SOME UNEXPECTED PHENOMENA IN THE FRANKE FILTRATION OF THE SPACE OF AUTOMORPHIC FORMS OF THE GENERAL LINEAR GROUP

NEVEN GRBAC AND HARALD GROBNER

Abstract. In his famous paper [11], J. Franke has defined a certain finite filtration of the space of automorphic forms of a general reductive group, which captures most of its internal representation theory. The purpose of this paper is to provide several concrete examples of yet unexpected phenomena, which occur in the Franke filtration for the general linear group. More precisely, we show that the degenerate Eisenstein series arising from the parabolic subgroups of the same rank are not necessarily contributing to the same quotient of the filtration, and that, even more, the Eisenstein series arising from the parabolic subgroups of higher relative rank may contribute to a deeper quotient of the filtration. These are the first structural counterexamples to an expectation, mentioned in [11].

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Introduction

In a nutshell. In this paper we undertake the study of the Franke filtration of the space of automorphic forms on the general linear group GL_n, defined over an algebraic number field. The main goal is to discover and describe through examples certain phenomena, which were not anticipated in

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the earlier work on the Franke filtration. We do not attempt to give a complete explicit description of the Franke filtration in the case of GL_n, as it is combinatorially very demanding and does not provide further insight in the phenomena considered here.

Context. In order to put our results in a larger framework, we shall use a more general setting in the introduction: Let G be a connected reductive linear algebraic group defined over an algebraic number field F. Consider the space $\mathcal{A}$ of K_∞-finite automorphic forms on $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of F, and K_∞ a fixed choice of a maximal compact subgroup of the archimedean part G_∞ of $G(\mathbb{A})$, as defined in [6]. It carries a natural structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module, where $\mathfrak{g}_\infty$, denotes the real Lie algebra of G_∞, and $\mathbb{A}_f$ is the ring of finite adèles of F. The Franke filtration provides a way to approach this module structure of the space of automorphic forms.

The first step in the study of the space of automorphic forms is its decomposition along the cuspidal support, cf. [25], [30, Sect. III.2.6], [12, Sect. 1]. Given an associate class $\{P\}$ of parabolic F-subgroups of G, represented by the parabolic subgroup P, and an associate class $\varphi(\pi)$ of cuspidal automorphic representations of the Levi factors of the parabolic subgroups in $\{P\}$, represented by a cuspidal automorphic representation π of the Levi factor $L(\mathbb{A})$ of P, cf. [12, Sect. 1.2] [26, Sect. 1.3], let $\mathcal{A}_{\{P\},\varphi(\pi)}$ be the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module of automorphic forms with cuspidal support in $\varphi(\pi)$. See [30], [12], for a precise definition of this notion.

The Franke filtration was originally defined by Franke in [11, Sect. 6]. In this paper, we consider the Franke filtration of the modules $\mathcal{A}_{\{P\},\varphi(\pi)}$ with the given cuspidal support $\varphi(\pi)$ as in [17], [15], [14], [16]. This is a slight modification of the original approach of Franke, which only considers the associate class $\{P\}$ of parabolic subgroups in which the cuspidal support lies, and does not fix the associate class of cuspidal automorphic representations.

The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P\},\varphi(\pi)}$ is a finite descending filtration of the form

$$\mathcal{A}_{\{P\},\varphi(\pi)} = \mathcal{A}^0_{\{P\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{P\},\varphi(\pi)} \supsetneqq \cdots \supsetneqq \mathcal{A}^\ell_{\{P\},\varphi(\pi)} \supsetneqq \{0\}.$$

The main feature of this filtration is that the quotients of the filtration

$$\mathcal{A}^i_{\{P\},\varphi(\pi)}/\mathcal{A}^{i+1}_{\{P\},\varphi(\pi)},$$

where $i = 0, \ldots, \ell$ and we set $\mathcal{A}^{\ell+1}_{\{P\},\varphi(\pi)} = \{0\}$, are isomorphic to parabolically induced representations, or a direct sum or certain colimit of these. The isomorphisms are constructed using the main values of certain Eisenstein series and their derivatives evaluated at the complex parameter such that the cuspidal support is in the associate class $\varphi(\pi)$. The Eisenstein series are constructed from the sections of representations parabolically induced from the constituents of the discrete spectrum of the Levi factors of parabolic subgroups that contain an element of the associate class $\{P\}$. Thus, the degenerate Eisenstein series constructed starting from residual representations of the Levi factors must be used, and not only those constructed from a cuspidal automorphic representation. For full details of the construction see [11, Sect. 6].

The Franke filtration, with its rather explicit description of the successive quotients in terms of parabolically induced representations, has several very important applications. First of all, the fact that all the quotients of the filtration are spanned by the main values of Eisenstein series and their derivatives, implies that all automorphic forms on $G(\mathbb{A})$ are obtained as sums of Eisenstein series and their derivatives. In the number field case, this was first proved by Franke in [11, Cor. 1, p. 236] as a consequence of the construction of the filtration, although the function field case was known from [30, App. II].

Another important implication of the Franke filtration is the proof of the Borel–Harder conjecture on the cohomology of arithmetic groups, or, equivalently, the de Rham cohomology of locally symmetric spaces (see for instance [22, page 102] and [5, §6.9]). Borel's regularization theorems [4] show that the de Rham cohomology of a locally symmetric space, with coefficient in the local system constructed from a finite-dimensional algebraic representation, is isomorphic to the relative Lie algebra cohomology of the space of smooth functions of uniform moderate growth. The Borel–Harder conjecture claims that it is also isomorphic to the relative Lie algebra cohomology of the space of automorphic forms. The proof of the conjecture by Franke [11, Sect. 7.4] begins with the construction of an Eisenstein spectral sequence which computes the relative Lie algebra cohomology of the space of smooth functions of uniform moderate growth. The Franke filtration then implies that every cohomology class can be represented by an automorphic form, thus implying the Borel–Harder conjecture.

In the case of the general linear group, the Franke filtration was applied further in [11, Sect. 7.6] to obtain a rationality result for the summands in cohomology corresponding to spaces of automorphic forms supported in any associate class $\{P\}$ of parabolic subgroups. This extends the earlier result of Clozel [8], which provides the rationality for the case of the cuspidal summand, i.e. the summand supported in $\{G = GL_n\}$. As observed by Harder [21] and Clozel [8], see also [10], this rationality result for GL_n may be viewed as the generalization from GL_2 to GL_n of the Manin-Drinfel′d theorem, cf. [28], [9]. In [18], this result was generalized for regular Eisenstein cohomology of GL_n over a division algebra D/F. See [18, Thm. 7.23].

The filtration is also used by Franke [11, Sect. 7.7] to obtain a trace formula for Hecke operators on the de Rham cohomology with respect to a local system arising from a finite-dimensional representation. The formula is similar to and derived from Arthur's trace formula for L^2-cohomology [1]. The Goresky-MacPherson trace formula for Hecke operators on full cohomology was known earlier [13], but it was of different form and contained certain truncated Hecke correspondences.

Another important application of the Franke filtration is that it provides a key to an explicit description of automorphic cohomology, in particular, the summands in cohomology corresponding to spaces of automorphic forms with a given cuspidal support. This idea is pursued in [17], which provides a complete description of low-rank automorphic cohomology of a general connected reductive group in terms of the cohomology of the square-integrable automorphic representations of the Levi subgroups. Given Arthur's theory of global A-packets [2], [31], and Vogan-Zuckerman's theory of $A_{\mathfrak{q}}(\lambda)$-modules [32], this result of [17] reduces a full understanding of the low-rank automorphic cohomology of a general connected reductive group to an understanding of the cohomological cuspidal spectrum of GL_n. As a direct application, the main result of [17] implies new improved bounds on the degrees in cohomology in which the inclusion of the space of square-integrable forms into the space of all automorphic forms gives rise to the injective map in cohomology. In the special case of the trivial representation, i.e., the space of constant forms, the results of [17] improve the bounds obtained by Borel [3, Thm. 7.5].

The idea to use the Franke filtration for explicit computations of automorphic cohomology was first carried out in the case of the split symplectic group of rank two over a totally real field in [15] and, most recently, in the case of the unitary groups of rank one over $\mathbb{Q}$ in [16].

In view of all these important applications of the Franke filtration, it is clear that explicit description of the Franke filtration would be extremely useful.

Main results: Phenomenon I. In Remark 2 on [11, page 242], Franke explains why the contribution to the i-th quotient

$$\mathcal{A}^i_{\{P\},\varphi(\pi)}/\mathcal{A}^{i+1}_{\{P\},\varphi(\pi)},$$

of the filtration could not be in general determined by all the Eisenstein series arising from parabolic subgroups of the same rank. The problem occurs whenever the Eisenstein series have residues at points in the closure of the positive Weyl chamber which are not square-integrable.

However, in contrast to this general observation, in *loc. cit.* Franke also suggests that for the general linear group "*[The definition of the filtration in terms of the parabolic rank] also seems to work for GL_n because of the results of Mœglin and Waldspurger*".

In this paper we reconsider this thought and we show that the whole combinatorics of the Franke filtration for GL_n is substantially more complicated (and therefore, depending on the reader's taste, maybe also interesting): Partly inspired by the work of Hanzer–Muić [20] on the analytic properties of degenerate Eisenstein series on GL_n, we study certain examples of cuspidal supports for GL_n, which show that the Franke filtration exhibits phenomena, which imply that the rank of an Eisenstein series *cannot* determine the quotient of the filtration, to which they contribute.

More precisely, we consider cuspidal supports, which are not supports of a residual representation of GL_n. We begin with a simple lowest possible rank example in which two Eisenstein series arising from the parabolic subgroups of the same rank contribute to different quotients of the filtration. This example for the group GL_4 is studied in Theorem 5.1. More general higher rank examples of the same phenomenon are provided in Theorem 5.2.

In contrast to these examples, we also show that in the case of the cuspidal support of a residual representation of GL_n, the considered phenomenon never occurs. It seems that this was the case which Franke had in mind when making the comment in [11, Rmk. 2, p. 242]. We prove in Theorem 4.1 that the Franke filtration in this case may be arranged in such a way that the quotient of the filtration to which an Eisenstein series contributes is determined by the rank of the parabolic subgroup from which it arises.

Main results: Phenomenon II. Finally, we also give examples in which the Eisenstein series arising from the parabolic subgroup of a higher relative rank contribute to a deeper quotient of the filtration than those arising from the parabolic subgroup of lower relative rank. The lowest rank example of this phenomenon occurs for the group GL_6. We study this case in Theorem 6.1. The higher rank examples of the same phenomenon are provided in Theorem 6.2.

All the theorems, except Theorem 4.1 dealing with the case of cuspidal support of a residual representation, are stated for the cuspidal support in the Borel subgroup of GL_n. This simplifies the notation and exposition of the proofs, without losing any insight of the considered phenomenon, although the same results hold in the following more general setting: One may replace the Hecke characters χ in those theorems with a cuspidal automorphic representation of some GL_k, keep the same exponents, and work in the ambient group GL_{2mk}. The cuspidal support is then in the associate class of the parabolic subgroup with $2m$ diagonal blocks of size k.

The paper is structured as follows. We begin with preliminaries in Section 1, in which we explain the structure of the general linear group, its parabolic subgroups and the positive and obtuse Weyl chamber. In Section 2 the Franke filtration is defined and the required partial order is made explicit in the case of GL_n. The proof of an important lemma is the subject of Section 3. The remaining sections study examples of the Franke filtration for different types of the cuspidal support. The case of the cuspidal support of a residual representation is treated in Section 4. The examples in which Eisesntein series arising from parabolic subgroups of the same rank contribute to different quotients of the filtration are studied in Section 5. Finally, the examples in which the Eisenstein series arising from parabolic subgroups of higher relative rank contribute to deeper quotients of the filtration are studied in Section 6.

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1. Preliminaries

Let F be an algebraic number field. The set of archimedean places of F is denoted by V_∞. For a place v of F, let F_v denote the completion of F at the place v. For a non-archimedean place $v \notin V_\infty$, let $\mathcal{O}_v$ be the ring of integers in F_v. The ring of adèles of F is denoted by $\mathbb{A}$, the subring of finite adèles by $\mathbb{A}_f$, and the group of idèles of F by $\mathbb{I}$.

Throughout the paper, we let $G = GL_n$ be the general linear group over F. That is, for any F-algebra R, the group $G(R)$ of R-points of G is the general linear group $GL_n(R)$ of invertible elements in the algebra of all $n \times n$ matrices with entries in R.

We fix, once and for all, the Borel subgroup B of G such that $B(R)$ consists of upper-triangular matrices in $G(R)$ for any F-algebra R. Let T be the maximal F-split torus in G such that $T(R)$ consists of all diagonal matrices in $G(R)$, and let U be the unipotent radical of B. Then we have the Levi decomposition $B = TU$.

Let P be a standard parabolic F-subgroup of G with the Levi decomposition $P = LN$, where L is the Levi factor and N the unipotent radical. In this paper all parabolic F-subgroups are standard, unless otherwise specified. Then $P(R)$ consists of block-upper-triangular matrices, and $L(R)$ of block-diagonal matrices in $G(R)$, for any F-algebra R. The standard parabolic subgroups in G are in one-to-one correspondence with ordered partitions of n into positive integers. Given such a partition $(n_1, \ldots, n_k)$, where $\sum_{i=1}^k n_i = n$, the corresponding standard parabolic subgroup $P = P_{(n_1,\ldots,n_k)}$ is such that $P(R)$ consists of all block-upper-triangular matrices with blocks of sizes $n_1, \ldots, n_k$ along the diagonal. Hence, if $P = LN$ is the Levi decomposition, we have

$$L(R) = \{\operatorname{diag}(l_1, \ldots, l_k) \in G(R) \,:\, l_i \in GL_{n_i}(R)\} \cong GL_{n_1} \times \cdots \times GL_{n_k}.$$

Two parabolic subgroups are called associate if their Levi factors are F-conjugate. Conjugate parabolic subgroups are clearly associate. If P corresponds to the ordered partition $(n_1, \ldots, n_k)$ and P' to the ordered partition $(n'_1, \ldots, n'_l)$, then P is associate to P' if and only if partition $(n'_1, \ldots, n'_l)$ is a permutation of partition $(n_1, \ldots, n_k)$.

Given a standard parabolic F-subgroup P of G, let $X^*(P)$ be the $\mathbb{Z}$-module of F-rational characters of P. Let $\check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$, and let $\check{\mathfrak{a}}_{P,\mathbb{C}}$ denote its complexification. If $P = P_{(n_1,\ldots,n_k)}$ corresponds to the ordered partition $(n_1, \ldots, n_k)$ of n, then the space $\check{\mathfrak{a}}_{P,\mathbb{C}}$ is isomorphic to $\mathbb{C}^k$. It may be identified with a space of complex characters of $L(\mathbb{A})$ as follows. Given a k-tuple $(s_1, \ldots, s_k) \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, where $s_i \in \mathbb{C}$, the corresponding character of $L(\mathbb{A})$ is defined by the assignment

$$\operatorname{diag}(l_1, \ldots, l_k) \mapsto |\det l_1|^{s_1} \ldots |\det l_k|^{s_k}$$

for any $\operatorname{diag}(l_1, \ldots, l_k) \in L(\mathbb{A})$, where $|\cdot|$ denotes the normalized absolute value on the group of idéles $\mathbb{I}$. Throughout the paper det stands for the determinant on the algebra of matrices of appropriate size.

