P_2 contributes to the deeper filtration step. The colimit is calculated using Theorem A.2. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(1, \chi_1 \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1,1), \chi_1 \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

10.4 $\chi_1 = \mathbf{1}$ and $\chi_2 \neq \mathbf{1}$
This substep is the same as Step 10.3, and the result is obtained by replacing the roles of χ_1 and χ_2. The point is that the representatives of the cuspidal support are conjugate, so that they belong to the same associate class.

10.5 $\chi_1 = \chi_2 = \mathbf{1}$
Although in this substep there are two singular hypeplanes $\mathfrak{S}_2$ and $\mathfrak{S}_4$ for the Eisenstein series, the iterated residues vanish, as recalled in Theorem 6.3, so that the pole is at most of order one. Taken along any of the two singular hyperplanes, the residues of Eisenstein series span the same residual representation of the Levi factor of P_2. The stabilizer along $\mathfrak{S}'_1$ stabilizes π^u. Thus, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \mathbf{1} \otimes \mathbf{1}, (1,1)) \quad \text{and} \quad (P_2, \mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}, 1),$$

and the only non-trivial morphism is the automorphism of the first triple given by w_1. The partial order is the same as in the previous subcases. The colimit is calculated using Theorem A.2 in the case of $m = 0$ and $W_0 = \{1, w_1\}$. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_2}\left(1, \mathbf{1} \otimes \mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})$$

$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \Big(I_B((1,1), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_1}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where the exponent w_1 stands for the space of invariant vectors of the intertwining operator $M(w_1)$.

Step 11: $\underline{s}_0 = (s_{0,1}, s_{0,2}) = T_4$, i.e., $s_{0,1} = 1/2$ and $s_{0,2} = 1/2$.
At point T_4 the singular hyperplane $\mathfrak{S}_3$ and the stabilizing hyperplane $\mathfrak{S}'_1$ intersect. Hence, the substeps depend on the conditions for these two hyperplanes in Table 6.1 and Table 6.2.

11.1 $\chi_1\chi_2 \neq \mathbf{1}$ and $\chi_1 \neq \chi_2$
In this substep the Eisenstein series is holomorphic, and the stabilization is such that π^u is not stabilized. Hence, the proof and the result is the same as in Step 7.1 with $t_0 = 1/2$.

11.2 $\chi_1\chi_2 \neq \mathbf{1}$ and $\chi_1 = \chi_2$
The Eisenstein series is again holomorphic, but the stabilization is such that π^u is stabilized. Hence, this substep is the same as Step 7.2 with $t_0 = 1/2$.

11.3 $\chi_1\chi_2 = \mathbf{1}$ and $\chi_1 \neq \chi_2$
In this step the Eisenstein series has a pole along $\mathfrak{S}_3$, which produces a triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with the parabolic subgroup P_1 as in Step 4.2 with $s_{0,1} = s_{0,2} = 1/2$. The stabilization along $\mathfrak{S}'_1$ does not stabilize π^u, so that there are two triples with the parabolic subgroup B which are conjugate by w_1. It turns out that the triple with the parabolic subgroup P_1 also has non-trivial stabilization by the Weyl group element w_{212}. Thus, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi \otimes \chi^{-1}, (1/2, 1/2)), \quad (B, \chi^{-1} \otimes \chi, (1/2, 1/2)),$$

$$(P_1, \chi \circ \det, 0) \quad \text{and} \quad \left(P_1, \chi^{-1} \circ \det, 0\right),$$

where we set $\chi = \chi_1 = \chi_2^{-1}$, and the only non-trivial morphisms are the isomorphisms given by w_1 between the first pair of triples and the isomorphisms given by w_{212} between the second pair of triples. The partial order required in definition of the filtration implies that the triples with the parabolic subgroup P_1 contribute to deeper filtration steps. The colimits in the definition of the Franke filtration are calculated according to Theorem A.2 in the case of $m = 1$, $W_0 = \{1\}$ and either $w_{0,1} = w_1$ or $w_{0,1} = w_{212}$. All in all, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_{P_1}(0, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1/2, 1/2), \chi \otimes \chi^{-1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

11.4 $\chi_1\chi_2 = \mathbf{1}$ and $\chi_1 = \chi_2$
This substep is proved along the same lines as the proof in Step 11.3. The only difference is that the stabilization along $\mathfrak{S}'_1$ stabilizes π^u, so that the two pairs of triples with non-trivial morphisms are replaced with only two triples which have non-trivial automorphisms. More precisely, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are

$$(B, \chi \otimes \chi, (1/2, 1/2)) \quad \text{and} \quad (P_1, \chi \circ \det, 0)\,,$$

where we set $\chi = \chi_1 = \chi_2 = \chi_2^{-1}$, and the only non-trivial morphisms are the automorphism of the first triple given by w_1 and the automorphism of the second triple given by w_{212}. The partial order implies that the triple with the parabolic subgroup P_1 contributes to a deeper filtration step. The colimits are calculated using Theorem A.2 in the case of $m = 0$ and either $W_0 = \{1, w_1\}$ or $W_0 = \{1, w_{212}\}$. Hence, the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\varphi(\pi)$ is the two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \Big(I_{P_1}(0, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\Big)^{w_{212}}$$
$$\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong \Big(I_B((1/2, 1/2), \chi \otimes \chi^{-1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\Big)^{w_1}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where the exponent w_{212} in the first line denotes the space of invariant vectors of the intertwining operator $M(w_{212})$, and the exponent w_1 in the second line denotes the space of invariant vectors of the intertwining operator $M(w_1)$.

8.3. Proof of Theorem 7.8

The Eisenstein series associated to π^u are holomorphic at $\underline{s}_0 = O(0, 0)$ for any cuspidal support $\pi^u = \chi_1 \otimes \chi_2$. However, all four stabilizing hyperplanes pass through O. Hence, depending on the properties of the cuspidal support π^u, we obtain different instances of Theorem A.2 which determine the results.

In part (a) of Theorem 7.8, according to Table 6.2, the condition for stabilization of π^u is not satisfied along any of the stabilizing hyperplanes. Hence, there are eight different triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with a non-trivial isomorphism between each pair of triples, but no non-trivial automorphisms. Thus, we apply Theorem A.2 with $m = 7$ and $W_0 = \{1\}$, and obtain the claim.

In parts (b), (c), (d) and (e) of Theorem 7.8, the conditions for stabilization of π^u are satisfied along only one of the four stabilizing hyperplanes. Thus, the number of triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ is reduced to four, but there is a non-trivial automorphism on each of them, and two isomorphisms between every pair of different triples. Hence, we apply Theorem A.2 with $m = 3$ and $W_0 = \{1, w_0\}$, where w_0 is the non-trivial element in the stabilizer of π^u as indicated in Table 6.2. The element w_0 depends on the case of Theorem 7.8 which is considered. The result is then obtained as the spaces of invariant vectors for $M(w_0)$, as claimed.

In part (f) of Theorem 7.8, the cuspidal support π^u is stabilized along stabilizing hyperplanes $\mathfrak{S}'_2$ and $\mathfrak{S}'_4$. The stabilizer is thus generated by w_2 and w_{121}, as given in Table 6.2. There are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with four isomorphisms between them, and three non-trivial automorphisms on each triple. These non-trivial automorphisms are given by w_2, w_{121} and w_{1212}. Hence, we again apply Theorem A.2, but this time with $m = 1$ and $W_0 = \{1, w_2, w_{121}, w_{1212}\}$. The result follows.

Finally, in part (g) of Theorem 7.8, the condition for stabilization of π^u, given in Table 6.2, is satisfied for all four stabilizing hyperplanes. Hence, there is only one triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$, which has all elements of W as automorphisms. Hence, we apply Theorem A.2 with $m = 0$ and $W_0 = W$, and since W is generated by w_1 and w_2 the result follows.

CHAPTER 9

Properties and features of the filtration

In this final chapter we point out clearly and explain the underlying reasons for the properties and features of the Franke filtration, which can be observed in the case of the symplectic group of rank two. Our account comprises the following points:

- the role of the choice of the function T in the explicit description of the filtration,
- how the colimits in the description of the filtration take care of vanishing and equal contributions arising from functional equations of Eisenstein series,
- the mechanism in the definition of the filtration which solves the difficulties arising from the poles of Eisenstein series in the case of square-integrable residues,
- the similar mechanism in the case of the poles of Eisenstein series with residues that are not square-integrable.

In what follows, we refer to the theorems and the proofs in Chapter 7 and Chapter 8, which exhibit the phenomena listed above.