Let Z be the center of G. The subspace of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ corresponding to characters trivial on the center $Z(\mathbb{A})$ is denoted by $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$. For $P = P_{(n_1,\dots,n_k)}$, a k-tuple $(s_1,\dots,s_k)$ is in $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$ if and only if $n_1s_1 + n_2s_2 + \cdots + n_ks_k = 0$. The real subspace $\check{\mathfrak{a}}^G_P$ is given as the intersection of $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$ and $\check{\mathfrak{a}}_P$. The relative rank of a parabolic subgroup P is the dimension of the space $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$. Thus, $P = P_{(n_1,\dots,n_k)}$ is of relative rank $k-1$.

The positive Weyl chamber defined by the parabolic subgroup $P = P_{(n_1,\dots,n_k)}$ is the open cone in $\check{\mathfrak{a}}_P$ consisting of all $(s_1,\dots,s_k) \in \check{\mathfrak{a}}_P$ such that

$$s_1 > \cdots > s_k.$$

The closure of the positive Weyl chamber is given by

$$s_1 \geq \cdots \geq s_k.$$

We also need the notion of the obtuse Weyl chamber in the case of the Borel subgroup B. The positive obtuse Weyl chamber in $\check{\mathfrak{a}}_B$ is the open cone dual to the positive Weyl chamber. It consists of all $(s_1,\dots,s_n) \in \check{\mathfrak{a}}_B$ such that

$$s_1 + s_2 + \cdots + s_j > \frac{j}{n}(s_1 + s_2 + \cdots + s_n), \quad \text{for } j = 1,\dots,n-1.$$

The negative obtuse Weyl chamber is given by the reversed inequalities, and the closed negative obtuse Weyl chamber for G is given by

$$s_1 + s_2 + \cdots + s_j \leq \frac{j}{n}(s_1 + s_2 + \cdots + s_n), \quad \text{for } j = 1,\dots,n-1.$$

Since the sum on the right-hand side of these inequalities is zero for $(s_1,\dots,s_n) \in \check{\mathfrak{a}}^G_B$, the intersection of the closure of the negative obtuse Weyl chamber and $\check{\mathfrak{a}}^G_B$ is given by

$$\begin{aligned} s_1 &\leq 0 \\ s_1 + s_2 &\leq 0 \\ &\cdots \\ s_1 + s_2 + \cdots + s_{n-1} &\leq 0. \end{aligned}$$

This last set of inequalities will play a role in the definition of the Franke filtration for G.

The restriction of characters gives rise to the inclusion of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ into $\check{\mathfrak{a}}_{B,\mathbb{C}}$. We denote this inclusion by ι for any parabolic subgroup P. If $P = P_{(n_1,\dots,n_k)}$, then the inclusion takes $(s_1,\dots,s_k) \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ to

$$\iota(s_1,\dots,s_k) = (s_1,\dots,s_1,s_2,\dots,s_2,\dots,s_k,\dots,s_k) \in \check{\mathfrak{a}}_{B,\mathbb{C}},$$

where s_i occurs n_i times.

Let W be the Weyl group of G with respect to T. It is isomorphic to the symmetric group $\mathfrak{S}_n$ on n letters. For a parabolic subgroup $P = LN$, let W_L denote the Weyl group of the Levi factor L. If $P = P_{(n_1,\dots,n_k)}$, the Weyl group W_L is isomorphic to the product

$$W_L \cong \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}.$$

In each right coset in $W_L \backslash W$, there is a unique element of minimal length, the so-called minimal coset representative or the Kostant representative (cf. [24], [7]). We denote by W^P the set of such minimal coset representatives.

Let $G_\infty = \prod_{v \in V_\infty} G(F_v)$. The real Lie algebra of G_∞ is denoted by $\mathfrak{g}_\infty$. For a place $v \notin V_\infty$ of F, we fix a maximal compact subgroup $K_v = G(\mathcal{O}_v)$ of $G(F_v)$. For a real (resp. complex) place v of

F, we fix the group $K_v = O(n)$ (resp. $K_v = U(n)$) of $n \times n$ orthogonal (resp. unitary) matrices as a maximal compact subgroup of $G(F_v) = GL_n(\mathbb{R})$ (resp. $G(F_v) = GL_n(\mathbb{C})$). Then we fix a maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ as the product over all places of the fixed maximal compact subgroups K_v. This choice of K is in good position with respect to the fixed Borel subgroup B in the sense of [30, Sect. I.1.4]. We denote the archimedean part of the maximal compact subgroup by $K_\infty = \prod_{v\in V_\infty} K_v$. It is a maximal compact subgroup of G_∞.

2. The Franke filtration for $G = GL_n$

We retain the notation of the previous section. Let $\mathcal{A} = \mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ be the space of all automorphic forms[1] on $G(\mathbb{A})$, in the sense of [6], of central character ω. The space of automorphic forms carries a natural $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module structure. It exhibits a direct sum decomposition along the cuspidal support.

Given a cuspidal automorphic representation[2] π of the Levi factor $L(\mathbb{A})$ of a parabolic subgroup P of G, such that π restricted to the center of G acts as the central character ω, let $\mathcal{A}_{\{P\},\varphi(\pi)}$ be the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-submodule of the space of automorphic forms $\mathcal{A}$, supported in the associate class $\varphi(\pi)$ represented by π. We recall below the definition of $\mathcal{A}_{\{P\},\varphi(\pi)}$ following [12, Sect. 1]. For more details, we refer to *loc. cit.* or [30, Sect. III.2.6]. In particular, $\mathcal{A}_{\{P\},\varphi(\pi)}$ is a direct summand of $\mathcal{A}$ according to the results of [30, Sect. III.2.6]:

Let $P = P_{(n_1,\dots,n_k)}$, where $(n_1, \dots, n_k)$ is an ordered partition of n, so that the Levi factor L of P is isomorphic to the product

$$L \cong GL_{n_1} \times \cdots \times GL_{n_k}.$$

Then, the cuspidal support π may always be chosen in the form

$$\pi \cong \pi_1 |\det|^{s_1} \otimes \cdots \otimes \pi_k |\det|^{s_k},$$

where π_i is a unitary cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$, with the central character ω_i, such that the product $\omega_1 \dots \omega_k = \omega$, and $\underline{s}_0 = (s_1, \dots, s_k) \in \check{\mathfrak{a}}_P^G$ is in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P$ defined by P, i.e., $s_1 \geq \cdots \geq s_k$. Throughout the paper, the notation $\otimes$ stands for the outer tensor product of representations of different groups, which is by definition a representation of the direct product of these groups on the tensor product of the spaces of their representations.

Let $\pi^u \cong \pi_1 \otimes \cdots \otimes \pi_k$ be the unitary cuspidal automorphic representation of $L(\mathbb{A})$, which is the unitary part of the cuspidal support π in the form as above. For $\underline{s} \in \check{\mathfrak{a}}_{P,\mathbb{C}}^G$, let $I(\underline{s}, \pi^u)$ denote the representation parabolically induced from π^u twisted by a character of $L(\mathbb{A})$ corresponding to $\underline{s}$. Given an appropriate section $f_{\underline{s}}$ of the induced representations $I(\underline{s}, \pi^u)$ and $g \in G(\mathbb{A})$, one may define the Eisenstein series $E(f_{\underline{s}}, g)$ associated to π^u. It is defined as the analytic continuation from the cone of (absolute and locally uniform) convergence of the series

$$E(f_{\underline{s}}, g) = \sum_{\gamma\in P(F)\backslash G(F)} f_{\underline{s}}(\gamma g).$$

[1] We always assume that the automorphic forms are normalized in such way that they are trivial on the identity component of the archimedean part of the center Z of G. This assumption is not restrictive, as explained in [23, page 121].

[2] For convenience, we will not distinguish between a square-integrable automorphic representation, its smooth limit-Fréchet-space completion or its (non-smooth) Hilbert space completion in the L^2-spectrum. See [19] for a detailed account of these questions.

For the properties of such Eisenstein series, we refer to [30, Chap. IV] and [25, Sect. 7]. In particular, the poles of $E(f_{\underline{s}}, g)$ all lie along a locally finite family of singular hyperplanes in $\check{\mathfrak{a}}^G_{P,\mathbb{C}}$. Hence, given $\underline{s}_0 \in \check{\mathfrak{a}}^G_P$ from the cuspidal support π, there is a polynomial Q in $\underline{s}$ such that $Q(\underline{s})E(f_{\underline{s}}, g)$ is holomorphic around $\underline{s} = \underline{s}_0$. The space of automorphic forms $\mathcal{A}_{\{P\},\varphi(\pi)}$ supported in the associate class $\varphi(\pi)$ is then defined as the span of all coefficients, which are functions on $G(\mathbb{A})$, of the Taylor expansion of $Q(\underline{s})E(f_{\underline{s}}, g)$ around $\underline{s} = \underline{s}_0$.

The Franke filtration is originally defined in [11, Sect. 6]. In this paper we consider a slight modification which takes into account the cuspidal support as in [12, Sect. 1]. In other words, we describe the Franke filtration of a direct summand $\mathcal{A}_{\{P\},\varphi(\pi)}$ of $\mathcal{A}$. More precisely, we now describe the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P\},\varphi(\pi)}$ with the cuspidal support in the associate class $\varphi(\pi)$ of the cuspidal automorphic representation π of $L(\mathbb{A})$ as above. Consider the set $\mathcal{M}_{\{P\},\varphi(\pi)}$ of triples $(R, \Pi, \underline{z})$, where

- R is a parabolic subgroup of G which contains a parabolic subgroup associate to P,
- Π is a discrete spectrum unitary automorphic representation of the Levi factor $L_R(\mathbb{A})$ of R,
- $\underline{z}$ is in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}^G_R$ defined by R such that the representation Π of $L_R(\mathbb{A})$ twisted by the character of $L_R(\mathbb{A})$ corresponding to $\underline{z}$ has cuspidal support in the associate class $\varphi(\pi)$ represented by π.

Let $\mathcal{M}^j_{\{P\},\varphi(\pi)}$ be the subset of $\mathcal{M}_{\{P\},\varphi(\pi)}$ which contains all triples $(R, \Pi, \underline{z})$ with R of relative rank j.

The set $\mathcal{M}^j_{\{P\},\varphi(\pi)}$ is turned into a groupoid with the triples $(R, \Pi, \underline{z}) \in \mathcal{M}^j_{\{P\},\varphi(\pi)}$ as objects, and morphisms defined as follows. Given a pair of triples $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ in $\mathcal{M}^j_{\{P\},\varphi(\pi)}$, the set of morphisms is defined as the set of all $w \in W^R$ such that $w(L_R) = L_{R'}$, $w(\Pi) = \Pi'$ and $w(\underline{z}) = \underline{z}'$.

The functor $M_{\{P\},\varphi(\pi)}$ from the groupoid $\mathcal{M}^j_{\{P\},\varphi(\pi)}$ to the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules is defined on the objects as

$$M_{\{P\},\varphi(\pi)}\big((R, \Pi, \underline{z})\big) := I(\underline{z}, \Pi) \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \tag{2.1}$$

where

$$I(\underline{z}, \Pi) := \operatorname{Ind}^{G(\mathbb{A})}_{R(\mathbb{A})} (\Pi \otimes \underline{z})$$

is the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module parabolically induced from the discrete spectrum automorphic representation Π of $L_R(\mathbb{A})$ twisted by the character of $L_R(\mathbb{A})$ corresponding to $\underline{z} \in \check{\mathfrak{a}}_R$, and $S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$ is the symmetric algebra of $\check{\mathfrak{a}}^G_{R,\mathbb{C}}$, with the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module structure defined as in [11, page 218]. For a morphism w, the functor $M_{\{P\},\varphi(\pi)}(w)$ is defined as in [11, page 234].

Since the set $\mathcal{M}_{\{P\},\varphi(\pi)}$ is finite, there is a finite set of possible $\underline{z}$ in the triples. Let $S_{\{P\},\varphi(\pi)}$ be the finite subset of $\check{\mathfrak{a}}_B$ which consists of all such $\underline{z}$, viewed as elements of $\check{\mathfrak{a}}_B$ via the inclusion ι of $\check{\mathfrak{a}}_R$ into $\check{\mathfrak{a}}_B$. An integer-valued function $T_{\{P\},\varphi(\pi)}$ on the finite set $S_{\{P\},\varphi(\pi)}$ is chosen in such a way that

$$T_{\{P\},\varphi(\pi)}(\underline{t}) > T_{\{P\},\varphi(\pi)}(\underline{t}')$$

whenever $\underline{t}$ and $\underline{t}'$ in $S_{\{P\},\varphi(\pi)}$ are such that $\underline{t} \neq \underline{t}'$ and $\underline{t} - \underline{t}'$ lies in the closed negative obtuse Weyl chamber in $\check{\mathfrak{a}}^G_B$. In that case, we write

$$\underline{t} \succ \underline{t}'$$

for the partial order so obtained on $S_{\{P\},\varphi(\pi)}$. Explicitly in coordinates, if $\underline{t} = (t_1, \dots, t_n)$ and $\underline{t}' = (t'_1, \dots, t'_n)$, then the condition $\underline{t} \succ \underline{t}'$ is equivalent to the inequalities

$$\begin{aligned} t_1 &\le t'_1 \\ t_1 + t_2 &\le t'_1 + t'_2 \\ &\dots \\ t_1 + \dots + t_{n-1} &\le t'_1 + \dots + t'_{n-1} \end{aligned} \tag{2.2}$$

of partial sums, provided $\underline{t} \neq \underline{t}'$. These inequalities are the condition (8) in [11, page 233] made explicit for the case of G. The function $T_{\{P\},\varphi(\pi)}$ is not unique, but different choices give rise either to filtrations with the same quotients, or possibly several consecutive quotients of a filtration may be replaced with one filtration quotient isomorphic to their direct sum in the other filtration.

The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P\},\varphi(\pi)}$ is a finite descending filtration

$$\dots \supseteq \mathcal{A}^i_{\{P\},\varphi(\pi)} \supseteq \mathcal{A}^{i+1}_{\{P\},\varphi(\pi)} \supseteq \dots \tag{2.3}$$

where $i \in \mathbb{Z}$, but only finitely many inclusions are proper. The quotients of the filtration are isomorphic to

$$\mathcal{A}^i_{\{P\},\varphi(\pi)} / \mathcal{A}^{i+1}_{\{P\},\varphi(\pi)} \cong \bigoplus_{j=0}^{k-1} \operatorname*{colim}_{\substack{(R,\Pi,\underline{z}) \in \mathcal{M}^j_{\{P\},\varphi(\pi)} \\ T_{\{P\},\varphi(\pi)}(\iota(\underline{z}))=i}} M_{\{P\},\varphi(\pi)}(R, \Pi, \underline{z}) \tag{2.4}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. The colimit is taken over the full subcategory of $\mathcal{M}^j_{\{P\},\varphi(\pi)}$, consisting of objects satisfying $T_{\{P\},\varphi(\pi)}(\iota(\underline{z})) = i$, defined as in [27]. The functor $M_{\{P\},\varphi(\pi)}$ is defined by equation (2.1), so that the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules on the right-hand side are parabolically induced representations.

The construction of the isomorphism between the quotients of the Franke filtration and the induced representation on the right-hand side of (2.4) is based on the main values of the derivatives of the degenerate Eisenstein series arising from the discrete spectrum automorphic representation Π at the value $\underline{z}$ of its complex parameter. Since the Eisenstein series in question may have a pole at $\underline{z}$, the map realizing the isomorphism is well-defined only as an element of the quotient. For more details see [11, page 235].

3. An important lemma

The description of the filtration in the examples below relies on the construction of the residual spectrum for the general linear group by Mœglin and Waldspurger in [29]. Recall that, according to *loc. cit.*, if $n = km$, then for a unitary cuspidal automorphic representation σ of $GL_m(\mathbb{A})$ the induced representation

$$\operatorname{Ind}^{G(\mathbb{A})}_{P_{(m,m,\dots,m)}(\mathbb{A})} \left(\sigma|\det|^{\frac{k-1}{2}} \otimes \sigma|\det|^{\frac{k-3}{2}} \otimes \dots \otimes \sigma|\det|^{-\frac{k-1}{2}} \right)$$

has a unique irreducible constituent, denoted $J(k,\sigma)$, which is isomorphic to a summand in the spectral decomposition of the discrete spectrum of $G(\mathbb{A})$. For $k = 1$, we take $J(1,\sigma) = \sigma$. For $m = 1$ and σ a unitary Hecke character χ of $\mathbb{I}$, we have

$$J(n,\chi) \cong \chi \circ \det,$$

which is a character of $G(\mathbb{A})$. Conversely, every unitary discrete spectrum automorphic representation of $G(\mathbb{A})$ arises in this way.

A segment in the cuspidal support is the tensor product of the form

$$\sigma|\det|^{b}\otimes\sigma|\det|^{b-1}\otimes\cdots\otimes\sigma|\det|^{a+1}\otimes\sigma|\det|^{a}, \tag{3.1}$$

where σ is a unitary cuspidal automorphic representation of $GL_m(\mathbb{A})$, real numbers a and b are such that $b-a$ is a non-negative integer, and the exponents of $|\det|$ are decreasing from b to a by one, so that there are $b-a+1$ factors in the tensor product. We write

$$\Delta(\sigma,[a,b])$$

for the segment above. The number of factors $b-a+1$ is referred to as the length of the segment. According to [29], whenever there is a segment as above (up to permutation of factors) in the cuspidal support, there is a discrete spectrum representation of $GL_{(b-a+1)m}(\mathbb{A})$ isomorphic to

$$J(b-a+1,\sigma)|\det|^{\frac{a+b}{2}} \tag{3.2}$$

with the cuspidal support in that segment. We apply this observation many times in the arguments below.

We begin with a simple fact that given a cuspidal support π, the triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$ are parameterized by partitions of π into disjoint segments (allowing segments of length one).

Lemma 3.1. *Let $\pi\cong\pi_1|\det|^{s_1}\otimes\cdots\otimes\pi_k|\det|^{s_k}$ be a cuspidal automorphic representation of the Levi factor $L(\mathbb{A})$ of a standard parabolic subgroup P of G, where π_i are unitary cuspidal automorphic representations of the general linear groups of appropriate size, and $s_1\geq\cdots\geq s_k$ are real numbers. Then, the set $\mathcal{M}_{\{P\},\varphi(\pi)}$ is in finite-to-one correspondence with the set of all partitions of the cuspidal support π into segments. Given a partition into segments*

$$\Delta_i=\Delta(\sigma_i,[a_i,b_i]),$$

with $i=1,\ldots,l$, of the cuspidal support π, ordered in such a way that

$$\frac{a_i+b_i}{2}\geq\frac{a_{i+1}+b_{i+1}}{2},$$

for $i=1,\ldots,l-1$, and where σ_i is a cuspidal automorphic representation of $GL_{m_i}(\mathbb{A})$, a corresponding triple $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ is given as follows. The standard parabolic subgroup R corresponds to the ordered partition

$$((b_1-a_1+1)m_1,\ldots,(b_l-a_l+1)m_l),$$

the discrete spectrum representation Π of $L_R(\mathbb{A})$ is given as the tensor product

$$\Pi\cong J(b_1-a_1+1,\sigma_1)\otimes\cdots\otimes J(b_l-a_l+1,\sigma_l),$$

and the element $\underline{z}$ in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^G$ defined by R is given as

$$\underline{z}=\left(\frac{a_1+b_1}{2},\ldots,\frac{a_l+b_l}{2}\right).$$

Other triples $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$, corresponding to the same partition into segments, are obtained by permuting the consecutive segments for which the values of $\frac{a_i+b_i}{2}$ are equal.

Proof. The proof follows directly from the results of [29] mentioned above and the definition of the Franke filtration, in particular, the set of triples $\mathcal{M}_{\{P\},\varphi(\pi)}$, in Section 2. □

The main lemma below is concerned with the comparison, with respect to the partial order defining the filtration, of two triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$ of a special form. More precisely, we are interested in the triples such that one of them can be obtained from the other by making the union of two disjoint segments in the cuspidal support into a single larger segment.

Lemma 3.2. *Let $\pi \cong \pi_1|\det|^{s_1} \otimes \cdots \otimes \pi_k|\det|^{s_k}$ be a cuspidal automorphic representation of the Levi factor $L(\mathbb{A})$ of a standard parabolic subgroup P of G, where π_i are unitary cuspidal automorphic representations of the general linear groups of appropriate size, and $s_1 \geq \cdots \geq s_k$ are real numbers. Suppose that π contains a segment of the form*

$$\Delta(\sigma, [x-a-b+1, x]) = \sigma|\det|^{x} \otimes \sigma|\det|^{x-1} \otimes \cdots \otimes \sigma|\det|^{x-a-b+1},$$

of length $a+b$, where a and b are positive integers, x is a real number, and σ is a unitary cuspidal automorphic representation of $GL_m(\mathbb{A})$. Then, there exist triples $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ such that

- *R and R' are standard parabolic subgroups of G containing P corresponding, respectively, to the ordered partitions*
$$(m_1, \ldots, m_w, (a+b)m, m_{w+1}, \ldots, m_l)$$
and
$$(m_1, \ldots, m_u, am, m_{u+1}, \ldots, m_v, bm, m_{v+1}, \ldots, m_l)$$
of n, where $(m_1, \ldots, m_l)$ is an ordered partition of $n-(a+b)m$.
- *Π and Π' are (unitary) discrete spectrum representations of the Levi factors*
$$L_R \cong GL_{m_1} \times \cdots \times GL_{m_w} \times GL_{(a+b)m} \times GL_{m_{w+1}} \times \cdots \times GL_{m_l},$$
and
$$L_{R'} \cong GL_{m_1} \times \cdots \times GL_{m_u} \times GL_{am} \times GL_{m_{u+1}} \times \cdots \times GL_{m_v} \times GL_{bm} \times GL_{m_{v+1}} \times \cdots \times GL_{m_l}$$
given, respectively, as the tensor products
$$\Pi \cong \Pi_1 \otimes \cdots \otimes \Pi_w \otimes J(a+b, \sigma) \otimes \Pi_{w+1} \otimes \cdots \otimes \Pi_l,$$
and
$$\Pi' \cong \Pi_1 \otimes \cdots \otimes \Pi_u \otimes J(a, \sigma) \otimes \Pi_{u+1} \otimes \cdots \otimes \Pi_v \otimes J(b, \sigma) \otimes \Pi_{v+1} \otimes \cdots \otimes \Pi_l$$
where Π_i is a unitary discrete spectrum representation of $GL_{m_i}(\mathbb{A})$, for $i = 1, \ldots, l$.
- *$\underline{z}$ and $\underline{z}'$ are in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^G$ and $\check{\mathfrak{a}}_{R'}^G$ defined by R and R', respectively, given as*
$$\underline{z} = \left(z_1, \ldots, z_w, x - \frac{a+b-1}{2}, z_{w+1}, \ldots, z_l\right).$$
and
$$\underline{z}' = \left(z_1, \ldots, z_u, x - \frac{a-1}{2}, z_{u+1}, \ldots, z_v, x - a - \frac{b-1}{2}, z_{v+1}, \ldots, z_l\right),$$
where $z_1, \ldots, z_l$ are real numbers such that Π and Π' twisted by the character corresponding to $\underline{z}$ and $\underline{z}'$, respectively, are supported in the associate class $\varphi(\pi)$ represented by π.

The positions u, v and w of diagonal blocks arising from the given segment are determined by the non-decreasing property of the sequences $\underline{z}$ and $\underline{z}'$, i.e., by the condition

$$z_u \geq x - \frac{a-1}{2} > z_{u+1},$$

and similarly for v and w. The triples $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ satisfy

$$\iota(\underline{z}) \succ \iota(\underline{z}'),$$

with respect to the partial order defining the filtration.

Proof. The existence of the triples of the form as in the lemma is clear. The two triples correspond, in the sense of Lemma 3.1, to the same partition of the cuspidal support π into segments, except that for $(R, \Pi, \underline{z})$ the given segment $\Delta(\sigma, [x-a-b+1, x])$ is taken in the partition as a whole, and for $(R', \Pi', \underline{z}')$ it is split into two segments $\Delta(\sigma, [x-a+1, x])$ and $\Delta(\sigma, [x-a-b+1, x-a])$. It remains to show that $\iota(\underline{z}) \succ \iota(\underline{z}')$.

Since

$$x - \frac{a-1}{2} > x - \frac{a+b-1}{2} > x - a - \frac{b-1}{2}$$

the positions u, v and w satisfy

$$u \leq w \leq v,$$

with possible equalities, and allowing degenerate cases $u = 0$ and $v = l$.

Writing $\iota(\underline{z}) = (\zeta_1, \ldots, \zeta_n)$ and $\iota(\underline{z}') = (\zeta'_1, \ldots, \zeta'_n)$ in coordinates, we have

$$\zeta_i \leq \zeta'_i \quad \text{for } i = 1, \ldots, M_w + am,$$

and

$$\zeta_i \geq \zeta'_i \quad \text{for } i = M_w + am + 1, \ldots, n,$$

where $M_j = m_1 + \cdots + m_j$, $j = 1, \ldots, l$, and $M_0 = 0$. In more details,

- for $1 \leq i \leq M_u$, we have $\zeta_i = \zeta'_i$, because they are both equal to the same z_j with $1 \leq j \leq u$;
- for $M_u + 1 \leq i \leq M_w$, we have $\zeta_i = z_j$ for some j with $u+1 \leq j \leq w$, and either $\zeta'_i = x - \frac{a-1}{2}$ or $\zeta'_i = z_{j'}$ with $j \geq j'$, so that $\zeta_i \leq \zeta'_i = x - \frac{a-1}{2}$ by the definition of u in the first case, and $\zeta_i = z_j \leq z_{j'} = \zeta'_i$ by the non-increasing property of $\underline{z}$ and $\underline{z}'$ in the second case;
- for $M_w + 1 \leq i \leq M_w + am$, we have $\zeta_i = x - \frac{a+b-1}{2}$, and either $\zeta'_i = x - \frac{a-1}{2}$ or $\zeta'_i = z_j$ with $u + 1 \leq j \leq w$, so that $\zeta_i \leq \zeta'_i$ is obvious in the first case, and follows from the definition of w in the second case;
- for $M_w + am + 1 \leq i \leq M_w + (a+b)m$, we have $\zeta_i = x - \frac{a+b-1}{2}$, and either $\zeta'_i = z_j$ with $w + 1 \leq j \leq v$ or $\zeta'_i = x - a - \frac{b-1}{2}$, so that $\zeta_i \geq \zeta'_i$ by the definition of w in the first case, and $\zeta_i \geq \zeta'_i$ is obvious in the second case;
- for $M_w + (a+b)m + 1 \leq i \leq M_v + (a+b)m$, we have $\zeta_i = z_j$ with $w+1 \leq j \leq v$, and either $\zeta'_i = z_{j'}$ with $j \leq j'$ or $\zeta'_i = x - a - \frac{b-1}{2}$, so that $\zeta_i = z_j \geq z_{j'} = \zeta'_i$ by the non-increasing property of $\underline{z}$ and $\underline{z}'$ in the first case, and $\zeta_i = z_j \geq x - a - \frac{b-1}{2}$ by the definition of v in the second case;
- for $M_v + (a+b)m + 1 \leq i \leq n$, we have $\zeta_i = \zeta'_i$ as they are both equal to the same z_j with $v + 1 \leq j \leq l$.

Therefore, the partial sums of $\iota(\underline{z})$ and $\iota(\underline{z}')$ satisfy

$$\zeta_1 + \cdots + \zeta_j \leq \zeta_1' + \cdots + \zeta_j'$$

for $j = 1, \ldots, M_w + am$, because each summand satisfies $\zeta_i \leq \zeta_i'$ for $i = 1, \ldots, M_w + am$. On the other hand, since $\underline{z}$ and $\underline{z}'$ lie in $\check{\mathfrak{a}}_R^G$ and $\check{\mathfrak{a}}_{R'}^G$, respectively, we have

$$\zeta_1 + \cdots + \zeta_n = \zeta_1' + \cdots + \zeta_n' = 0.$$

Hence, the remaining partial sums satisfy

$$\zeta_1 + \cdots + \zeta_j = -(\zeta_{j+1} + \cdots + \zeta_n) \leq -(\zeta_{j+1}' + \cdots + \zeta_n') = \zeta_1' + \cdots + \zeta_j'$$

for $j = M_w + am + 1, \ldots, n$, because $\zeta_i \geq \zeta_i'$ for $i = M_w + am + 1, \ldots, n$. These inequalities of partial sums imply $\iota(\underline{z}) \succ \iota(\underline{z}')$ as required. □

4. The Franke filtration in the case of the cuspidal support of a residual representation

In this section we consider the space of automorphic forms on $G(\mathbb{A})$ with the cuspidal support of a residual automorphic representation of $G(\mathbb{A})$. The goal is to show that in that case the Franke filtration can be arranged in such a way that the degenerate Eisenstein series contribute to the same quotient of the filtration if and only if they are arising from the parabolic subgroups of the same rank. This is in line with the Remark 2 in [11, page 242], and this fact is probably the point meant by Franke in that remark.

Let $n = km$, and let σ be a unitary cuspidal automorphic representation of $GL_m(\mathbb{A})$. The following theorem describes the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module of automorphic forms on $G(\mathbb{A})$, with the cuspidal support of the residual representation $J(k, \sigma)$, in the notation of Section 3.