9.1. The choice of the function T

In the definition of the Franke filtration of the space $\mathcal{A}_{\{P\},\varphi(\pi)}$ of automorphic forms with the cuspidal support in the associate class $(\{P\},\varphi(\pi))$, as given in Chapter 4, the contribution to different quotients of the filtration is determined by the function $T = T_{\{P\},\varphi(\pi)}$. However, the choice of the function T satisfying the partial order property of Section 4.2 is not at all unique.

In the cases in which the partial order $\prec$ in the definition of the filtration is a total order on $\mathcal{S}_{\{P\},\varphi(\pi)}$, the choice of T follows the total order, but the assigned integer values are not necessarily consecutive. However, the non-trivial quotients of the filtration are the same and appear in the same order. The only difference in the filtration arising from the choice of T in this case is the number of trivial quotients between the consecutive non-trivial ones.

The more interesting situation is the case in which there exist elements in $\mathcal{S}_{\{P\},\varphi(\pi)}$ which are incomparable in the partial order $\prec$ of Section 4.2. Depending on the choice of T, such elements may be assigned equal integers, or different integers in any order. If they are assigned equal integers, then the corresponding quotient of the filtration is a direct sum of contributions obtained from different Eisenstein series. On the other hand, if they are assigned different integers, the direct sum splits into successive quotients of the filtration.

The example of this phenomenon is provided by part (4k) of Theorem 7.7. It is the case of the cuspidal support in the associate class of the Borel subgroup and the character

$$\pi \cong |\cdot|^2 \otimes |\cdot|$$

of $T(\mathbb{A}) \cong \mathbb{I} \times \mathbb{I}$. Observe that the Franke filtration in the statement of the result in this case is of length three. The proof is given in Step 6.4 of Section 8.2. In the proof there are four triples

$$(B, \mathbf{1}\otimes\mathbf{1}, (2,1)), \quad (P_1, \mathbf{1}\circ\det, 3/2), \quad (P_2, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}, 2) \quad \text{and} \quad (G, \mathbf{1}_{G(\mathbb{A})}, 0)$$

in $\mathcal{M}_{\{B\},\varphi(\pi)}$, and there is a pair of incomparable triples because

$$\iota_{P_1}(3/2) = (3/2, 3/2) \quad \text{and} \quad \iota_{P_2}(2) = (2,0)$$

are incomparable. In the Step 6.4 of the proof, the choice of T is made in such a way that

$$T(\iota_{P_1}(3/2)) = T(\iota_{P_2}(2)) = 1.$$

This choice of T results in the Franke filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\}$$

of length three, where

$$\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} \cong \Big(I_{P_1}\left(3/2, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}})\Big) \bigoplus \Big(I_{P_2}\left(2, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\Big)$$

is a direct sum of two induced representations arising from the incomparable triples.

A different choice of T is

$$T(\iota_B(2,1)) = 0, \quad T(\iota_{P_2}(2)) = 1, \quad T(\iota_{P_1}(3/2)) = 2, \quad T(\iota_G(0)) = 3,$$

which would result in the Franke filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A'}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A'}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A'}^2_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A'}^3_{\{B\},\varphi(\pi)} \supsetneqq \{0\}$$

of length four, where the quotients of the filtration are isomorphic to

$$\begin{aligned}
\mathcal{A'}^3_{\{B\},\varphi(\pi)} &\cong \mathbf{1}_{G(\mathbb{A})} \\
\mathcal{A'}^2_{\{B\},\varphi(\pi)}/\mathcal{A'}^3_{\{B\},\varphi(\pi)} &\cong I_{P_1}\left(3/2, \mathbf{1}\circ\det\right) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\
\mathcal{A'}^1_{\{B\},\varphi(\pi)}/\mathcal{A'}^2_{\{B\},\varphi(\pi)} &\cong I_{P_2}\left(2, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) \otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}}) \\
\mathcal{A'}^0_{\{B\},\varphi(\pi)}/\mathcal{A'}^1_{\{B\},\varphi(\pi)} &\cong I_B((2,1), \mathbf{1}\otimes\mathbf{1}) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}).
\end{aligned}$$

Observe that the two middle quotients in this filtration are the two direct summands in the filtration in the statement of part (4k) of Theorem 7.7.

Another significantly different choice of T is the one with

$$T(\iota_B(2,1)) = 0, \quad T(\iota_{P_1}(3/2)) = 1, \quad T(\iota_{P_2}(2)) = 2, \quad T(\iota_G(0)) = 3,$$

which would also give rise to the filtration of length four, with the two middle quotients interchanged compared to the filtration of length four above.

In conclusion, the freedom of choice of T may possibly give rise to different quotients of the filtration. However, it is only a minor modification. A quotient which is a direct sum of summands arising from several triples can be split into several quotients of the filtration, as in the example above.

9.2. Functional equations of Eisenstein series and colimits

The colimits in the definition of the filtration take care of the equal contributions to the filtration which appear due to the functional equations of the Eisenstein series. Recall that the colimits are taken with respect to the functor M from the monoid $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ to the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. The objects in the monoid $\mathcal{M}^k_{\{P\},\varphi(\pi)}$ are triples $(R, \Pi, \underline{z})$, as in Section 4.2, with R of rank k. The morphisms from the triple $(R, \Pi, \underline{z})$ to the (possibly equal) triple $(R', \Pi', \underline{z}')$ are given by the Weyl group elements $w \in W(R)$ under which the Levi factors of R and R', the representations Π and Π', and $\underline{z}$ and $\underline{z}'$ are all conjugate, as in Section 4.2.

9.2.1. The functional equation of two different holomorphic Eisenstein series. As the clearest example of this phenomenon, consider the case of the cuspidal support in the associate class of the Borel subgroup with

$$\underline{s}_0 = (s_{0,1}, s_{0,2}) \in \mathfrak{S}'_2 \setminus \{T_2\}, \quad \text{i.e.,} \quad 0 = s_{0,2} < s_{0,1} \neq 1.$$

Hence, the cuspidal support is represented by the character

$$\pi \cong \chi_1 |\cdot|^{s_{0,1}} \otimes \chi_2$$

of $T(\mathbb{A})$, where $s_{0,1}$ is as above, and χ_1 and χ_2 are unitary Hecke characters of $\mathbb{I}$. This example, depending on the properties of χ_1 and χ_2, is treated in several theorems in Section 7.2, all of which are proved in Step 8 of the proof in Section 8.2. More precisely, it is considered in part (0b) of Theorem 7.1, part (1–1a) of Theorem 7.2, part (1–2b) of Theorem 7.3, part (1–3a) of Theorem 7.4, part (1–4b) of Theorem 7.5, part (2c) of Theorem 7.6 and part (4c) of Theorem 7.7. The filtration is always one-step filtration, but its description in terms of parabolic induction depends on the Hecke character χ_2.

Consider first the case of χ_2 non-trivial and non-quadratic Hecke character, i.e., $\chi_2^2 \neq \mathbf{1}$. In this case, the proof is given in Step 8.1 of Section 8.2. There are two different triples

$$(B, \chi_1 \otimes \chi_2, (s_{0,1}, 0)) \quad \text{and} \quad (B, \chi_1 \otimes \chi_2^{-1}, (s_{0,1}, 0))$$

in $\mathcal{M}_{\{P\},\varphi(\pi)} = \mathcal{M}^0_{\{P\},\varphi(\pi)}$, which are conjugate by the Weyl group element w_2, so that w_2 is the only morphism between these triples. As explained in the proof, the colimit in the definition of the Franke filtration implies that only one of the two triples contributes.

In this case the Eisenstein series constructed from these two triples are as follows. Let $E_{\pi^u}(f, \underline{s})$ be the Eisenstein series associated to the unitary character

$$\pi^u \cong \chi_1 \otimes \chi_2$$

of $T(\mathbb{A})$, as in Section 3.2, where f ranges over the space $\mathcal{W}_{\pi^u}$, and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}} \cong \mathbb{C}^2$ is the complex parameter. Similarly, let $E_{w_2(\pi^u)}(f', \underline{s})$ be the Eisenstein series associated to

$$w_2(\pi^u) = \chi_1 \otimes \chi_2^{-1},$$

where f' ranges over $\mathcal{W}_{w_2(\pi^u)}$. These two Eisenstein series arising from conjugate characters of $T(\mathbb{A})$ are related by the functional equation

$$E_{\pi^u}(f, s) = E_{w_2(\pi^u)}(A(\underline{s}, \pi^u, w_2)f, w_2(\underline{s})),$$

as in [**Lan76**], [**MW95**], where $A(\underline{s}, \pi^u, w_2)$ is the standard intertwining operator [**Sha10**] which intertwines, away from its poles, the induced representations

$$I_B(\underline{s}, \pi^u) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} (\chi_1 |\cdot|^{s_1} \otimes \chi_2 |\cdot|^{s_2})$$

and

$$I_B(w_2(\underline{s}), w_2(\pi^u)) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \left(\chi_1 |\cdot|^{s_1} \otimes \chi_2^{-1} |\cdot|^{-s_2}\right).$$

The standard intertwining operator $A(\underline{s}, \pi^u, w_2)$ is holomorphic at the value $\underline{s} = (s_{0,1}, 0)$ of its complex parameter. Moreover, since $A((s_{0,1}, 0), \pi^u, w_2)$ can be viewed as the standard intertwining operator associated to the character χ_2 of the torus in the group SL_2 at the value 0 of its complex parameter, it is an isomorphism. Therefore, taking the main values of (the derivatives of) the two equal Eisenstein series generates the same automorphic forms, and one of the two Eisenstein series should be discarded from the description of the Franke filtration. This is exactly what is achieved by taking the colimit, which discards one of the two triples in Step 8.1 of Section 8.2. This example shows how the colimit takes care of the functional equation in the case of two different triples with a morphism between them.