Theorem 4.1. *Let $P = P_{(m,\ldots,m)}$ be the standard parabolic subgroup of G corresponding to the ordered partition $(m, \ldots, m)$ of n, where $n = km$. Let*

$$\pi \cong \sigma|\det|^{\frac{k-1}{2}} \otimes \sigma|\det|^{\frac{k-3}{2}} \otimes \cdots \otimes \sigma|\det|^{-\frac{k-1}{2}}$$

be a cuspidal automorphic representation of the Levi factor $L(\mathbb{A}) = L_{(m,\ldots,m)}(\mathbb{A})$, where σ is a unitary cuspidal automorphic representation of $GL_m(\mathbb{A})$. The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{P\},\varphi(\pi)}$ of automorphic forms on $G(\mathbb{A})$ with the cuspidal support in the associate class of π can be defined as the k-step filtration

$$\mathcal{A}_{\{P\},\varphi(\pi)} = \mathcal{A}^0_{\{P\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{P\},\varphi(\pi)} \supsetneqq \cdots \supsetneqq \mathcal{A}^{k-1}_{\{P\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\mathcal{A}^{i-1}_{\{P\},\varphi(\pi)}/\mathcal{A}^{i}_{\{P\},\varphi(\pi)} \cong \bigoplus_{\substack{(R,\Pi,\underline{z})\in\mathcal{M}_{\{P\},\varphi(\pi)} \text{ such that} \\ \text{the relative rank of } R \text{ is } k-i}} I(\underline{z}, \Pi) \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, for $i = 1, \ldots, k$, where we take $\mathcal{A}^k_{\{P\},\varphi(\pi)} = \{0\}$, and the notation at the right-hand side is as in Section 2.

Proof. The cuspidal support as in the theorem, which is the cuspidal support of a residual representation of $G(\mathbb{A})$, is the segment

$$\Delta = \Delta\left(\sigma, \left[-\frac{k-1}{2}, \frac{k-1}{2}\right]\right).$$

Hence, according to Lemma 3.1, the triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$ are in this case in one-to-one correspondence with partitions of the segment Δ into disjoint subsegments, which are in one-to-one correspondence with ordered partitions of k into positive integers. More precisely, Lemma 3.1 implies that the ordered partition $(k_1, \dots, k_l)$ of k into positive integers corresponds to the triple $(R, \Pi, \underline{z}) \in \mathcal{M}_{\{P\},\varphi(\pi)}$ such that

- R is the standard parabolic subgroup of G corresponding to the ordered partition
$$(k_1 m, k_2 m, \dots, k_l m)$$
of n into positive integers;
- Π is the residual automorphic representation of the Levi factor of R isomorphic to
$$J(k_1, \sigma) \otimes J(k_2, \sigma) \otimes \cdots \otimes J(k_l, \sigma);$$
- $\underline{z} = (z_1, \dots, z_l)$ is the element of the space $\check{\mathfrak{a}}_R^G$ given by
$$z_i = \frac{k - K_{i-1} - K_i}{2},$$
where $K_j = k_1 + \cdots + k_j$, for $j = 1, \dots, l$, and we set $K_0 = 0$.

Observe that the relative rank of R equals $l-1$, where l is, as above, the number of subsegments, and that $\underline{z}$ is in the open positive Weyl chamber in $\check{\mathfrak{a}}_R^G$, so that the correspondence of Lemma 3.1 really is one-to-one in this case.

The crucial new ingredient in the proof is Lemma 4.2 below, which shows that if two triples $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ are such that R and R' are of the same (relative) rank, then $\iota(\underline{z})$ and $\iota(\underline{z}')$ are incomparable in the partial order of the Franke filtration defined in Section 2. Lemma 4.2 is proved below, and for the moment we take it for granted and finish the proof of the theorem.

However, we first show that if $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ are such that R is of lower relative rank than R', then it is not possible that $\iota(\underline{z}') \succ \iota(\underline{z})$. Suppose the contrary, i.e., R is of lower relative rank than R' and $\iota(\underline{z}') \succ \iota(\underline{z})$ holds. But then, repeatedly dividing the subsegments of $(R, \Pi, \underline{z})$ into disjoint union of two subsegments in any way, and applying Lemma 3.2, we can always end up with the triple $(R'', \Pi'', \underline{z}'')$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ such that R'' is of the same relative rank as R', and $\iota(\underline{z}) \succ \iota(\underline{z}'')$. By the transitivity property of the partial order, it would follow that $\iota(\underline{z}') \succ \iota(\underline{z}'')$, which is a contradiction with Lemma 4.2, because R' and R'' are of the same relative rank.

Hence, we may define the function $T_{\{P\},\varphi(\pi)}$ for the Franke filtration in terms of relative rank of R. Given any triple $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{P\},\varphi(\pi)}$ of relative rank $k-i$, where $i = 1, \dots, k$, we define

$$T_{\{P\},\varphi(\pi)}(\iota(\underline{z})) = i - 1.$$

This is possible, because triples with R of the same relative rank are incomparable by Lemma 4.2, and thus can be assigned the same value, while triples with R of different relative rank are ordered according to their relative ranks, where lower relative rank should be assigned higher values as proved above.

The general form of the quotients of the Franke filtration, given in (2.4), with $T_{\{P\},\varphi(\pi)}$ as just defined, now implies the theorem. Namely, the direct sum ranging over the relative rank of R in (2.4) contains only one summand, because all R in the triples with the same value of $T_{\{P\},\varphi(\pi)}$ are of the same relative rank. The colimit in (2.4) is, in fact, the direct sum, because the category does not contain any non-trivial morphisms as all $\underline{z}$ are in the appropriate open positive Weyl chamber. □

Lemma 4.2. *In the notation of Theorem 4.1 and its proof, if the triples* $(R,\Pi,\underline{z})$ *and* $(R',\Pi',\underline{z}')$ *in* $\mathcal{M}_{\{P\},\varphi(\pi)}$ *are such that* R *and* R' *are of the same relative rank, then* $\iota(\underline{z})$ *and* $\iota(\underline{z}')$ *are incomparable.*

Proof. The intuition behind this proof is that two broken lines inscribed in a convex curve, which is a parabola in our case, with the same endpoints, do not intersect away from the vertices only if one of their sets of vertices includes the other.

As explained in the proof of Theorem 4.1, the triples in $\mathcal{M}_{\{P\},\varphi(\pi)}$ are, according to Lemma 3.1, in one-to-one correspondence with ordered partitions of k into positive integers. Let $(R,\Pi,\underline{z})$ and $(R',\Pi',\underline{z}')$ correspond, respectively, to the ordered partitions

$$(k_1,\dots,k_l) \quad \text{and} \quad (k'_1,\dots,k'_l)$$

of k, where the lengths of the partitions are equal due to the fact that R and R' are of the same relative rank. Let

$$K_j = k_1+\dots+k_j \quad \text{and} \quad K'_j = k'_1+\dots+k'_j,$$

for $j=1,\dots,l$, so that $K_l = K'_l = k$, and we set $K_0 = K'_0 = 0$. Then, as in the proof of Theorem 4.1, writing $\underline{z}=(z_1,\dots,z_l)$ and $\underline{z}'=(z'_1,\dots,z'_l)$, we have

$$z_i = \frac{k-K_{i-1}-K_i}{2} \quad \text{and} \quad z'_i = \frac{k-K'_{i-1}-K'_i}{2},$$

for $i=1,\dots,l$. Let

$$N_j = K_j m = k_1 m+\dots+k_j m \quad \text{and} \quad N'_j = K'_j m = k'_1 m+\dots+k'_j m,$$

for $j=0,1,\dots,l$, so that $N_l = N'_l = n$, and $N_0 = N'_0 = 0$.

Since the partitions corresponding to the two triples are different, but of the same length, there exists an integer j_0 such that $0<j_0<l$ and N_{j_0} is different from all N'_j with $j=1,\dots,l$. Such j_0 may not be unique, and we fix any of the possible choices. Then, for the fixed choice of j_0, there exist unique integers i'_0 and j'_0 such that

$$N_{j_0} = N'_{j'_0} + i'_0, \quad \text{and } 0<i'_0<N'_{j'_0+1}-N'_{j'_0} = k'_{j'_0+1}m,$$

that is, N_{j_0} is strictly between $N'_{j'_0}$ and $N'_{j'_0+1}$.

Writing in coordinates $\iota(\underline{z})=(\zeta_1,\dots,\zeta_n)$ and $\iota(\underline{z}')=(\zeta'_1,\dots,\zeta'_n)$, we have

$$\zeta_i = z_j, \quad \text{for } i=N_{j-1}+1,\dots,N_j \text{ and } j=1,\dots,l,$$

$$\zeta'_i = z'_j, \quad \text{for } i=N'_{j-1}+1,\dots,N'_j \text{ and } j=1,\dots,l.$$

Consider the N_{j_0}-th partial sums of these two sequences. For $\iota(\underline{z})$, we obtain

$$\begin{aligned}
\zeta_1+\cdots+\zeta_{N_{j_0}} &= N_1z_1+(N_2-N_1)z_2+\cdots+(N_{j_0}-N_{j_0-1})z_{j_0} \\
&= m\left(k_1z_1+k_2z_2+\cdots+k_{j_0}z_{j_0}\right) \\
&= m\left[\left(\frac{k+1}{2}-1\right)+\left(\frac{k+1}{2}-2\right)+\cdots+\left(\frac{k+1}{2}-K_{j_0}\right)\right] \\
&= mK_{j_0}\cdot\frac{k-K_{j_0}}{2} \\
&= N_{j_0}\cdot\frac{n-N_{j_0}}{2m} \\
&= \left(N'_{j'_0}+i'_0\right)\cdot\frac{n-N'_{j'_0}-i'_0}{2m} \\
&= \frac{\left(N'_{j'_0}+i'_0\right)\cdot\left(n-N'_{j'_0}\right)}{2m}-\frac{i'_0\left(N'_{j'_0}+i'_0\right)}{2m},
\end{aligned}$$

where the sum $k_1z_1+k_2z_2+\cdots+k_{j_0}z_{j_0}$ in the second line equals exactly the sum of the first K_{j_0} exponents in the cuspidal support, and at the end of the calculation we used the relation $N_{j_0}=N'_{j'_0}+i'_0$. On the other hand, for $\iota(\underline{z}')$, the N_{j_0}-th partial sum equals

$$\begin{aligned}
\zeta'_1+\cdots+\zeta'_{N_{j_0}} &= \zeta'_1+\cdots+\zeta'_{N'_{j'_0}+i'_0} \\
&= N'_1z'_1+(N'_2-N'_1)z'_2+\cdots+(N'_{j'_0}-N'_{j'_0-1})z'_{j'_0}+i'_0z'_{j'_0+1} \\
&= m(k'_1z'_1+k'_2z'_2+\cdots+k'_{j'_0}z'_{j'_0})+i'_0z'_{j'_0+1} \\
&= m\left[\left(\frac{k+1}{2}-1\right)+\left(\frac{k+1}{2}-2\right)+\cdots+\left(\frac{k+1}{2}-K'_{j'_0}\right)\right]+i'_0\cdot\frac{k-K'_{j'_0}-K'_{j'_0+1}}{2} \\
&= mK'_{j'_0}\cdot\frac{k-K'_{j'_0}}{2}+i'_0\cdot\frac{k-K'_{j'_0}-K'_{j'_0+1}}{2} \\
&= N'_{j'_0}\cdot\frac{n-N'_{j'_0}}{2m}+i'_0\cdot\frac{n-N'_{j'_0}-N'_{j'_0+1}}{2m} \\
&= \frac{\left(N'_{j'_0}+i'_0\right)\cdot\left(n-N'_{j'_0}\right)}{2m}-\frac{i'_0N'_{j'_0+1}}{2m},
\end{aligned}$$

where again the sum $k'_1 z'_1 + k'_2 z'_2 + \cdots + k'_{j'_0} z'_{j'_0}$ equals the sum of the first $K'_{j'_0}$ exponents in the cuspidal support.[3] As $N_{j_0} = N'_{j'_0} + i'_0 < N'_{j'_0+1}$, we have that the strict inequality

$$\zeta_1 + \zeta_2 + \cdots + \zeta_{N_{j_0}} > \zeta'_1 + \zeta'_2 + \cdots + \zeta'_{N_{j_0}}$$

of N_{j_0}-th partial sums holds.

However, interchanging the roles of the two triples, the same argument as above shows that there exist partial sums of $\iota(\underline{z})$ and $\iota(\underline{z}')$ for which the opposite strict inequality holds. Therefore, $\iota(\underline{z})$ and $\iota(\underline{z}')$ are indeed incomparable as claimed. □

5. The Franke filtration in the case of degenerate Eisenstein series of the same rank contributing to different quotients of the filtration

In this section we consider some examples of spaces of automorphic forms such that the degenerate Eisenstein series, arising from parabolic subgroups of the same rank, contribute to different quotients of the Franke filtration. These are the first examples showing that the claim of Remark 2 of Franke [11, page 242] cannot be always achieved, i.e., the filtration cannot be defined based only on the rank of the degenerate Eisenstein series.

5.1. **The lowest rank example.** We begin with the example of the lowest possible rank in which the considered phenomenon occurs. We take $n = 4$, that is, the case of the general linear group GL_4 of rank three. Let $\mathcal{A} = \mathcal{A}(GL_4(F)\backslash GL_4(\mathbb{A}), \omega)$ be the $(\mathfrak{g}_\infty, K_\infty; GL_4(\mathbb{A}_f))$-module of automorphic forms on $GL_4(\mathbb{A})$ with the central character ω.

Let π be a cuspidal automorphic representation of $T(\mathbb{A}) \cong \mathbb{I}\times\mathbb{I}\times\mathbb{I}\times\mathbb{I}$ given as the tensor product

$$\pi \cong \chi|\,| \otimes \chi \otimes \chi \otimes \chi|\,|^{-1},$$

where χ is a unitary Hecke character of the group of idèles $\mathbb{I}$ such that $\chi^4 = \omega$. In other words, π is the unitary cuspidal automorphic representation $\chi\otimes\chi\otimes\chi\otimes\chi$ of $T(\mathbb{A})$ twisted by the character of $T(\mathbb{A})$ corresponding to the element $(1,0,0,-1) \in \check{\mathfrak{a}}_B^{GL_4}$. The Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$ is explicitly described in the following theorem.