9.2.2. The functional equation of a single holomorphic Eisenstein series. Consider now the case of χ_2 either trivial or quadratic Hecke character, i.e., $\chi_2^2 = \mathbf{1}$. In Step 8.2 of Section 8.2, which deals with this case, there is only one triple

$$(B, \chi_1 \otimes \chi_2, (s_{0,1}, 0))$$

in $\mathcal{M}_{\{P\},\varphi(\pi)} = \mathcal{M}^0_{\{P\},\varphi(\pi)}$, but now w_2 is a non-trivial automorphism of the triple. Taking the colimit in this case implies that the contribution of the triple consists of invariants for certain intertwining operator associated to w_2 acting on the induced representation

$$I_B((s_{0,1}, 0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}).$$

This behavior of the colimit is responsible for dealing with the functional equation of the Eisenstein series as explained below.

Let $E_{\pi^u}(f, \underline{s})$ be the Eisenstein series associated to the unitary Hecke character

$$\pi^u = \chi_1 \otimes \chi_2$$

of $T(\mathbb{A})$, where $f \in \mathcal{W}_{\pi^u}$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}} \cong \mathbb{C}^2$, as above. In this case, since $w_2(\pi^u) = \pi^u$, the functional equation is

$$E_{\pi^u}(f, \underline{s}) = E_{\pi^u}(A(\underline{s}, \pi^u, w_2)f, w_2(\underline{s})).$$

The standard intertwining operator $A(\underline{s}, \pi^u, w_2)$ at the value $\underline{s} = (s_{1,0}, 0)$ of its complex parameter is an involutive automorphism of the induced representation

$$I_B((s_{1,0}, 0), \pi^u) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} (\chi_1 |\cdot|^{s_{0,1}} \otimes \chi_2).$$

Hence, the induced representation is a direct sum of the ±1-eigenspaces for that operator. The functional equation of the Eisenstein series at the value $\underline{s} = (s_{0,1}, 0)$ reads

$$E_{\pi^u}(f, (s_{0,1}, 0)) = E_{\pi^u}(A((s_{0,1}, 0), \pi^u, w_2)f, (s_{0,1}, 0)).$$

It implies that for any $f = f^-$ in the -1-eigenspace for the intertwining operator, the Eisenstein series

$$E_{\pi^u}(f^-, (s_{0,1}, 0)) = 0.$$

This argument reveals that only the Eisenstein series constructed from functions $f = f^+$ that are invariant under the intertwining operator contribute non-trivially to the space of automorphic forms. Similar parity conditions arise from the derivatives of the functional equation evaluated at $\underline{s} = (s_{0,1}, 0)$, which imply that only invariants of certain intertwining operator associated to w_2 acting on the induced representation

$$I_B((s_{0,1}, 0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}).$$

contribute non-trivially to the space of automorpic forms. This is exactly what is obtained by taking the colimits, and exhibits how the colimits deal with the functional equation in the case of the triple with a non-trivial automorphism.

All other cases in which the functional equation must be considered are essentially reduced to the two cases explained above. The case of the cuspidal support represented by the character

$$\pi \cong \chi_1|\cdot|^{t_0} \otimes \chi_2|\cdot|^{t_0}, \quad \text{with } t_0 > 0, t_0 \neq 1, t_0 \neq 1/2$$

of $T(\mathbb{A})$, where χ_1 and χ_2 are unitary Hecke characters and $\underline{s}_0 = (t_0, t_0) \in \mathfrak{S}'_1 \setminus \{T_3, T_4\}$, is completely analogous. The condition on χ_2^2 is replaced with the condition on equality of χ_1 and χ_2, and the Weyl group element w_2 is replaced by w_1. The colimits deal with the functional equation of the Eisenstein series in this case in Step 7 of Section 8.2, and the corresponding parts of the theorems can be found in Tables 8.1 and 8.2.

9.2.3. The functional equations in the case of non-holomorphic Eisenstein series. There are also examples in which the filtration has more than one step. In such examples, there could be several Eisenstein series with functional equations, but contributing to different quotients of the filtration. Some of these Eisenstein series may be degenerate, that is, associated to a residual representation of a Levi factor, as mentioned at the end of Section 3.2. However, the colimit takes care of all these functional equations in the same way as in the two basic examples explained above. A particularly nice example of this phenomenon is given in part (2f) of Theorem 7.6 and part (4h) of Theorem 7.7. The cuspidal support in this case is represented by the character

$$\pi \cong \chi|\cdot|^{1/2} \otimes \chi|\cdot|^{1/2}$$

of $T(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$ such that $\chi^2 = \mathbf{1}$. It is treated in Step 11.4 of the proof in Section 8.2. There are only two triples

$$(B, \chi \otimes \chi, (1/2, 1/2)) \quad \text{and} \quad (P_1, \chi \circ \det, 0)$$

in $\mathcal{M}_{\{B\},\varphi(\pi)}$, and both of them admit a non-trivial automorphism, given by the Weyl group elements w_1 and w_{212}, respectively. As they contribute to different quotients of the filtration, the Eisenstein series associated to each of the triples is treated separately. The Eisenstein series associated to the first triple is just the Eisenstein series associated to $\pi^u = \chi \otimes \chi$. Its functional equation is taken into account in the same way as in the examples explained above. However, the other Eisenstein series is associated to the degenerate Eisenstein series $E_\Pi(f, s)$ associated to the residual representation $\Pi \cong \chi \circ \det$ of the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ of P_1, where f ranges over $\mathcal{W}_\Pi$ and $s \in \check{\mathfrak{a}}_{P_1,\mathbb{C}} \cong \mathbb{C}$ is the complex parameter. The degenerate Eisenstein series also satisfy the functional equation. In our case, it is the functional equation

$$E_\Pi(f,s) = E_\Pi(A(s,\Pi,w_{212})f,s),$$

where $A(s,\Pi,w_{212})$ is the standard intertwining operator that intertwines the induced representations

$$I_{P_1}(s,\Pi) = \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}\left((\chi\circ\det)|\det|^s\right).$$

and

$$I_{P_1}(w_{212}(s),\Pi) = \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}\left((\chi\circ\det)|\det|^{-s}\right).$$

As in the non-degenerate case, the operator at the value $s=0$ of its complex parameter becomes an involutive automorphism of the induced representation

$$I_{P_1}(0,\Pi) = \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}\left(\chi\circ\det\right).$$

Hence, by the same argument as in the non-degenerate case, the functional equation implies that one should take invariants for certain intertwining operator associated to w_{212} acting on

$$I_{P_1}(0,\chi\circ\det)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}),$$

which is exactly the result of taking the colimit in this case.

9.2.4. The case of several functional equations. The only case in which there are several functional equations relating more than two Eisenstein series happens in Theorem 7.8. In that theorem $\underline{s}_0=(0,0)$, and the cuspidal support is represented by the character

$$\pi \cong \chi_1\otimes\chi_2,$$

where χ_1 and χ_2 are unitary Hecke characters. Depending on the properties of these characters, there could be up to eight different Eisenstein series related by functional equations, but also, in the case of $\chi_1=\chi_2=\mathbf{1}$, there is just one Eisenstein series with seven functional equations. All these possibilities are essentially just a combination of the two basic examples explained above, and the colimits take care of all the functional equations. All these Eisenstein series are holomorphic at the value $\underline{s}_0=(0,0)$ of the complex parameter, so that the filtration is of length one in all these cases, but its explicit description in terms of parabolically induced representations depends on the functional equations.