Theorem 5.1. *The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; GL_4(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms on $GL_4(\mathbb{A})$ supported in the associate class $\varphi(\pi)$, represented by a cuspidal automorphic representation*

$$\pi \cong \chi|\,| \otimes \chi \otimes \chi \otimes \chi|\,|^{-1}$$

of $T(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$, is the length four filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^3_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

[3] The first named author discovered that their little daughter Lana, when she was 11 months and 5 days old, typed at this place in the text the following lines:

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/822340,332
2
2

Although nobody knows the meaning of these lines, it must be very deep and extremely important. We strongly believe that the understanding of these lines would lead to the full understanding of the Langlands program.

where the quotients of the filtration are isomorphic to

$$\begin{aligned}
\mathcal{A}^3_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_4(\mathbb{A})}_{P_{(3,1)}(\mathbb{A})}\left((\chi\circ\det)\otimes\chi\right)\otimes S\left(\check{\mathfrak{a}}^{GL_4}_{P_{(3,1)},\mathbb{C}}\right)\\
\mathcal{A}^2_{\{B\},\varphi(\pi)}/\mathcal{A}^3_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_4(\mathbb{A})}_{P_{(2,2)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{1/2}\otimes(\chi\circ\det)|\det|^{-1/2}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_4}_{P_{(2,2)},\mathbb{C}}\right)\\
\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_4(\mathbb{A})}_{P_{(2,1,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{1/2}\otimes\chi\otimes\chi|\,|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_4}_{P_{(2,1,1)},\mathbb{C}}\right)\\
&\qquad\bigoplus \mathrm{Ind}^{GL_4(\mathbb{A})}_{P_{(1,1,2)}(\mathbb{A})}\left(\chi|\,|\otimes\chi\otimes(\chi\circ\det)|\det|^{-1/2}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_4}_{P_{(1,1,2)},\mathbb{C}}\right)\\
\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \left(\mathrm{Ind}^{GL_4(\mathbb{A})}_{B(\mathbb{A})}\left(\chi|\,|\otimes\chi\otimes\chi\otimes\chi|\,|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_4}_{B,\mathbb{C}}\right)\right)^+
\end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; GL_4(\mathbb{A}_f))$-modules. The exponent $+$ on the right-hand side of the last quotient refers to the $+1$-eigenspace for the action of the non-trivial intertwining operator obtained using the functor $M_{\{B\},\varphi(\pi)}$ from the unique non-trivial automorphism of the corresponding triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$.

Proof. The Franke filtration is defined in terms of triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ and their morphisms, as explained in Section 2. To construct those triples, we start with the cuspidal support. It gives the triple

$$(B, \chi\otimes\chi\otimes\chi\otimes\chi, (1,0,0,-1)).$$

According to [29], as explained in Section 3, the discrete spectrum representations of Levi factors supported in $\varphi(\pi)$ are determined by the segments appearing in the cuspidal support. Besides segments of length one, these are

$$\begin{aligned}
&\chi|\,|\otimes\chi,\\
&\chi\otimes\chi|\,|^{-1},\\
&\chi|\,|\otimes\chi\otimes\chi|\,|^{-1}.
\end{aligned}$$

Using these segments, we may form five more triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$. More precisely,

$$\begin{aligned}
\mathcal{M}_{\{B\},\varphi(\pi)} = \{\ &(B, \chi\otimes\chi\otimes\chi\otimes\chi, (1,0,0,-1))\,,\\
&\left(P_{(2,1,1)}, (\chi\circ\det)\otimes\chi\otimes\chi, (1/2,0,-1)\right),\\
&\left(P_{(1,1,2)}, \chi\otimes\chi\otimes(\chi\circ\det), (1,0,-1/2)\right),\\
&\left(P_{(2,2)}, (\chi\circ\det)\otimes(\chi\circ\det), (1/2,-1/2)\right),\\
&\left(P_{(3,1)}, (\chi\circ\det)\otimes\chi, (0,0)\right),\\
&\left(P_{(1,3)}, \chi\otimes(\chi\circ\det), (0,0)\right)\ \}\,.
\end{aligned}$$

The only morphisms between these triples is the automorphism of the first triple given by the transposition of the inner two factors, and the isomorphisms between the last two triples interchanging the two factors.

The last entries of the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, viewed as elements in $\check{\mathfrak{a}}^{GL_4}_B$, give the set

$$\begin{aligned}
S_{\{B\},\varphi(\pi)} = \{\,&(1,0,0,-1), (1/2,1/2,0,-1), (1,0,-1/2,-1/2),\\
&(1/2,1/2,-1/2,-1/2), (0,0,0,0)\ \}\,.
\end{aligned}$$

The function $T_{\{B\},\varphi(\pi)}$ on the set $S_{\{B\},\varphi(\pi)}$ is chosen as in Table 5.1. The partial sums required for making a good choice of $T_{\{B\},\varphi(\pi)}$ are given in the same table. The inequalities between partial

R	Π	$\underline{z}$	$\iota(\underline{z}) = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$	ζ_1	$\zeta_1 + \zeta_2$	$\zeta_1 + \zeta_2 + \zeta_3$	$T_{\{B\},\varphi(\pi)}(\iota(\underline{z}))$
B	$\chi \otimes \chi \otimes \chi \otimes \chi$	$(1, 0, 0, -1)$	$(1, 0, 0, -1)$	1	1	1	0
$P_{(2,1,1)}$	$(\chi \circ \det) \otimes \chi \otimes \chi$	$(1/2, 0, -1)$	$(1/2, 1/2, 0, -1)$	1/2	1	1	1
$P_{(1,1,2)}$	$\chi \otimes \chi \otimes (\chi \circ \det)$	$(1, 0, -1/2)$	$(1, 0, -1/2, -1/2)$	1	1	1/2	1
$P_{(2,2)}$	$(\chi \circ \det) \otimes (\chi \circ \det)$	$(1/2, -1/2)$	$(1/2, 1/2, -1/2, -1/2)$	1/2	1	1/2	2
$P_{(3,1)}$	$(\chi \circ \det) \otimes \chi$	$(0, 0)$	$(0, 0, 0, 0)$	0	0	0	3
$P_{(1,3)}$	$\chi \otimes (\chi \circ \det)$	$(0, 0)$	$(0, 0, 0, 0)$	0	0	0	3

TABLE 5.1. The definition of $T_{\{B\},\varphi(\pi)}(R, \Pi, \underline{z})$ along with the partial sums for $\iota(\underline{z})$ required for comparison. The sum $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$ is omitted, because it is always zero due to condition $\underline{s} \in \check{\mathfrak{a}}_B^{GL_4}$.

sums as in (2.2) impose conditions on values of $T_{\{B\},\varphi(\pi)}$, and we choose for those values consecutive integers from 0 to 3. The two triples with the value 1 of $T_{\{B\},\varphi(\pi)}$ are in fact incomparable, and could be given two different values. Our choice of $T_{\{B\},\varphi(\pi)}$ puts them in the same quotient of the filtration, but they form a direct sum, so that they could make two consecutive quotients in any order.

Finally, having found the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ and their morphisms, and having defined the function $T_{\{B\},\varphi(\pi)}$, we are in position to describe the quotients of the Franke filtration in this example. According to equation (2.4), the quotients of the filtration are certain colimits of induced representations obtained from the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$. For the quotients $\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)}$ and $\mathcal{A}^2_{\{B\},\varphi(\pi)}/\mathcal{A}^3_{\{B\},\varphi(\pi)}$, the claim of the theorem easily follows, as there are no non-trivial isomorphisms between the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with values 1 and 2 of the function $T_{\{B\},\varphi(\pi)}$, so that the colimit is just a direct sum. For the space $\mathcal{A}^3_{\{B\},\varphi(\pi)}$, there are two triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with value 3 of the function $T_{\{B\},\varphi(\pi)}$, and there is a unique isomorphism between them, so that the colimit is isomorphic to one of them. This gives the claim for $\mathcal{A}^3_{\{B\},\varphi(\pi)}$. Finally, for the quotient $\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)}$, there is a unique triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with value 0 of the function $T_{\{B\},\varphi(\pi)}$, but it has a non-trivial automorphism. Hence, the colimit in this case is the $+1$-eigenspace of the intertwining operator obtained from that non-trivial automorphism by applying the functor $M_{\{B\},\varphi(\pi)}$, as in the statement of the theorem. See also [27] or [14] for the calculation of these colimits. □

The main point of the example of the Franke filtration in Theorem 5.1 is that there are two different quotients of the filtration arising from degenerate Eisenstein series associated to discrete spectrum automorphic representations of the Levi factors of parabolic subgroups of the same rank. More precisely, the quotient $\mathcal{A}^3_{\{B\},\varphi(\pi)}$ arises from the degenerate Eisenstein series, associated to the representation $(\chi \circ \det) \otimes \chi$ of the Levi factor $L_{(3,1)}(\mathbb{A}) \cong GL_3(\mathbb{A}) \times \mathbb{I}$ of relative rank one, at the value zero of its complex parameter. This Eisenstein series is holomorphic at the value zero. On the other hand, the quotient $\mathcal{A}^2_{\{B\},\varphi(\pi)}/\mathcal{A}^3_{\{B\},\varphi(\pi)}$ arises from the degenerate Eisenstein series, associated to the representation $(\chi \circ \det) \otimes (\chi \circ \det)$ of the Levi factor $L_{(2,2)}(\mathbb{A}) \cong GL_2(\mathbb{A}) \times GL_2(\mathbb{A})$ also of relative rank one, at the value $(1/2, -1/2)$ of its complex parameter. By direct calculation, or invoking [20, Thm. 5-2], we see that this Eisenstein series has a simple pole at the value $(1/2, -1/2)$, but the residues are not square-integrable. That is the underlying reason for the need of another degenerate Eisenstein series, coming from a parabolic subgroup of the same rank, contributing to a lower quotient of the filtration.

5.2. **More general results.** The cases considered here are a generalization of the previous example to the case of $n = 2m \geq 4$, that is, the general linear group $G = GL_{2m}$ of odd rank $2m-1$. We consider the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of the automorphic forms supported in the associate class $\varphi(\pi)$ of cuspidal automorphic representations of the torus $T(\mathbb{A}) \cong \mathbb{I} \times \cdots \times \mathbb{I}$, where $\mathbb{I}$ appears $2m$ times as a factor, represented by

$$\pi \cong \chi|\,|^{m-1} \otimes \chi|\,|^{m-2} \otimes \cdots \otimes \chi|\,| \otimes \chi \otimes \chi \otimes \chi|\,|^{-1} \otimes \cdots \otimes \chi|\,|^{-(m-2)} \otimes \chi|\,|^{-(m-1)},$$

where χ is a unitary Hecke character of $\mathbb{I}$. In other words, π is the unitary character $\chi \otimes \cdots \otimes \chi$ of $T(\mathbb{A})$, twisted by the character of $T(\mathbb{A})$ corresponding to

$$(m-1, m-2, \ldots, 1, 0, 0, -1, \ldots, -(m-2), -(m-1)) \in \check{\mathfrak{a}}_B^G.$$

As above, we consider the automorphic forms of the fixed central character ω, so that χ should be such that $\chi^{2m} = \omega$. The following theorem describes the part of the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$ in which the considered phenomenon occurs. The rest of the filtration is not described explicitly, because the description is combinatorially demanding and does not provide further insight in the considered phenomenon.

Theorem 5.2. *The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of the automorphic forms on $G(\mathbb{A})$ supported in the associate class $\varphi(\pi)$, represented by a cuspidal automorphic representation*

$$\pi \cong \chi|\,|^{m-1} \otimes \chi|\,|^{m-2} \otimes \cdots \otimes \chi|\,| \otimes \chi \otimes \chi \otimes \chi|\,|^{-1} \otimes \cdots \otimes \chi|\,|^{-(m-2)} \otimes \chi|\,|^{-(m-1)},$$

of $T(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$, is of the form

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneq \cdots \supsetneq \mathcal{A}^{\ell-1}_{\{B\},\varphi(\pi)} \supsetneq \mathcal{A}^{\ell}_{\{B\},\varphi(\pi)} \supsetneq \{0\},$$

where the last two quotients of the filtration are isomorphic to

$$\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)} \cong \mathrm{Ind}_{P_{(2m-1,1)}(\mathbb{A})}^{G(\mathbb{A})} \left((\chi \circ \det) \otimes \chi\right) \otimes S\left(\check{\mathfrak{a}}^G_{P_{(2m-1,1)},\mathbb{C}}\right)$$

$$\mathcal{A}^{\ell-1}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)} \cong \mathrm{Ind}_{P_{(m,m)}(\mathbb{A})}^{G(\mathbb{A})} \left((\chi \circ \det)|\det|^{\frac{m-1}{2}} \otimes (\chi \circ \det)|\det|^{-\frac{m-1}{2}}\right) \otimes S\left(\check{\mathfrak{a}}^G_{P_{(m,m)},\mathbb{C}}\right)$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. The length $\ell + 1$ of the filtration is not given explicitly, as it depends on certain choices in the definition of the filtration.

Proof. As in the proof of Theorem 5.1, we first need to find the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, and this boils down to finding all segments in the cuspidal support

$$\pi \cong \chi|\,|^{m-1} \otimes \chi|\,|^{m-2} \otimes \cdots \otimes \chi|\,| \otimes \chi \otimes \chi \otimes \chi|\,|^{-1} \otimes \cdots \otimes \chi|\,|^{-(m-2)} \otimes \chi|\,|^{-(m-1)}.$$

On the other hand, according to Lemma 3.1, this task is reduced to finding all partitions of the sequence of exponents

$$(m-1, m-2, \ldots, 1, 0, 0, -1, \ldots, -(m-2), -(m-1))$$

into subsequences of consecutive integers.

Since our aim is to describe only the two deepest quotients of the filtration, we will first find all the triples $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with R a maximal proper parabolic subgroup, and then show that the remaining triples are either incomparable or contribute to shallower filtration steps than those with a maximal parabolic subgroup R. Observe that there is no residual automorphic representation of the full group $G(\mathbb{A})$ supported in $\varphi(\pi)$, so that there is no triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with $R = G$.

The discrete spectrum automorphic representation of the Levi factor of a maximal proper parabolic subgroup corresponds to a partition of the sequence of exponents of the cuspidal support into two subsequences. Since 0 in the only integer appearing twice in the sequence of exponents, it should appear once in both subsequences. Then the remaining exponents can be divided between the two subsequences of consecutive integers in only two ways: either all of them belong to one of the subsequences, or the positive exponents belong to one of the subsequences and the negative exponents to the other. Hence, the triples $(R, \Pi, \underline{z})$, with R a maximal parabolic subgroup, are given as follows

$$\begin{aligned}
&\big(P_{(2m-1,1)}, (\chi \circ \det) \otimes \chi, (0,0)\big) \\
&\big(P_{(1,2m-1)}, \chi \otimes (\chi \circ \det), (0,0)\big) \\
&\left(P_{(m,m)}, (\chi \circ \det) \otimes (\chi \circ \det), \left(\frac{m-1}{2}, -\frac{m-1}{2}\right)\right)
\end{aligned}$$

and the only non-trivial morphisms between them are the isomorphisms between the first two triples, given by the interchange of the two factors.

For these three triples the inclusion $\iota(\underline{z})$ of $\underline{z}$ into $\check{\mathfrak{a}}_B^G$ is given as

$$\begin{aligned}
\iota(0,0) &= (0, \dots, 0), \\
\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right) &= \left(\frac{m-1}{2}, \dots, \frac{m-1}{2}, -\frac{m-1}{2}, \dots, -\frac{m-1}{2}\right),
\end{aligned}$$

where $\frac{m-1}{2}$ and $-\frac{m-1}{2}$ in the second line both appear m times.