More precisely, in the case of the cuspidal support represented by the character

$$\pi = \pi^u \cong \chi_1\otimes\chi_2$$

of $T(\mathbb{A})$, where $\chi_1\neq\chi_2$ and $\chi_1\neq\chi_2^{-1}$ and $\chi_1^2\neq\mathbf{1}$ and $\chi_2^2\neq\mathbf{1}$, which is handled in part (a) of Theorem 7.8, there are eight different triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, given by the action of the Weyl group, i.e.,

$$\mathcal{M}_{\{B\},\varphi(\pi)} = \{(B, w(\chi_1\otimes\chi_2),(0,0)) \,:\, w\in W\}.$$

There is a unique isomorphism between each pair of different triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, but there are no non-trivial automorphisms. The isomorphisms are given by the conjugate action of the appropriate Weyl group element as in Table 3.1.

On the other hand, for each triple, there is an Eisenstein series $E_w(f,\underline{s})$ associated to the unitary character $w(\chi_1\otimes\chi_2)$ obtained from $\pi^u\cong\chi_1\otimes\chi_2$ by conjugation by the element w of the Weyl group. Here f ranges over the space $\mathcal{W}_{w(\pi^u)}$ and $\underline{s}=(s_1,s_2)\in\check{\mathfrak{a}}_{B,\mathbb{C}}\cong\mathbb{C}^2$ is the complex parameter. Each pair of these Eisenstein series admits a functional equation, so that we have essentially seven independent functional equations given by

$$E_w(f, \underline{s}) = E_1(A(\underline{s}, \chi_1 \otimes \chi_2, w)f, w(\underline{s})),$$

where $A(\underline{s}, \chi_1 \otimes \chi_2, w)$ is the standard intertwining operator between $I_B(\underline{s}, \pi^u)$ and $I_B(w(\underline{s}), w(\pi^u))$, and the Eisenstein series on the right-hand side corresponds to the identity element of the Weyl group. Since all the Eisenstein series and the standard intertwining operators are holomorphic at the value $\underline{s} = \underline{s}_0 = (0,0)$ of their complex parameter, these functional equations and their derivatives evaluated at $\underline{s} = (0,0)$ imply that the contributions to the space of automorphic forms of all eight Eisenstein series are the same. Therefore, the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in $\pi = \chi_1 \otimes \chi_2$ in this case is obtained by the derivatives at $\underline{s} = (0,0)$ of only one of the Eisenstein series considered above, and thus isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)} \cong I_B((0,0), \chi_1 \otimes \chi_2) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}).$$

But this is exactly the result of taking the colimit over the eight triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with an isomorphism between each pair of different triples. This shows how the colimit takes care of the functional equations in this particular case.

On the other extreme, in the case of the cuspidal support in the trivial character $\pi = \pi^u = \mathbf{1} \otimes \mathbf{1}$ of $T(\mathbb{A})$, there is only one triple

$$(B, \mathbf{1} \otimes \mathbf{1}, (0,0))$$

in $\mathcal{M}_{\{B\},\varphi(\pi)}$, but it admits seven non-trivial automorphisms, one for each non-trivial element of the Weyl group. This case is just one instance of part (g) of Theorem 7.8. Consider the Eisenstein series $E(f, \underline{s})$, associated to $\pi^u = \mathbf{1} \otimes \mathbf{1}$, where f ranges over the space $\mathcal{W}_{\pi^u}$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{\{B\},\mathbb{C}} \cong \mathbb{C}^2$ is the complex parameter. This Eisenstein series admits seven functional equations

$$E(f, \underline{s}) = E(A(\underline{s}, \mathbf{1} \otimes \mathbf{1}, w)f, w(\underline{s})),$$

where w is a non-trivial element of the Weyl group, and $A(\underline{s}, \mathbf{1}\otimes\mathbf{1}, w)$ is the standard intertwining operator between $I_B(\underline{s}, \mathbf{1}\otimes\mathbf{1})$ and $I_B(w(\underline{s}), \mathbf{1}\otimes\mathbf{1})$. The Eisenstein series and the standard intertwining operators are holomorphic at the value $\underline{s} = (0,0)$ of their complex parameter. Since the standard intertwining operator $A(\underline{s}, \mathbf{1} \otimes \mathbf{1}, w)$ at $\underline{s} = (0,0)$ is an involution, it follows that the Eisenstein series is zero for f in the -1-eigenspace of that operator. Thus, in order to obtain non-zero automorphic forms, one should take in the construction of the Eisenstein series only those f that are invariant under the action of standard intertwining operators $A(\underline{s}, \mathbf{1} \otimes \mathbf{1}, w)$ at $\underline{s} = (0,0)$ for all non-trivial elements w in the Weyl group. Taking also the derivatives of Eisenstein series, it follows that the space $\mathcal{A}_{\{B\},\varphi(\mathbf{1}\otimes\mathbf{1})}$ of automorphic forms supported in $\pi \cong \mathbf{1} \otimes \mathbf{1}$ is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\mathbf{1}\otimes\mathbf{1})} \cong \left(I_B((0,0), \mathbf{1} \otimes \mathbf{1}) \otimes S(\check{\mathfrak{a}}_{\{B\},\mathbb{C}})\right)^W,$$

where W stands for the invariants for the action of intertwining operators associated to all elements of the Weyl group. Since W is generated by the simple reflections w_1 and w_2, it is sufficient to take invariants for these two intertwining operators, as stated in Theorem 7.8. But this is exactly the result of taking the colimit, so that the colimit discards the zero contribution of the Eisenstein series and its derivatives arising from the functional equations.

The other cases in Theorem 7.8 are just a combination of the two extremes explained above. In these cases, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ admit non-trivial automorphisms, and there are isomorphisms between each pair of different triples. On the other hand, the corresponding functional equations of different Eisenstein series imply that we should consider only one of them, because their contributions to the space of automorpic forms are the same, just as in the first case above. However, the functional equations of one of the considered Eisenstein series imply that one should also take invariants for certain intertwining operators to avoid zero contributions of the Eisenstein series as in the second case considered above. Both issues are neatly solved by taking the colimit.

9.2.5. The functional equations of Eisenstein series supported in a maximal parabolic subgroup. The functional equations of Eisenstein series, and the corresponding colimits, play a role also in the case of the cuspidal support in a maximal proper parabolic subgroup treated in Chapter 5. The non-trivial phenomena occur for the cuspidal support with $\underline{s}_0 = 0$, i.e., $\pi \cong \pi^u$ in Theorem 5.4 and Theorem 5.5.

In that case, if $w(\pi^u) \ncong \pi^u$, then the Eisenstein series associated to π^u and $w(\pi^u)$ are related by the functional equation, and thus, only one of them contributes to the space of automorphic forms. This is controlled by the colimit, as in this case there are two triples

$$(P_i, \pi^u, 0) \quad \text{and} \quad (P_i, w(\pi^u), 0)$$

in $\mathcal{M}_{\{P_i\},\varphi(\pi^u)}$, with an isomorphism between them. The situation is completely analogous to the first case considered in this section with $\chi_2^2 \neq \mathbf{1}$.

On the other hand, if $w(\pi^u) \cong \pi^u$, then the situation is analogous to the second case considered in this section, in which $\chi_2^2 = \mathbf{1}$. Namely, the Eisenstein series associated to π^u admits a functional equation, which implies that for certain choice of the function f in $\mathcal{W}_{\pi^u}$ the Eisenstein series vanishes. The choice of the appropriate functions is again governed by the colimit. In this case, the only triple in $\mathcal{M}_{\{P_i\},\varphi(\pi^u)}$ is

$$(P_i, \pi^u, 0),$$

which admits a non-trivial automorphism given by w. Thus, the colimit consists of the invariants for the corresponding intertwining operator on the induced representation. This can be observed in Theorem 5.4 and Theorem 5.5, in which such invariants appear.

9.3. Eisenstein series with square-integrable residues

If the Eisenstein series is not holomorphic at the relevant value of its complex parameter, then the main value of its derivatives is well-defined only up to the automorphic representation spanned by the coefficients in the principal part of the Laurent series. If these coefficients are square-integrable automorphic forms on the Levi factor of a parabolic subgroup of higher rank, then they span a residual representation of such Levi factor. This residual representation occurs among the triples defining the Franke filtration in Chapter 4, and the partial order assigns it to a deeper quotient of the filtration. In such a way the main values of the derivatives of the Eisenstein series are well-defined as elements of the quotient of the filtration.

9.3.1. The pole of order one. Examples of this phenomenon occur in several theorems of Section 7.2. The simplest cases are those in which the Eisenstein series in question has a pole of order one at the relevant value of its complex parameter. This happens in part (1–1c) of Theorem 7.2, part (1–2c) of Theorem 7.3, part (1–3b) of Theorem 7.4, parts (1–4c) and (1–4d) of Theorem 7.5, parts (2d), (2e) and (2f) of Theorem 7.6 and parts (4d), (4e), (4f) and (4g) of Theorem 7.7. In some of these theorems, the functional equations of the Eisenstein series also play a role, as explained in Section 9.2.

As a convenient example, consider the cuspidal support represented by the character

$$\pi \cong \chi|\cdot|^{t_0+1/2} \otimes \chi|\cdot|^{t_0-1/2},$$

of $T(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$, and

$$\underline{s}_0 = (t_0 + 1/2, t_0 - 1/2)$$

lies on the part $\widehat{\mathfrak{S}}_1$ of the singular hyperplane $\mathfrak{S}_1$, i.e., $1/2 < t_0 \neq 3/2$, in order to avoid possible poles of higher order. This case is treated in part (1–1c) of Theorem 7.2, part (2d) of Theorem 7.6 and part (4d) of Theorem 7.7, and proved in Step 2.2 of the proof in Section 8.2.

In this case, the Franke filtration is a two-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\begin{aligned} \mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong I_{P_1}(t_0, \chi \circ \det) \otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong I_B(\underline{s}_0, \chi \otimes \chi) \otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}}) \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. We explain below the underlying reason in terms of Eisenstein series responsible for the necessity of two different quotients of the filtration in this example.

Let $E(f, \underline{s})$ be the Eisenstein series associated to the unitary character $\pi^u \cong \chi \otimes \chi$ of $T(\mathbb{A})$, where f ranges over $\mathcal{W}_{\pi^u}$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}} \cong \mathbb{C}^2$ is the complex parameter. It has a pole of order one along the singular hyperplane $\mathfrak{S}_1$, so that the main values of its derivatives are not well-defined. The residues, on the other hand, are square-integrable as automorphic forms on the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ of P_1, and span the residual representation isomorphic to

$$(\chi \circ \det)|\det|^{t_0}$$

of $L_1(\mathbb{A})$. The degenerate Eisenstein series $E_\Pi(f', s')$ associated to the unitary residual representation $\Pi \cong \chi \circ \det$ of the Levi factor $L_1(\mathbb{A})$ of P_1 is holomorphic at $s' = t_0$, which is the relevant value of the complex parameter for the given cuspidal support. The coefficients in the Taylor expansion of $E_\Pi(f', s')$ span the automorphic representation which contains the automorphic forms obtained as residues of $E(f, \underline{s})$ along $\mathfrak{S}_1$. Therefore, the contribution of the degenerate Eisenstein series $E_\Pi(f', s')$ should be assigned to a deeper quotient of the filtration than the contribution of the Eisenstein series $E(f, \underline{s})$, so that the main values of the derivatives of the latter are well-defined as elements of the quotient. Since the Eisenstein series $E(f, \underline{s})$ and $E_\Pi(f', s')$ correspond to the triples

$$(B, \chi \otimes \chi, (t_0 + 1/2, t_0 - 1/2)) \quad \text{and} \quad (P_1, \chi \circ \det, t_0)$$

in $\mathcal{M}_{\{B\},\varphi(\pi)}$, respectively, the partial order implies

$$\iota_{P_1}(t_0) = (t_0, t_0) \succ \iota_B((t_0 + 1/2, t_0 - 1/2)) = (t_0 + 1/2, t_0 - 1/2).$$

Thus, the Eisenstein series $E_\Pi(f', s')$ contributes to a deeper quotient of the filtration than the Eisenstein series $E(f, \underline{s})$, exactly as required. This explains how the Franke filtration deals with the problem of defining the main values of the derivatives of the Eisenstein series in the case of Eisenstein series which has a pole of order one at the relevant value of the complex parameter and the residues are square-integrable.

9.3.2. The pole of order two. The Franke filtration for the symplectic group of rank two also exhibits examples in which the Eisenstein series required for the definition of the filtration has a pole of order two with square-integrable residues. This is the case in part (2g) of Theorem 7.6 and part (4k) of Theorem 7.7, which are proved, respectively, in Step 9.7 and Step 6.4 of the proof in Section 8.2. For simplicity of exposition we explain here part (2g) of Theorem 7.6. The other example is analogous, and is already considered in regard to the freedom of choice of the function T in Section 9.1.

Consider the cuspidal support represented by the character

$$\pi \cong \chi|\cdot| \otimes \chi$$

of $T(\mathbb{A})$, where χ is a unitary non-trivial quadratic Hecke character of $\mathbb{I}$, i.e., $\chi^2 = \mathbf{1}$, but $\chi \neq \mathbf{1}$, as in part (2g) of Theorem 7.6. Let $E(f, \underline{s})$ be the Eisenstein series associated to $\pi^u \cong \chi \otimes \chi$, where f ranges over $\mathcal{W}_{\pi^u}$ and $\underline{s} = (s_1, s_2) \in \check{\mathfrak{a}}_{B,\mathbb{C}}$ is the complex parameter. There are two singular hyperplanes for the Eisenstein series $E(f, \underline{s})$ passing through $\underline{s} = \underline{s}_0 = (1, 0)$, which is the relevant point for the considered cuspidal support. Although the pole along each of the singular hyperplanes is of order one, the residue of $E(f, \underline{s})$ along each of them has a pole of order one at $\underline{s} = \underline{s}_0 = (1, 0)$. Thus, there are two terms in the principal part of the Laurent series of $E(f, \underline{s})$ around $\underline{s}_0 = (1, 0)$ along a generic line. The main values (of the derivatives) of $E(f, \underline{s})$ are well-defined only up to the automorphic representation spanned by the coefficients in the principal part of the Laurent series. These coefficients must be assigned to a deeper quotient of the filtration, so that the main values of the Eisenstein series $E(f, \underline{s})$ are well-defined as elements in the quotient.

This is achieved in two steps. In the first step, consider the residues of the Eisenstein series $E(f, \underline{s})$ along the singular hyperplane $\mathfrak{S}_1$. The pole along $\mathfrak{S}_1$ is of order one, and the residues span the residual representation

$$(\chi \circ \det)|\det|^{\frac{s_1+s_2}{2}}, \quad \text{where } (s_1, s_2) \in \mathfrak{S}_1, \text{ i.e., } s_1 - s_2 = 1,$$

of the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ of P_1. This is along the same lines as in the example of the pole of order one with square-integrable residues, which is elaborated above. Hence, consider the degenerate Eisenstein series $E_\Pi(f', s')$ associated to $\Pi = \chi \circ \det$, where f ranges over $\mathcal{W}_\Pi$ and $s' \in \check{\mathfrak{a}}_{P_1,\mathbb{C}} \cong \mathbb{C}$ is the complex parameter. In our case, the relevant value of its complex parameter is $s' = 1/2$, because it corresponds to $\frac{s_1+s_2}{2}$ at $(s_1, s_2) = (1, 0)$. However, according to [**Kim95**], the degenerate Eisenstein series $E_\Pi(f', s')$ has a pole of order one at $s' = 1/2$, and thus, the main values of its derivatives are well-defined only up to the representation spanned by the residues. The residues are square integrable, and they span the residual representation $J(\chi)$ of $G(\mathbb{A})$, as in [**Kim95**], which is also recalled in Theorem 6.3. Let $\mathcal{A}^2_{\{B\},\varphi(\pi)}$ denote the subrepresentation of the space of automorphic forms isomorphic to residual representation $J(\chi)$, obtained as the span of residues of $E_\Pi(f',s')$ at $s'=1/2$. Then, the main values of the derivatives of $E_\Pi(f',s')$ at $s'=1/2$ are well defined as elements of the quotient of the space of automorphic forms by $\mathcal{A}^2_{\{B\},\varphi(\pi)}$.

The span of these main values, together with the forms in $\mathcal{A}^2_{\{B\},\varphi(\pi)}$, form the subspace of the space of automorphic forms supported in the associate class of π, which we denote by $\mathcal{A}^1_{\{B\},\varphi(\pi)}$. Then, there is a filtration

$$\mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where taking the main values of the derivatives of the Eisenstein series yields that the quotients are isomorphic to

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong J(\chi) \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong I_{P_1}(1/2,\chi\circ\det)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \end{aligned}$$

as $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-modules. By the construction, the space $\mathcal{A}^1_{\{B\},\varphi(\pi)}$ of automorphic forms contains the coefficients of the principal part of the Laurent expansion around $\underline{s}_0=(1,0)$ of the original Eisenstein series $E(f,\underline{s})$. Therefore, the main values of the derivatives of $E(f,\underline{s})$ at $\underline{s}=\underline{s}_0=(1,0)$ are well-defined as elements of the quotient of the space of automorphic forms by the subspace $\mathcal{A}^1_{\{B\},\varphi(\pi)}$. This implies that the quotient is isomorphic to

$$\mathcal{A}_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} \cong I_B((1,0),\chi\otimes\chi)\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})$$

as a $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-module.