Let $(R, \Pi, \underline{z})$ be any triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ Write $\iota(\underline{z}) = (\zeta_1, \dots, \zeta_n) \in \check{\mathfrak{a}}_B^G$ in coordinates. Since $\underline{z}$ is lying in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^G$, we have

$$\zeta_1 \geq \cdots \geq \zeta_n \quad \text{and} \quad \zeta_1 + \cdots + \zeta_n = 0.$$

Thus, $\zeta_1 \geq 0$ and $\zeta_n \leq 0$, and the partial sums $\zeta_1 + \cdots + \zeta_j$ are first increasing starting with $\zeta_1 \geq 0$ and then decreasing until reaching zero at the end. This implies that all the partial sums $\zeta_1 + \cdots + \zeta_j$ are non-negative, and at least one of them is positive, unless $\iota(\underline{z}) = (0, \dots, 0)$ which may happen for the considered cuspidal support only if $R = P_{(2m-1,1)}$ or $R = P_{(1,2m-1)}$. In other words, triples with $\iota(\underline{z}) = (0, \dots, 0)$ contribute to the deepest quotient of the filtration, and we may define

$$T_{\{B\},\varphi(\pi)}(0, \dots, 0) = \ell,$$

for some sufficiently large positive integer ℓ, and then necessarily require that $T_{\{B\},\varphi(\pi)}(\iota(\underline{z})) < \ell$ for the third entry $\underline{z}$ of all $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ other than those two with $R = P_{(2m-1,1)}$ and $R = P_{(1,2m-1)}$.

We now consider a triple $(R', \Pi', \underline{z}')$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ such that R' is not a maximal parabolic subgroup. Writing again $\iota(\underline{z}') = (\zeta_1', \dots, \zeta_n')$ in coordinates, we have

$$\zeta_1' \geq \cdots \geq \zeta_n' \quad \text{and} \quad \zeta_1' + \cdots + \zeta_n' = 0.$$

Then this triple corresponds to a partition of the sequence of exponents of the cuspidal support in more than two subsequences of consecutive integers. If the subsequence containing the largest exponent $m-1$ ends with some integer $l>0$, then the first entry of $\iota(\underline{z}')$ is given as in (3.2) by

$$\zeta'_1 = \frac{(m-1)+l}{2} > \frac{m-1}{2},$$

so that $\iota(\underline{z}')$ is either incomparable to $\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)$, or

$$\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right) \succ \iota(\underline{z}')$$

On the other hand, the same conclusion is obtained if the sequence containing the least exponent $-(m-1)$ starts with an integer $-l$, where $l>0$, because then the partial sum

$$\zeta'_1 + \cdots + \zeta'_{n-1} = -\zeta'_n = -\frac{-(m-1)-l}{2} = \frac{(m-1)+l}{2} > \frac{m-1}{2}.$$

Hence, the only remaining possibility is that the subsequence which contains $m-1$ also contains 0, and the same for $-(m-1)$. But then, there are only two subsequences, and that cannot happen since R' is not maximal. In this way, we have proved that, for any triple $(R', \Pi', \underline{z}')$ with R' not maximal, $\iota(\underline{z}')$ is either incomparable to $\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)$, or

$$\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right) \succ \iota(\underline{z}').$$

Hence, we may define

$$T_{\{B\},\varphi(\pi)}\left(\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)\right) = \ell - 1$$

and require $T_{\{B\},\varphi(\pi)}(\iota(\underline{z}')) < \ell - 1$ for the third entry $\underline{z}'$ of all $(R', \Pi', \underline{z}')$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ such that R' is not maximal. The freedom of choice for $T_{\{B\},\varphi(\pi)}$ would allow $T_{\{B\},\varphi(\pi)}(\iota(\underline{z}')) = \ell - 1$ for $\iota(\underline{z}')$ incomparable with $\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)$, but we choose that $T_{\{B\},\varphi(\pi)}$ takes value $\ell - 1$ only at $\iota\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)$. Otherwise, the filtration quotient $\mathcal{A}^{\ell-1}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$ would have more than just one direct summand indicated in the statement of the theorem.

Finally, according to (2.4), the module $\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$ is isomorphic to the colimit of the two induced representations given by triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with $R = P_{(2m-1,1)}$ and $R = P_{(1,2m-1)}$. Since there is a unique non-trivial isomorphism between them, the colimit is isomorphic to one of these representations, as claimed in the theorem. With our choice of $T_{\{B\},\varphi(\pi)}$, there is only one triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$ contributing to the quotient $\mathcal{A}^{\ell-1}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$, with no non-trivial automorphisms. It is the triple with $R = P_{(m,m)}$, and the quotient is isomorphic to the induced representation coming from that triple, as claimed. □

The main point of Theorem 5.2 is that we again have two different quotients of the filtration arising from the degenerate Eisenstein series associated to discrete spectrum automorphic representations of the Levi factors of parabolic subgroups of the same rank. The degenerate Eisenstein series contributing to $\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$ is associated to the representation $(\chi \circ \det) \otimes \chi$ of the Levi factor $L_{(2m-1,1)}(\mathbb{A})$ of the relative rank one parabolic subgroup $P_{(2m-1,1)}$. It is holomorphic at the point of interest, which is $(0,0)$. The quotient $\mathcal{A}^{\ell-1}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$ is determined by the degenerate Eisenstein series associated to the representation $(\chi \circ \det) \otimes (\chi \circ \det)$ of the Levi factor $L_{(m,m)}(\mathbb{A})$ of another relative rank one parabolic subgroup $P_{(m,m)}$. According to [20, Thm. 5-2], this Eisenstein series has a simple pole at the point $\left(\frac{m-1}{2}, -\frac{m-1}{2}\right)$, but the residues are not square-integrable. As in the previous low rank example, this non-square-integrability of residues is the reason for these two degenerate Eisenstein series, although associated to the parabolic subgroups of the same rank, cannot contribute to the same quotient of the filtration.

6. The Franke filtration in the case of degenerate Eisenstein series of higher relative rank contributing to deeper quotients of the filtration

In this section, we consider cases in which ordering of contributions to the quotients of the Franke filtration according to the relative rank of the degenerate Eisenstein series is reversed, compared to the case of cuspidal support of a residual representation in Section 4. In other words, degenerate Eisenstein series arising from parabolic subgroups of higher relative rank contribute to some deeper quotients of the filtration. These examples show once more that the Franke filtration for the general linear group cannot be defined based only on the rank of degenerate Eisenstein series, mentioned in Remark 2 [11, page 242]. The filtration is much more involved already in the examples below.

6.1. **The lowest rank example.** The phenomenon considered in this section occurs in the case of the general linear group GL_6 of rank five, i.e., $n = 6$. Let $\mathcal{A} = \mathcal{A}(GL_6(F)\backslash GL_6(\mathbb{A}), \omega)$ be the $(\mathfrak{g}_\infty, K_\infty; GL_6(\mathbb{A}_f))$-module of automorphic forms on $GL_6(\mathbb{A})$ of central character ω.

Let π be a cuspidal automorphic representation of the torus $T(\mathbb{A}) \cong \mathbb{I}\times\mathbb{I}\times\mathbb{I}\times\mathbb{I}\times\mathbb{I}\times\mathbb{I}$ isomorphic to the tensor product

$$\pi \cong \chi|\,|^{3/2} \otimes \chi|\,|^{1/2} \otimes \chi|\,|^{1/2} \otimes \chi|\,|^{-1/2} \otimes \chi|\,|^{-1/2} \otimes \chi|\,|^{-3/2},$$

where χ is a unitary Hecke character of $\mathbb{I}$ such that $\chi^6 = \omega$. Thus, π is the unitary cuspidal automorphic representation $\chi \otimes \chi \otimes \chi \otimes \chi \otimes \chi \otimes \chi$ of $T(\mathbb{A})$, twisted by the character of $T(\mathbb{A})$ corresponding to

$$(3/2, 1/2, 1/2, -1/2, -1/2, -3/2) \in \check{\mathfrak{a}}_B^{GL_6}.$$

The following theorem explicitly describes the Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms supported in the associate class $\varphi(\pi)$ of π.

Theorem 6.1. *The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; GL_6(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of automorphic forms on $GL_6(\mathbb{A})$, supported in the associate class $\varphi(\pi)$, represented by a cuspidal automorphic representation*

$$\pi \cong \chi|\,|^{3/2} \otimes \chi|\,|^{1/2} \otimes \chi|\,|^{1/2} \otimes \chi|\,|^{-1/2} \otimes \chi|\,|^{-1/2} \otimes \chi|\,|^{-3/2}$$

of $T(\mathbb{A})$, where χ is a Hecke character of $\mathbb{I}$, is the length eight filtration

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^2_{\{B\},\varphi(\pi)} \supsetneqq \quad \cdots \quad \supsetneqq \mathcal{A}^7_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the quotients of the filtration are isomorphic to

$$\begin{aligned}
\mathcal{A}^7_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(4,2)}(\mathbb{A})}\left((\chi\circ\det)\otimes(\chi\circ\det)\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(4,2)},\mathbb{C}}\right)\\
\mathcal{A}^6_{\{B\},\varphi(\pi)}/\mathcal{A}^7_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,4,1)}(\mathbb{A})}\left(\chi|\,|^{\frac12}\otimes(\chi\circ\det)\otimes\chi|\,|^{-\frac12}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,4,1)},\mathbb{C}}\right)\\
\mathcal{A}^5_{\{B\},\varphi(\pi)}/\mathcal{A}^6_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(3,3)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{\frac12}\otimes(\chi\circ\det)|\det|^{-\frac12}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(3,3)},\mathbb{C}}\right)\\
\mathcal{A}^4_{\{B\},\varphi(\pi)}/\mathcal{A}^5_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(3,2,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{\frac12}\otimes(\chi\circ\det)\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(3,2,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(3,1,2)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{\frac12}\otimes\chi|\,|^{\frac12}\otimes(\chi\circ\det)|\det|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(3,2,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(2,3,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|\otimes(\chi\circ\det)|\det|^{-\frac12}\otimes\chi|\,|^{-\frac12}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(2,3,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,2,3)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes(\chi\circ\det)\otimes(\chi\circ\det)|\det|^{-\frac12}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,2,3)},\mathbb{C}}\right)\\
\mathcal{A}^3_{\{B\},\varphi(\pi)}/\mathcal{A}^4_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(2,2,2)}(\mathbb{A})}\left((\chi\circ\det)|\det|\otimes(\chi\circ\det)\otimes(\chi\circ\det)|\det|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(2,2,2)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(3,1,1,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{\frac12}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(3,1,1,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,1,1,3)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes(\chi\circ\det)|\det|^{-\frac12}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,1,1,3)},\mathbb{C}}\right)\\
&\quad\bigoplus\left(\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,2,2,1)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes(\chi\circ\det)\otimes(\chi\circ\det)\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,2,2,1)},\mathbb{C}}\right)\right)^+\\
\mathcal{A}^2_{\{B\},\varphi(\pi)}/\mathcal{A}^3_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(2,2,1,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|\otimes(\chi\circ\det)\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(2,2,1,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(2,1,1,2)}(\mathbb{A})}\left((\chi\circ\det)|\det|\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes(\chi\circ\det)|\det|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(2,1,1,2)},\mathbb{C}}\right)\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,1,2,2)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes\chi|\,|^{\frac12}\otimes(\chi\circ\det)\otimes(\chi\circ\det)|\det|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,1,2,2)},\mathbb{C}}\right)\\
\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)} &\cong \left(\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(2,1,1,1,1)}(\mathbb{A})}\left((\chi\circ\det)|\det|\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(2,1,1,1,1)},\mathbb{C}}\right)\right)^+\\
&\quad\bigoplus\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,1,2,1,1)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes\chi|\,|^{\frac12}\otimes(\chi\circ\det)\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,1,2,1,1)},\mathbb{C}}\right)\\
&\quad\bigoplus\left(\mathrm{Ind}^{GL_6(\mathbb{A})}_{P_{(1,1,1,1,2)}(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes(\chi\circ\det)|\det|^{-1}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{P_{(1,1,1,1,2)},\mathbb{C}}\right)\right)^+\\
\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)} &\cong \left(\mathrm{Ind}^{GL_6(\mathbb{A})}_{B(\mathbb{A})}\left(\chi|\,|^{\frac32}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{\frac12}\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac12}\otimes\chi|\,|^{-\frac32}\right)\otimes S\left(\check{\mathfrak{a}}^{GL_6}_{B,\mathbb{C}}\right)\right)^{+,+}
\end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; GL_6(\mathbb{A}_f))$-modules. The exponent $+$ on the right-hand side of the quotient of the filtration refers to the $+1$-eigenspace for the action of the non-trivial intertwining operator obtained using the functor $M_{\{B\},\varphi(\pi)}$ from the unique non-trivial automorphism of the corresponding triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$. The exponent $+,+$ on the right-hand side of the last quotient refers to the intersection of $+1$-eigenspaces for the action of the non-trivial intertwining operators obtained using the functor $M_{\{B\},\varphi(\pi)}$ from the three non-trivial automorphisms of the corresponding triple in $\mathcal{M}_{\{B\},\varphi(\pi)}$.