In conclusion, we have explained the underlying reasons, in terms of the analytic properties of the Eisenstein series, for the existence of the three different quotients of the filtration in the considered example. On the other hand, for the considered cuspidal support, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ defining the filtration as in Chapter 4 are

$$(B,\chi\otimes\chi,(1,0)),\quad (P_1,\chi\circ\det,1/2),\quad \text{and}\quad (G,J(\chi),0).$$

The first triple corresponds to the Eisenstein series $E(f,\underline{s})$, the second triple correspond to the degenerate Eisenstein series $E_\Pi(f',s')$, and the last triple is already a residual representation of $G(\mathbb{A})$. In the partial order required for the definition of the filtration, we have

$$\iota_G(0)=(0,0)\succ\iota_{P_1}(1/2)=(1/2,1/2)\succ\iota_B(1,0)=(1,0),$$

so that the contributions of the triples are ordered precisely in a way governed by the analytic properties of the Eisenstein series required in the construction.

9.3.3. The pole of the Eisenstein series supported in a maximal parabolic subgroup. The same feature of the Franke filtration occurs in the case of the cuspidal support in a maximal proper parabolic subgroup considered in Chapter 5. In that case, if the Eisenstein series associated to π^u has a pole at the relevant value $s=s_0\geq 0$ of its complex parameter, then the residues are always square-integrable automorphic forms on $G(\mathbb{A})$. This is a general fact, which holds for the cuspidal support in a maximal proper parabolic subgroup of any reductive linear algebraic group over a number field. It was already observed by Franke in Remark 2 of

[**Fra98**, page 242-243], and follows directly from the Langlands square-integrability criterion [**Lan76**].

In the Franke filtration, the partial order always assigns the residual representations of the full group $G(\mathbb{A})$ into the deepest quotient of the filtration. This is due to the fact that any residual representation Π of $G(\mathbb{A})$ appears in $\mathcal{M}_{\{P\},\varphi(\pi)}$ as the triple

$$(G, \Pi, 0),$$

and we have

$$\iota_G(0) = (0,0) \succ \iota_R(\underline{z})$$

in the partial order defining the filtration for any other triple $(R, \Pi', \underline{z})$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$. Therefore, the main values of the derivatives of the Eisenstein series associated to π^u are well-defined as the elements of the quotient of the space of automorphic forms by the space of automorphic forms isomorphic to the residual representation Π.

This can be observed in Theorem 5.4 and Theorem 5.5, in which the space $\mathcal{L}_{\{P_i\},\varphi(\pi)}$ is the space of square-integrable automorphic forms spanned by the residues of Eisenstein series associated to π^u, and the quotient $\mathcal{A}_{\{P_i\},\varphi(\pi)}/\mathcal{L}_{\{P_i\},\varphi(\pi)}$ is obtained as main values of the derivatives of such Eisenstein series.

9.4. Eisenstein series with non-square-integrable residues

Unlike in the previous examples, the non-square-integrable residues of Eisenstein series do not span a residual representation of the Levi factor of a higher rank parabolic subgroup, because automorphic forms in residual representations must be square-integrable. Nevertheless, these non-square-integrable residues should not contribute to the same filtration step as the main values of the Eisenstein series, because main values are well-defined only up to the representation spanned by these non-square-integrable residues. Hence, there must exist another Eisenstein series of the same or lower rank, which contributes to a deeper quotient of the filtration and is holomorphic at the relevant value of the complex parameter, whose Taylor coefficients span a representation containing the non-square-integrable residues in question.

This fact was already observed by Franke in Remark 2 on page 242-243 of [**Fra98**], in which he writes[1]: “One gets problems with this approach [referring to the idea of ordering contributions to the quotients of the filtration according to the rank of parabolic subgroups in triples] in the rank two case if there are Eisenstein series from a maximal parabolic subgroup whose residue at a point in the positive Weyl chamber is not square integrable. This never happens for cuspidal Eisenstein series, and for residual Eisenstein series the only example of this kind which I know, and which I will explain in more detail below, is the example of G_2 described in the appendix in Langlands’ book.”

This paper provides another example of the same phenomenon, apparently not anticipated by Franke, in the case of the symplectic group of rank two. It occurs in part (4j) of Theorem 7.7, which is proved in Step 9.8 of the proof in Section 8.2. The cuspidal support is represented by the character

$$\pi \cong |\cdot| \otimes \mathbf{1}$$

[1]The text in the square-brackets are additional explanations by the author.

of $T(\mathbb{A})$. Recall that in this case the Franke filtration of the space $\mathcal{A}_{\{B\},\varphi(\pi)}$ is the three-step filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where

$$\begin{aligned} \mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \left(I_{P_2}\left(0, \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right)\otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}})\right)^{w_{121}} \\ \mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong I_{P_1}\left(1/2, \mathbf{1}\circ\det\right)\otimes S(\check{\mathfrak{a}}_{P_1,\mathbb{C}}) \\ \mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \left(I_B((1,0),\mathbf{1}\otimes\mathbf{1})\otimes S(\check{\mathfrak{a}}_{B,\mathbb{C}})\right)^{w_2} \end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. Observe that there are two quotients of the filtration arising from parabolic subgroups of the same rank P_1 and P_2, which cannot be rearranged as in Section 9.1 in a way that they form a direct sum in the same quotient of the filtration. Here is the explanation of such behavior.

There is a residual representation isomorphic to

$$(\mathbf{1}\circ\det)|\det|^{1/2}$$

of the Levi factor $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$ of the parabolic subgroup P_1, with cuspidal support in the associate class of π. Let $E_{\Pi_1}(f,s)$ be the degenerate Eisenstein series associated to the unitary residual representation $\Pi_1 \cong \chi\circ\det$ of $L_1(\mathbb{A}) \cong GL_2(\mathbb{A})$, where f ranges over the space $\mathcal{W}_{\Pi_1}$ and $s \in \check{\mathfrak{a}}_{P_1,\mathbb{C}} \cong \mathbb{C}$ is the complex parameter. According to the results of [**Kim95**], recalled in Section 6.2, this Eisenstein series has a pole of order one at the value $s = 1/2$ of its complex parameter. However, the residues of $E_{\Pi_1}(f,s)$ at $s = 1/2$ are not square-integrable automorphic forms.

The calculations under point (ii) on page 144 of [**Kim95**], implicitly contain the description of the automorphic representation spanned by these non-square-integrable residues. It turns out that the representation spanned by the residues of $E_{\Pi_1}(f,s)$ at $s = 1/2$ is isomorphic to the image of the normalized standard intertwining operator $N((1,0),\mathbf{1}\otimes\mathbf{1},w_2w_1)$. Such normalized operators are obtained from the standard intertwining operators appearing in the functional equations of Section 9.2 using certain ratio of automorphic L-functions, as in [**Sha10**], [**Kim04**]. The reason for normalization is that the standard intertwining operators may have poles at the relevant values of the complex parameter, and the pole is captured by the automorphic L-functions in the normalizing factor. Another important property of the normalized intertwining operator is that it may be decomposed according to the decomposition of the Weyl group element into a product of simple reflections [**Sha90**]. In our case, the intertwining operators attached to the Weyl group elements w_1 and w_2 are essentially intertwining operators for groups GL_2 and SL_2, respectively. Hence, the image of

$$N = N((1,0),\mathbf{1}\otimes\mathbf{1},w_2w_1) = N((0,1),\mathbf{1}\otimes\mathbf{1},w_2)N((1,0),\mathbf{1}\otimes\mathbf{1},w_1)$$

can be described through the diagram

$$\begin{array}{ccccccc}
\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\left(|\cdot|\otimes\mathbf{1}\right) & & & & & & \\
\downarrow^{N_1} & & \searrow^{N_1} & & & & \\
\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\left(\mathbf{1}\otimes|\cdot|\right) & \supset & \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}\left((\mathbf{1}\circ\det)|\det|^{1/2}\right) & & & & \\
\downarrow^{N_2} & & \searrow^{N_2} & & \downarrow^{N_2} & & \searrow^{N_2} \\
\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\left(\mathbf{1}\otimes|\cdot|^{-1}\right) & \supset & & \mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}\left(\mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right) & & \supset & \mathrm{Im}(N)
\end{array}$$

where N_1 and N_2 denote the normalized intertwining operators associated to w_1 and w_2 in the decomposition of N above. The first column in the diagram represents the composition of operators N_1 and N_2 at the level of induced representations from the Borel subgroup. The second column represents the images of the operators N_1 and N_2, and the image of the composition $N = N_2N_1$ in the last column of the diagram is a subrepresentation of the image of N_2. The images of N_1 and N_2 are well-known from the theory of intertwining operators for GL_2 and SL_2, respectively. In other words, the residues of the Eisenstein series $E_{\Pi_1}(f,s)$ span the automorphic representation isomorphic to

$$\mathrm{Im}\, N((1,0),\mathbf{1}\otimes\mathbf{1},w_2w_1) \subseteq \mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}\left(\mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right).$$

This automorphic representation must contribute to a deeper quotient of the filtration than the main values (of the derivatives) of the Eisenstein series $E_{\Pi_1}(f,s)$.

That is achieved by considering another degenerate Eisenstein series. There is a residual representation isomorphic to

$$\Pi_2 = \mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}$$

of the Levi factor $L_2(\mathbb{A}) \cong \mathbb{I}\otimes SL_2(\mathbb{A})$ of P_2, with the cuspidal support in the character

$$w_1(\pi) = \mathbf{1}\otimes|\cdot|$$

that is associate to π. Let $E_{\Pi_2}(f',s')$ be the degenerate Eisenstein series associated to the residual representation Π_2, where f' ranges over $\mathcal{W}_{\Pi_2}$ and $s' \in \check{\mathfrak{a}}_{P_2,\mathbb{C}} \cong \mathbb{C}$ is the complex parameter. It is holomorphic at the value $s' = 0$ of its complex parameter, so that the main values of its derivatives are well-defined and span the automorphic representation isomorphic to the space of invariant vectors under certain intertwining operator associated to the Weyl group element w_{212} in the induced representation

$$\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}\left(\mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})}\right)\otimes S(\check{\mathfrak{a}}_{P_2,\mathbb{C}}).$$

The invariants must be taken in order to handle the functional equation

$$E_{\Pi_2}(f',s') = E_{\Pi_2}(A(s',\Pi_2,w_{212})f',-s')$$

of the degenerate Eisenstein series, as in Section 9.2. Nevertheless, the automorphic representation spanned by the non-square-integrable residues of the degenerate Eisenstein series $E_{\Pi_1}(f,s)$ at $s = 1/2$ is a constituent, as an abstract representation, of the automorphic representation spanned by the main values of the derivatives of the degenerate Eisenstein series $E_{\Pi_2}(f',s')$.

The actual inclusion of the space of automorphic forms obtained as non-square-integrable residues, described as an abstract representations above, follows from the functional equations

$$\begin{aligned} E(f,(s_1,s_2)) &= E(A((s_1,s_2),\mathbf{1}\otimes\mathbf{1},w_1)f,(s_2,s_1)) \\ &= E(A((s_1,s_2),\mathbf{1}\otimes\mathbf{1},w_2w_1)f,(s_2,-s_1)), \end{aligned}$$

after taking the iterated residues, first along the singular hyperplane $\mathfrak{S}_1$, and then at the point on $\mathfrak{S}_1$ corresponding to the value $\underline{s}=(s_1,s_2)=(1,0)$ of the complex parameter. The Eisenstein series $E(f,\underline{s})$ is the Eisenstein series associated to the trivial character $\mathbf{1}\otimes\mathbf{1}$ of $T(\mathbb{A})$, and $A((s_1,s_2),\mathbf{1}\otimes\mathbf{1},w)$ denotes the standard intertwining operator associated to the Weyl group element w.

In view of the behavior of these Eisenstein series, it is now clear that the contribution in the Franke filtration of the Eisenstein series $E_{\Pi_2}(f',s')$ at $s'=0$ must be in a deeper quotient of the filtration than the contribution of the Eisenstein series $E_{\Pi_1}(f,s)$ at $s=1/2$. This is assured by the definition of the partial order on the set of triples. The Eisenstein series $E_{\Pi_1}(f,s)$ and $E_{\Pi_2}(f',s')$ correspond to the triples

$$(P_1,\Pi_1=\mathbf{1}\circ\det,1/2) \quad \text{and} \quad (P_2,\Pi_2=\mathbf{1}\otimes\mathbf{1}_{SL_2(\mathbb{A})},0),$$

respectively. As already observed in Step 9.8 of the proof in Section 8.2, we have

$$\iota_{P_2}(0)=(0,0)\succ\iota_{P_1}(1/2)=(1/2,1/2)$$

in the partial order defining the filtration. Therefore, the contribution of the second triple, that is, the Eisenstein series $E_{\Pi_2}(f',s')$ at $s'=0$ is assigned to a deeper filtration step than the contribution of the first triple, that is, the Eisenstein series $E_{\Pi_1}(f,s)$, precisely as required. This explains the underlying reason for the existence of two different quotients of the filtration arising from the Eisenstein series associated to the parabolic subgroups of the same rank.

In groups of higher rank, the phenomenon described here often occurs. There are even examples in which the order of contributions is reversed, so that the Eisenstein series associated to a parabolic subgroup of lower rank contributes to a deeper quotient of the filtration than the Eisenstein series associated to a parabolic subgroup of higher rank. In the case of the general linear group, the recent paper [**GG22**] reveals and studies such phenomena.

However, as already explained at the end of Section 9.3 and observed in the quoted Remark 2 of [**Fra98**, page 242-243], this phenomenon never occurs in the case of cuspidal support in a maximal proper parabolic subgroup, even for groups of higher rank, because in that case the residues at the relevant value of the complex parameter of the Eisenstein series are always square-integrable.

APPENDIX A

Calculation of colimits

For convenience of the reader, we provide in this appendix the explicit calculation of colimits required in the proofs in Chapter 8. We first recall the definition of the colimit. It can be found in any standard reference on the subject, such as [**Mac71**], but we state it in the context of the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules.

Let $\mathcal{M}$ be a finite groupoid, that is, a category with a finite number of objects and morphisms, such that all morphisms in $\mathcal{M}$ are isomorphisms. Let $\mathcal{C}$ be the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. Let M be a covariant functor from $\mathcal{M}$ to $\mathcal{C}$. In this setting, we may define the colimit of the functor M as follows.

Definition A.1. The colimit of the functor M from the groupoid $\mathcal{M}$ to the category $\mathcal{C}$ as above, consists of an object C in $\mathcal{C}$ together with a family of morphisms $(\phi_X)_{X\in\mathcal{M}}$ in $\mathcal{C}$, where

$$\phi_X : M(X) \to C,$$

such that

- for every morphism $w : X_1 \to X_2$ in the category $\mathcal{M}$, where X_1 and X_2 are (possibly equal) objects in $\mathcal{M}$, the following diagram commutes

$$\begin{array}{ccc} M(X_1) & \xrightarrow{M(w)} & M(X_2) \\ {\scriptstyle \phi_{X_1}} \searrow & \circlearrowleft & \swarrow {\scriptstyle \phi_{X_2}} \\ & C & \end{array}$$

  in category $\mathcal{C}$, i.e., $\phi_{X_2} \circ M(w) = \phi_{X_1}$, and
- the following universal property holds: if C' is another object in $\mathcal{C}$ and $(\phi'_X)_{X\in\mathcal{M}}$ a family of morphisms $\phi'_X : M(X) \to C'$ satisfying the condition as above, then there exists a unique morphism $u : C \to C'$ such that, for every object X in $\mathcal{M}$, the following diagram commutes

$$\begin{array}{ccc} & M(X) & \\ {\scriptstyle \phi_X} \swarrow & \circlearrowleft & \searrow {\scriptstyle \phi'_X} \\ C & \xrightarrow{u} & C' \end{array}$$

  in category $\mathcal{C}$, i.e., $u \circ \phi_X = \phi'_X$ for every object X in $\mathcal{M}$.

The colimit is unique by the universal property, and we write

$$C = \operatorname*{colim}_{X\in\mathcal{M}} M(X)$$

for the object C in the definition of the colimit.

Since the colimit is unique, it is sufficient to make a construction of the colimit in any particular case required. That is, one must define an object C, together with a family of morphisms ϕ_X, and check the two conditions in the definition of the colimit. We make such construction in the most general case required in the paper.