Proof. The proof of this theorem is a direct application of the definition of the filtration, as in the proof of Theorem 5.1. According to Lemma 3.1, the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$ are in the correspondence

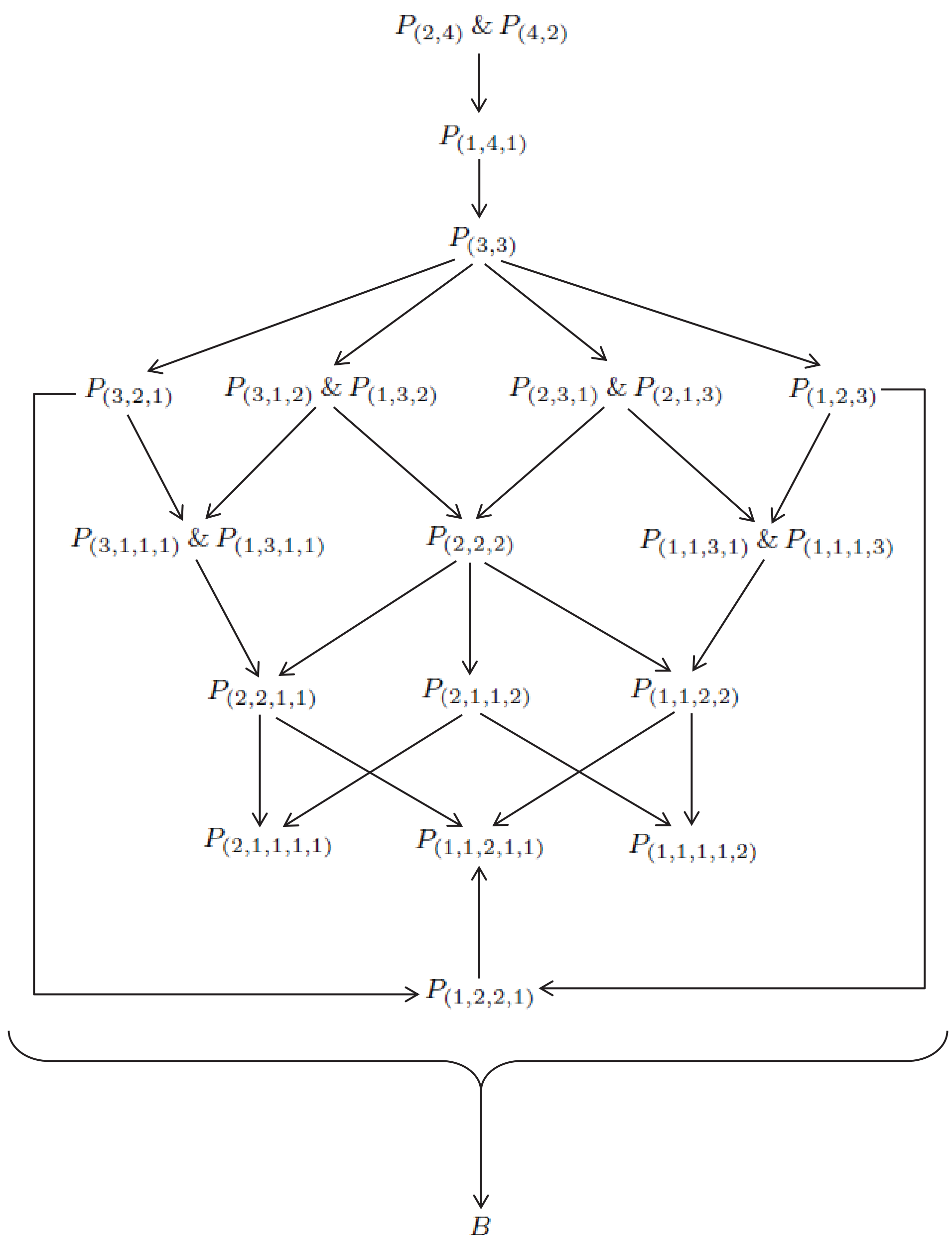

FIGURE 6.1. Partial order on triples $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ required for the Franke filtration. In the figure a triple is represented by its parabolic subgroup R. The notation $R\&R'$ means that the triples $(R, \Pi, \underline{z})$ and $(R', \Pi', \underline{z}')$ satisfy $\iota(\underline{z}) = \iota(\underline{z}')$, and thus cannot be distinguished by the partial order. Arrows point from larger triples towards smaller triples. Big curly bracket above B denotes that the triple with B is smaller than all other triples.

R	Π	$\underline{z}$	$\iota(\underline{z})$	$T_{\{B\},\varphi(\pi)}(\iota(\underline{z}))$
B	$\chi\otimes\chi\otimes\chi\otimes\chi\otimes\chi\otimes\chi$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	0
$P_{(2,1,1,1,1)}$	$(\chi\circ\det)\otimes\chi\otimes\chi\otimes\chi\otimes\chi$	$(1,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	$(1,1,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	1
$P_{(1,1,2,1,1)}$	$\chi\otimes\chi\otimes(\chi\circ\det)\otimes\chi\otimes\chi$	$(\frac{3}{2},\frac{1}{2},0,-\frac{1}{2},-\frac{3}{2})$	$(\frac{3}{2},\frac{1}{2},0,0,-\frac{1}{2},-\frac{3}{2})$	1
$P_{(1,1,1,1,2)}$	$\chi\otimes\chi\otimes\chi\otimes\chi\otimes(\chi\circ\det)$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-1)$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-1,-1)$	1
$P_{(2,2,1,1)}$	$(\chi\circ\det)\otimes(\chi\circ\det)\otimes\chi\otimes\chi$	$(1,0,-\frac{1}{2},-\frac{3}{2})$	$(1,1,0,0,-\frac{1}{2},-\frac{3}{2})$	2
$P_{(2,1,1,2)}$	$(\chi\circ\det)\otimes\chi\otimes\chi\otimes(\chi\circ\det)$	$(1,\frac{1}{2},-\frac{1}{2},-1)$	$(1,1,\frac{1}{2},-\frac{1}{2},-1,-1)$	2
$P_{(1,1,2,2)}$	$\chi\otimes\chi\otimes(\chi\circ\det)\otimes(\chi\circ\det)$	$(\frac{3}{2},\frac{1}{2},0,-1)$	$(\frac{3}{2},\frac{1}{2},0,0,-1,-1)$	2
$P_{(1,2,2,1)}$	$\chi\otimes(\chi\circ\det)\otimes(\chi\circ\det)\otimes\chi$	$(\frac{3}{2},0,0,-\frac{3}{2})$	$(\frac{3}{2},0,0,0,0,-\frac{3}{2})$	3
$P_{(3,1,1,1)}$	$(\chi\circ\det)\otimes\chi\otimes\chi\otimes\chi$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	3
$P_{(1,3,1,1)}$	$\chi\otimes(\chi\circ\det)\otimes\chi\otimes\chi$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$	3
$P_{(1,1,3,1)}$	$\chi\otimes\chi\otimes(\chi\circ\det)\otimes\chi$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	3
$P_{(1,1,1,3)}$	$\chi\otimes\chi\otimes\chi\otimes(\chi\circ\det)$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	3
$P_{(2,2,2)}$	$(\chi\circ\det)\otimes(\chi\circ\det)\otimes(\chi\circ\det)$	$(1,0,-1)$	$(1,1,0,0,-1,-1)$	3
$P_{(3,2,1)}$	$(\chi\circ\det)\otimes(\chi\circ\det)\otimes\chi$	$(\frac{1}{2},0,-\frac{3}{2})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,-\frac{3}{2})$	4
$P_{(3,1,2)}$	$(\chi\circ\det)\otimes\chi\otimes(\chi\circ\det)$	$(\frac{1}{2},\frac{1}{2},-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-1,-1)$	4
$P_{(1,3,2)}$	$\chi\otimes(\chi\circ\det)\otimes(\chi\circ\det)$	$(\frac{1}{2},\frac{1}{2},-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-1,-1)$	4
$P_{(2,3,1)}$	$(\chi\circ\det)\otimes(\chi\circ\det)\otimes\chi$	$(1,-\frac{1}{2},-\frac{1}{2})$	$(1,1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	4
$P_{(2,1,3)}$	$(\chi\circ\det)\otimes\chi\otimes(\chi\circ\det)$	$(1,-\frac{1}{2},-\frac{1}{2})$	$(1,1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	4
$P_{(1,2,3)}$	$\chi\otimes(\chi\circ\det)\otimes(\chi\circ\det)$	$(\frac{3}{2},0,-\frac{1}{2})$	$(\frac{3}{2},0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	4
$P_{(1,4,1)}$	$\chi\otimes(\chi\circ\det)\otimes\chi$	$(\frac{1}{2},0,-\frac{1}{2})$	$(\frac{1}{2},0,0,0,0,-\frac{1}{2})$	6
$P_{(3,3)}$	$(\chi\circ\det)\otimes(\chi\circ\det)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	5
$P_{(4,2)}$	$(\chi\circ\det)\otimes(\chi\circ\det)$	$(0,0)$	$(0,0,0,0,0,0)$	7
$P_{(2,4)}$	$(\chi\circ\det)\otimes(\chi\circ\det)$	$(0,0)$	$(0,0,0,0,0,0)$	7

TABLE 6.1. The triples $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ and the definition of the choice of $T_{\{B\},\varphi(\pi)}(\iota(\underline{z}))$ based on the partial order given in Fig. 6.1.

with partitions into segments of the sequence of exponents

$$(3/2,1/2,1/2,-1/2,-1/2,-3/2)$$

appearing in the cuspidal support π. These are all listed in Table 6.1.

The groupoid $\mathcal{M}_{\{B\},\varphi(\pi)}$ has non-trivial morphisms. More precisely, the pairs of triples with parabolic subgroups

$$P_{(3,1,1,1)} \text{ and } P_{(1,3,1,1)}$$
$$P_{(1,1,3,1)} \text{ and } P_{(1,1,1,3)}$$
$$P_{(3,1,2)} \text{ and } P_{(1,3,2)}$$
$$P_{(2,3,1)} \text{ and } P_{(2,1,3)}$$
$$P_{(4,2)} \text{ and } P_{(2,4)}$$

have unique isomorphisms between each other. The triples with parabolic subgroups

$$P_{(2,1,1,1,1)}$$
$$P_{(1,1,1,1,2)}$$
$$P_{(1,2,2,1)}$$

have a unique non-trivial automorphism, and the triple with the Borel subgroup B has three non-trivial automorphisms, one of which is the composition of the other two.

The partial order $\succ$, required in the definition of the Franke filtration in Section 2, is given in Figure 6.1. It is determined by the partial sums of the inclusions $\iota(\underline{z})$ into $\check{\mathfrak{a}}_B^{GL_6}$ of the third component $\underline{z}$ of the triples in $\mathcal{M}_{\{B\},\varphi(\pi)}$, see equation (2.2) and Table 6.1. The choice of the function $T_{\{B\},\varphi(\pi)}$ in Table 6.1 is made so that the condition of Section 2 is satisfied.

The description of the quotients of the Franke filtration given in the statement of the theorem now follows from the description in (2.4) of Section 2. In some of the cases the colimit of (2.4) is non-trivial, because there are non-trivial isomorphisms in $\mathcal{M}_{\{B\},\varphi(\pi)}$. These cases are mentioned above. In the quotient $\mathcal{A}^7_{\{B\},\varphi(\pi)}$, there are two triples with an isomorphism between each other, so that the colimit is isomorphic to the image of $M_{\{B\},\varphi(\pi)}$ on one of them. In the quotient $\mathcal{A}^4_{\{B\},\varphi(\pi)}/\mathcal{A}^5_{\{B\},\varphi(\pi)}$, there are two such pairs of triples, so we get contribution to the direct sum from one member of each pair. In the quotient $\mathcal{A}^3_{\{B\},\varphi(\pi)}/\mathcal{A}^4_{\{B\},\varphi(\pi)}$, we again have two pairs of triples with isomorphisms, but also a triple with a non-trivial automorphism. The two pairs of triples contribute as in the previous quotient, while the triple with a non-trivial automorphism contributes with the $+1$-eigenspace of the corresponding automorphism on the parabolically induced representation. In the quotient $\mathcal{A}^1_{\{B\},\varphi(\pi)}/\mathcal{A}^2_{\{B\},\varphi(\pi)}$, there are two triples with a non-trivial automorphism, so that each of them contributes with the $+1$-eigenspace. Finally, in the quotient $\mathcal{A}^0_{\{B\},\varphi(\pi)}/\mathcal{A}^1_{\{B\},\varphi(\pi)}$, there are three non-trivial automorphisms of the same triple. One of these automorphisms is a composition of the other two. In that case, the direct computation of the colimit implies that the contribution is given by the intersection of $+1$-eigenspaces of the corresponding automorphisms on the parabolically induced representation. See [27] or [14] for details regarding computation of the colimits. □

6.2. **More general results in the case of n even.** Let $n = 2m \geq 6$, that is, we consider the general linear group $G = GL_{2m}$ of odd rank $2m-1$. The results presented here are a generalization of the previous example in the case of any even $n \geq 6$. It also generalizes the results of Theorem 5.2. The cuspidal support is still in the Borel subgroup, but there is a whole segment of exponents appearing twice in the cuspidal support, and not only the exponent zero as in Theorem 5.2. However, some of the arguments are quite similar to the previous cases, and thus, we omit some details, but emphasize the differences.

Let $\mathcal{A}_{\{B\},\varphi(\pi)}$ be the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module of automorphic forms, with a fixed central character ω, supported in the associate class $\varphi(\pi)$, represented by a cuspidal automorphic representation π of $T(\mathbb{A}) \cong \mathbb{I} \times \cdots \times \mathbb{I}$, where the group of idèles $\mathbb{I}$ appears $2m$ times as a factor, given as the tensor product

$$\begin{aligned}\pi \cong & \chi|\,|^{m-\frac{\alpha+1}{2}} \otimes \chi|\,|^{m-\frac{\alpha-1}{2}} \otimes \cdots \otimes \chi|\,|^{\frac{\alpha+3}{2}} \otimes \chi|\,|^{\frac{\alpha+1}{2}} \\ & \otimes \chi|\,|^{\frac{\alpha-1}{2}} \otimes \chi|\,|^{\frac{\alpha-1}{2}} \otimes \cdots \otimes \chi|\,|^{-\frac{\alpha-1}{2}} \otimes \chi|\,|^{-\frac{\alpha-1}{2}} \\ & \otimes \chi|\,|^{-\frac{\alpha+1}{2}} \otimes \chi|\,|^{-\frac{\alpha+3}{2}} \otimes \cdots \otimes \chi|\,|^{-\left(m-\frac{\alpha-1}{2}\right)} \otimes \chi|\,|^{-\left(m-\frac{\alpha+1}{2}\right)},\end{aligned}$$

where χ is a unitary Hecke character of $\mathbb{I}$ such that $\chi^{2m} = \omega$, and α is an integer such that $2 \leq \alpha \leq m-1$. Thus, π is the unitary character $\chi \otimes \cdots \otimes \chi$ of $T(\mathbb{A})$, twisted by the character of $T(\mathbb{A})$ corresponding to

$$\begin{aligned}&\Bigg(m-\frac{\alpha+1}{2}, m-\frac{\alpha-1}{2}, \ldots, \frac{\alpha+3}{2}, \frac{\alpha+1}{2}, \\ &\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \ldots, -\frac{\alpha-1}{2}, -\frac{\alpha-1}{2}, \\ &-\frac{\alpha+1}{2}, -\frac{\alpha+3}{2}, \ldots, -\left(m-\frac{\alpha-1}{2}\right), -\left(m-\frac{\alpha+1}{2}\right)\Bigg) \in \check{\mathfrak{a}}_B^G.\end{aligned}$$

The case $\alpha = 1$ is omitted because it gives Theorem 5.2. In the theorem below we additionally suppose that

$$\alpha \leq \frac{m+1}{2}.$$

This is a simplifying technical assumption, under which we may explicitly describe a large part of the Franke filtration of $\mathcal{A}_{\{B\},\varphi(\pi)}$, and the considered phenomenon may be easily observed. The cases with $\alpha > \frac{m+1}{2}$ are combinatorially more demanding.

For the statement of the theorem, we require more notation. In the setting as above, consider the set of triples $(R, \Pi, \underline{z}) \in \mathcal{M}_{\{B\},\varphi(\pi)}$ such that R contains a diagonal block of size $2m-\alpha$. According to Lemma 3.1, such triples correspond to partitions of the cuspidal support π into segments which contain the segment

$$\Delta = \Delta\left(\chi, \left[-\left(m-\frac{\alpha+1}{2}\right), m-\frac{\alpha+1}{2}\right]\right).$$

The segment Δ is the longest possible segment in the cuspidal support π.