THEOREM A.2. *Let $\mathcal{M}$ be a finite groupoid with objects $X_0, X_1, \ldots, X_m$, where $m \geq 0$. Let W_i denote the group of automorphisms of X_i in category $\mathcal{M}$. Suppose that for every pair of objects (X_i, X_j) in $\mathcal{M}$ there exists a morphism $w_{i,j}$ from X_i to X_j such that the set of morphisms from X_i to X_j is the coset $w_{i,j}W_i = W_j w_{i,j}$. Observe that then W_i and W_j are isomorphic as they are conjugate by $w_{i,j}$. For $i = j$, we choose $w_{i,i}$ to be the identity morphism on X_i. We may and will choose $w_{i,j}$ in such a way that $w_{j,i} = w_{i,j}^{-1}$.*

Let M be a covariant functor from $\mathcal{M}$ to the category $\mathcal{C}$ of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. Then, the colimit of M can be described as

$$\begin{aligned}\operatorname*{colim}_{X_i \in \mathcal{M}} M(X_i) &\cong M(X_0)/S \\ &\cong \left\{ \sum_{w \in W_0} M(w)\xi \,:\, \xi \in M(X_0) \right\} \\ &\cong M(X_0)^{W_0} \\ &\cong M(X_0)^T,\end{aligned}$$

where S is the submodule of $M(X_0)$ generated by

$$\{x - M(w)x \,:\, x \in M(X_0), w \in W_0\},$$

T is a set of generators for W_0, and $M(X_0)^{W_0}$, respectively $M(X_0)^T$, denotes the submodule of $M(X_0)$ consisting of all vectors invariant under $M(w)$ for all $w \in W_0$, respectively for all $w \in T$. The morphisms ϕ_{X_i} are defined as

$$\phi_{X_i} = \varepsilon \circ M(w_{i,0}),$$

where $M(w_{i,0}) = M(w_{0,i}^{-1})$ is a morphism from $M(X_i)$ to $M(X_0)$, and ε is the canonical epimorphism from $M(X_0)$ to the quotient $M(X_0)/S$.

PROOF. For the object C defined as the quotient $M(X_0)/S$ in the theorem, and the morphisms ϕ_{X_i} we must prove that the two conditions in Definition A.1 of colimit are satisfied. Let ε denote the quotient map from $M(X_0)$ to the quotient $M(X_0)/S$.

For the first condition, let w be any morphism from X_i to X_j, where X_i and X_j are (possibly equal) objects in $\mathcal{M}$. Then we must prove that

$$\phi_{X_j} \circ M(w) = \phi_{X_i},$$

where $\phi_{X_l} = \varepsilon \circ M(w_{l,0})$ for $l = 0, 1, \ldots, m$, as in the theorem. Inserting this into the desired equation we obtain

$$\varepsilon \circ M(w_{j,0}) \circ M(w) = \varepsilon \circ M(w_{i,0})$$

which can be rewritten as

$$\varepsilon \circ (M(w_{i,0}) - M(w_{j,0}w)) = 0.$$

Hence, it is enough to show that the image of the morphism in brackets is in the kernel S of ε, that is,

$$M(w_{i,0})y - M(w_{j,0}w)y \in S$$

for all $y \in M(X_i)$. But

$$M(w_{i,0})y - M(w_{j,0}w)y = M(w_{i,0})y - M(w_{j,0}ww_{0,i})M(w_{i,0})y,$$

so that if we set $x = M(w_{i,0})y \in M(X_0)$ and $w' = w_{j,0}ww_{0,i} \in W_0$, it is clear that this last expression is in S.

For the second condition, i.e., the universal property, we must prove the existence and uniqueness of the morphism u. More precisely, let C' be an object in $\mathcal{C}$, and ϕ'_{X_i}, where $i = 0, 1, \ldots, m$, a family of morphisms from $M(X_i)$ to C', indexed by objects of $\mathcal{M}$, such that the first condition in Definition A.1 of the colimit is satisfied. That is,

$$\phi'_{X_j} \circ M(w) = \phi'_{X_i},$$

for any morphism w from X_i to X_j in category $\mathcal{M}$. In particular, for any $w \in W_0$, we have

$$\phi'_{X_0} \circ M(w) = \phi'_{X_0},$$

which implies that ϕ'_{X_0} is trivial on the kernel S of the quotient map ε.

Then, we define the morphism u from C to C' as

$$u(\varepsilon(x)) := \phi'_{X_0}(x),$$

where $x \in M(X_0)$. This is well-defined, because any element of C is of the form $\varepsilon(x)$, and if $x' \in M(X_0)$ is such that $\varepsilon(x) = \varepsilon(x')$, then $x - x' \in S$ and, since ϕ'_{X_0} is trivial on S, we have $\phi'_{X_0}(x) = \phi'_{X_0}(x')$. To show that the diagram in the universal property commutes, we make a direct computation

$$\begin{aligned} u \circ \phi_{X_i} &= u \circ \varepsilon \circ M(w_{i,0}) \\ &= \phi'_{X_0} \circ M(w_{i,0}) \\ &= \phi'_{X_i}, \end{aligned}$$

where we use the definition of ϕ_{X_i} in the first line, the defining relation of u in the second line, and the assumed property of the family ϕ'_{X_i} in the last line. Thus, we have proved the existence of u.

For the uniqueness, suppose that for the same C' and the same family ϕ'_{X_i}, there is another morphism u' from C to C' such that the diagram in the universal property commutes, i.e.,

$$u' \circ \phi_{X_i} = \phi'_{X_i},$$

for all X_i, $i = 0, 1, \ldots, m$. Then, we may write

$$\begin{aligned} u'(\varepsilon(x)) &= u'(\phi_{X_0}(x)) \\ &= \phi'_{X_0}(x) \\ &= u(\varepsilon(x)), \end{aligned}$$

where we use the definition of ϕ_{X_0} in the first line, the universal property for u' in the second line, and the definition of u in the last line. Since an arbitrary element of C is of the form $\varepsilon(x)$, where $x \in M(X_0)$, it follows that $u = u'$, and thus, u is unique.

It remains to show that the other three descriptions of the colimit in the theorem are indeed isomorphic to $M(X_0)/S$. For the moment, let

$$V = \left\{ \sum_{w \in W_0} M(w)\xi \; : \; \xi \in M(X_0) \right\}.$$

We prove the first isomorphism directly using the first isomorphism theorem. If we define an intertwining map $\Psi : M(X_0) \to V$ by the formula

$$\Psi(\xi) = \sum_{w \in W_0} M(w)\xi,$$

then it is clear that the image of Ψ is V. For the kernel, we have that $S \subseteq \mathrm{Ker}\Psi$, because

$$\begin{aligned}\Psi(x - M(w_0)x) &= \sum_{w \in W_0} M(w)\,(x - M(w_0)x) \\ &= \sum_{w \in W_0} M(w)x - \sum_{w \in W_0} M(ww_0)x \\ &= \sum_{w \in W_0} M(w)x - \sum_{w \in W_0} M(w)x \\ &= 0,\end{aligned}$$

for all $x \in M(X_0)$ and all $w_0 \in W_0$. On the other hand, $\mathrm{Ker}\Psi \subseteq S$, because given any $y \in \mathrm{Ker}\Psi$, we have

$$y = -\sum_{w \in W_0, w \neq 1} M(w)y,$$

which may be rearranged as

$$y = \frac{1}{|W_0|} \sum_{w \in W_0, w \neq 1} (y - M(w)y) \in S,$$

where $|W_0|$ is the cardinality of W_0. Thus, $\mathrm{Ker}\Psi = S$, so that the first isomorphism is proved.

For the second isomorphism, it is clear that all vectors in V are invariant under $M(w_0)$ for all $w_0 \in W_0$, because

$$\begin{aligned}M(w_0) \sum_{w \in W_0} M(w)\xi &= \sum_{w \in W_0} M(w_0 w)\xi \\ &= \sum_{w \in W_0} M(w)\xi.\end{aligned}$$

Conversely, if $y \in M(X_0)$ is invariant under $M(w)$ for all $w \in W_0$, then

$$|W_0| y = \sum_{w \in W_0} M(w)y,$$

which implies that for $\xi = \frac{1}{|W_0|} y$ we have

$$y = \sum_{w \in W_0} M(w)\xi \in V.$$

This shows the second isomorphism. The third isomorphism is obvious. □

All the colimits required in the paper can be explicitly determined using Theorem A.2. In fact, most of the time, only two special cases are required in the proofs. The first of these cases is the case of $m = 0$, so that there is only one object X_0 in $\mathcal{M}$, with a non-trivial automorphism w_0 in W_0, i.e., $W_0 = \{1, w_0\}$. In this case, Theorem A.2 implies that the colimit is isomorphic to the invariants in X_0 for the automorphism w_0. The second case is the case of $m = 1$, so that there are two objects X_0 and X_1 in $\mathcal{M}$, without non-trivial automorphisms, i.e., $W_0 = \{1\}$, but with a non-trivial isomorphism $w_{0,1} = w_{1,0}^{-1}$ between them. In this case, the colimit is isomorphic to X_0. The only exception of this rule is the proof of Theorem 7.8 given in Section 8.3. The special cases of Theorem A.2 required in that proof are mentioned there.

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