Having fixed the segment Δ, the partition of the cuspidal support into segments is reduced to the partition into segments of the remaining segment

$$\Delta' = \Delta\left(\chi, \left[-\frac{\alpha-1}{2}, \frac{\alpha-1}{2}\right]\right)$$

in the cuspidal support π, which is the support of a residual representation of $GL_\alpha(\mathbb{A})$ and we may use the results of Section 4. Let

$$\Delta_i = \Delta(\chi, [a_i, b_i]), \quad i = 1, \ldots, k,$$

be the partition of the segment Δ' into disjoint subsegments, ordered in such a way that the mid-points of Δ_i and Δ_{i+1} satisfy

$$\frac{a_i + b_i}{2} > \frac{a_{i+1} + b_{i+1}}{2}$$

for $i = 1, \ldots, k-1$. If α_i denotes the length of the segment Δ_i, then $(\alpha_1, \ldots, \alpha_k)$ form an ordered partition of α into positive integers. Conversely, any ordered partition of α into positive integers gives rise to a partition of Δ' into disjoint segments.

Let i_+ denote the largest integer $1 \leq i \leq k$ such that $\frac{a_i+b_i}{2}$ is non-negative. Since $\frac{a_1+b_1}{2} \geq 0$, such i_+ always exists. We consider separately two cases depending on the value of $\frac{a_{i_+}+b_{i_+}}{2}$. Observe that the mid-point of the segment Δ is zero.

In the case of $\frac{a_{i_+}+b_{i_+}}{2} > 0$, the only segment in the partition with the mid-point zero is Δ. Hence, the segments in the partition of the cuspidal support π in this case are ordered in a unique way as

$$\Delta_1, \ldots, \Delta_{i_+}, \Delta, \Delta_{i_++1}, \ldots, \Delta_k,$$

and the corresponding triple $(R, \Pi, \underline{z}) \in \mathcal{M}_{\{B\},\varphi(\pi)}$ is unique. As in Lemma 3.1, it is given as

- $R = P_{(\alpha_1,\ldots,\alpha_{i_+},2m-\alpha,\alpha_{i_++1},\ldots,\alpha_k)}$ is of relative rank k,
- $\Pi = \chi \circ \det \otimes \chi \circ \det \otimes \cdots \otimes \chi \circ \det$, where det is the determinant on the algebra of matrices of the appropriate size,
- $\underline{z} = \left(\frac{a_1+b_1}{2}, \ldots, \frac{a_{i_+}+b_{i_+}}{2}, 0, \frac{a_{i_++1}+b_{i_++1}}{2}, \ldots, \frac{a_k+b_k}{2}\right)$.

In the case of $\frac{a_{i_+}+b_{i_+}}{2} = 0$, the segments in the partition with the mid-point zero are Δ and Δ_{i+}. Hence, the segments in the partition of the cuspidal support π in this case can be ordered in two ways

$$\Delta_1, \ldots, \Delta_{i_+-1}, \Delta, \Delta_{i_+}, \Delta_{i_++1}, \ldots, \Delta_k, \quad \text{and} \quad \Delta_1, \ldots, \Delta_{i_+-1}, \Delta_{i_+}, \Delta, \Delta_{i_++1}, \ldots, \Delta_k.$$

The two corresponding triples in the correspondence of Lemma 3.1 are given as follows

- $R' = P_{(\alpha_1,\ldots,\alpha_{i_+-1},2m-\alpha,\alpha_{i_+},\alpha_{i_++1},\ldots,\alpha_k)}$ is of relative rank k,
- $\Pi' = \chi \circ \det \otimes \chi \circ \det \otimes \cdots \otimes \chi \circ \det$, where det is the determinant on the algebra of matrices of the appropriate size,
- $\underline{z}' = \left(\frac{a_1+b_1}{2}, \ldots, \frac{a_{i_+-1}+b_{i_+-1}}{2}, 0, 0, \frac{a_{i_++1}+b_{i_++1}}{2}, \ldots, \frac{a_k+b_k}{2}\right)$,

and

- $R'' = P_{(\alpha_1,\ldots,\alpha_{i_+-1},\alpha_{i_+},2m-\alpha,\alpha_{i_++1},\ldots,\alpha_k)}$ is of relative rank k,
- $\Pi'' = \chi \circ \det \otimes \chi \circ \det \otimes \cdots \otimes \chi \circ \det$, where det is the determinant on the algebra of matrices of the appropriate size,
- $\underline{z}'' = \left(\frac{a_1+b_1}{2}, \ldots, \frac{a_{i_+-1}+b_{i_+-1}}{2}, 0, 0, \frac{a_{i_++1}+b_{i_++1}}{2}, \ldots, \frac{a_k+b_k}{2}\right)$.

For the purpose of stating the theorem below, for each partition of the segment Δ', we choose one of the triples $(R, \Pi, \underline{z}) \in \mathcal{M}_{\{B\},\varphi(\pi)}$ corresponding as above to that partition with the segment Δ of length $2m - \alpha$ added. The set of chosen triples is denoted

$$\mathcal{M}'_{\{B\},\varphi(\pi)}.$$

In the notation as above, the choice must be made only if $\frac{a_{i_+}+b_{i_+}}{2} = 0$, and in that case there are two possible choices. The results do not depend on the choices made.

Theorem 6.2. *The Franke filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\{B\},\varphi(\pi)}$ of the automorphic forms on $G(\mathbb{A})$, where $2m \geq 6$, supported in the associate class $\varphi(\pi)$, represented by a cuspidal* *automorphic representation*

$$\begin{aligned}\pi \cong & \chi|\,|^{m-\frac{\alpha+1}{2}} \otimes \chi|\,|^{m-\frac{\alpha-1}{2}} \otimes \cdots \otimes \chi|\,|^{\frac{\alpha+3}{2}} \otimes \chi|\,|^{\frac{\alpha+1}{2}} \\ & \otimes \chi|\,|^{\frac{\alpha-1}{2}} \otimes \chi|\,|^{\frac{\alpha-1}{2}} \otimes \cdots \otimes \chi|\,|^{-\frac{\alpha-1}{2}} \otimes \chi|\,|^{-\frac{\alpha-1}{2}} \\ & \otimes \chi|\,|^{-\frac{\alpha+1}{2}} \otimes \chi|\,|^{-\frac{\alpha+3}{2}} \otimes \cdots \otimes \chi|\,|^{-\left(m-\frac{\alpha-1}{2}\right)} \otimes \chi|\,|^{-\left(m-\frac{\alpha+1}{2}\right)},\end{aligned}$$

of $T(\mathbb{A})$, where χ is a unitary Hecke character of $\mathbb{I}$ and $2 \leq \alpha \leq \frac{m+1}{2}$, is of the form

$$\mathcal{A}_{\{B\},\varphi(\pi)} = \mathcal{A}^0_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\pi)} \supsetneqq \cdots \supsetneqq \mathcal{A}^{\ell-\alpha}_{\{B\},\varphi(\pi)} \supsetneqq \mathcal{A}^{\ell-\alpha+1}_{\{B\},\varphi(\pi)} \supsetneqq \cdots \supsetneqq \mathcal{A}^{\ell}_{\{B\},\varphi(\pi)} \supsetneqq \{0\},$$

where the last $\alpha+1$ quotients of the filtration are isomorphic to

$$\begin{aligned}\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{G(\mathbb{A})}_{P_{(2m-\alpha,\alpha)}(\mathbb{A})}\left((\chi\circ\det)\otimes(\chi\circ\det)\right)\otimes S\left(\check{\mathfrak{a}}^G_{P_{(2m-\alpha,\alpha)},\mathbb{C}}\right) \\ \mathcal{A}^{\ell-\alpha+i}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell-\alpha+i+1}_{\{B\},\varphi(\pi)} &\cong \bigoplus_{\substack{(R,\Pi,\underline{z})\in\mathcal{M}'_{\{B\},\varphi(\pi)} \text{ such that} \\ \text{the relative rank of } R \text{ is } \alpha-i+1}} I(\underline{z},\Pi)\otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}), \qquad \textit{for } i=1,\ldots,\alpha-1 \\ \mathcal{A}^{\ell-\alpha}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell-\alpha+1}_{\{B\},\varphi(\pi)} &\cong \mathrm{Ind}^{G(\mathbb{A})}_{P_{(m,m)}(\mathbb{A})}\left((\chi\circ\det)|\det|^{\frac{m-\alpha}{2}}\otimes(\chi\circ\det)|\det|^{-\frac{m-\alpha}{2}}\right)\otimes S\left(\check{\mathfrak{a}}^G_{P_{(m,m)},\mathbb{C}}\right)\end{aligned}$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, where $\mathcal{M}'_{\{B\},\varphi(\pi)}$ denotes the set of triples defined above. The length $\ell+1$ of the filtration is not given explicitly, as it depends on certain choices in the definition of the filtration.

Proof. As in the proof of Theorem 5.2, we consider the subsequences of the sequence of exponents

$$\begin{aligned}&\Bigg(m-\frac{\alpha+1}{2}, m-\frac{\alpha-1}{2}, \ldots, \frac{\alpha+3}{2}, \frac{\alpha+1}{2}, \\ &\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \ldots, -\frac{\alpha-1}{2}, -\frac{\alpha-1}{2}, \\ &-\frac{\alpha+1}{2}, -\frac{\alpha+3}{2}, \ldots, -\left(m-\frac{\alpha-1}{2}\right), -\left(m-\frac{\alpha+1}{2}\right)\Bigg) \in \check{\mathfrak{a}}^G_B.\end{aligned}$$

appearing in the cuspidal support. These exponents are not always integers and we need subsequences which are segments in the sense introduced in Section 3.

Since there is no residual representation of $G(\mathbb{A})$ supported in $\varphi(\pi)$, there is no triple $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with $R = G$. For the triples $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with R a maximal proper parabolic subgroup, we observe that the exponents $\frac{\alpha-1}{2}, \ldots, -\frac{\alpha-1}{2}$ are the only exponents appearing twice in the sequence, so each of the two segments must contain them. The remaining exponents are divided between the two segments in such a way that either all of them are in one segment, or positive exponents are in one segment and negative in the other. This gives three triples

$$\begin{aligned}&\left(P_{(2m-\alpha,\alpha)}, (\chi\circ\det)\otimes(\chi\circ\det), (0,0)\right) \\ &\left(P_{(\alpha,2m-\alpha)}, (\chi\circ\det)\otimes(\chi\circ\det), (0,0)\right) \\ &\left(P_{(m,m)}, (\chi\circ\det)\otimes(\chi\circ\det), \left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)\right)\end{aligned}$$

in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with R maximal. The only non-trivial morphisms between them are the isomorphisms between the first two triples, given by the interchange of the two factors.

The same argument as in the proof of Theorem 5.2 shows that $T_{\{B\},\varphi(\pi)}$ may be chosen so that it takes value ℓ at $\iota(\underline{z}) = (0,\dots,0)$ for the third entry $\underline{z} = (0,0)$ of the two triples with $R = P_{(2m-\alpha,\alpha)}$ and $R = P_{(\alpha,2m-\alpha)}$, and is less than ℓ at the third entry of all other triples. The claim for $\mathcal{A}^{\ell}_{\{B\},\varphi(\pi)}$ then follows as in the proof of Theorem 5.2.

We now show that $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$ is either incomparable to, or contributes to a deeper filtration step than any other triple $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with R not containing the diagonal block of size $2m-\alpha$. This is achieved by a similar argument as in the proof of Theorem 5.2. Let $\iota(\underline{z}) = (\zeta_1,\dots,\zeta_n)$ in coordinates. If the largest exponent $m - \frac{\alpha+1}{2}$ belongs to the segment which ends with some $l > -\frac{\alpha-1}{2}$, then the residual representation supported in that segment is isomorphic to

$$(\chi \circ \det)|\det|^{\frac{m-\frac{\alpha+1}{2}+l}{2}}.$$

Hence, the first entry ζ_1 of $\iota(\underline{z})$ is at least equal to the exponent in that residual representation, so that

$$\zeta_1 \geq \frac{m - \frac{\alpha+1}{2} + l}{2} > \frac{m - \frac{\alpha+1}{2} - \frac{\alpha-1}{2}}{2} = \frac{m-\alpha}{2}.$$

This shows that $\iota(\underline{z})$ cannot contribute to a deeper filtration step than $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$, unless $-\frac{\alpha-1}{2}$ is in the same segment as $m - \frac{\alpha+1}{2}$. On the other hand, considering the segment in which $-\left(m - \frac{\alpha+1}{2}\right)$ belongs, and comparing the partial sum $\zeta_1 + \cdots + \zeta_{n-1} = -\zeta_n$ implies the same conclusion for $\frac{\alpha-1}{2}$ and $-\left(m - \frac{\alpha+1}{2}\right)$. Since we now study only triples with R not containing the diagonal block of size $2m-\alpha$, it is not possible that $m - \frac{\alpha+1}{2}$ and $-\left(m - \frac{\alpha+1}{2}\right)$ are in the same segment. Hence, the triple $(R,\Pi,\underline{z})$ with R not containing the diagonal block of size $2m-\alpha$, which may possibly contribute to a deeper filtration step than the triple with $R = P_{(m,m)}$, must correspond to the partition in two segments. But the partition in two segments gives back $R = P_{(m,m)}$. Thus, the triple with $R = P_{(m,m)}$ contributes to the deepest filtration step among all triples with R not containing the diagonal block of size $2m-\alpha$.

It remains to compare $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$ to the triples $(R,\Pi,\underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ such that R contains the diagonal block of size $2m-\alpha$. These were studied just above the statement of the theorem. In the correspondence of Lemma 3.1, these triples correspond to partitions of the cuspidal support π into Δ and any partition of Δ' into subsegments. Since Δ' is the cuspidal support of a residual representation of $GL_\alpha(\mathbb{A})$, we may use the results and techniques of Section 4. The same argument as in the proof of Theorem 4.1 and Lemma 4.2 shows that the values of the function $T_{\{B\},\varphi(\pi)}$ on $\iota(\underline{z})$ for triples $(R,\Pi,\underline{z})$ with R containing the diagonal block of size $2m-\alpha$ may be chosen in terms of the (relative) rank of R. More precisely, triples with R of lower relative rank contribute to deeper filtration steps, and triples with R of the same rank may be arranged to contribute to the same quotient of the filtration.

Thus, due to transitivity of the partial order, it is sufficient to compare $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$ to $\iota(\underline{z})$ of the triple with R of highest relative rank among those containing the diagonal block of size $2m-\alpha$, i.e., the triple with

- $R = P_{(1,\dots,1,2m-\alpha,1,\dots,1)}$,
- $\Pi = \chi \otimes \cdots \otimes \chi \otimes \chi \circ \det \otimes \chi \otimes \cdots \otimes \chi$,
- $\underline{z} = \left(\frac{\alpha-1}{2}, \frac{\alpha-3}{2}, \dots, \frac{\alpha-1}{2} - \left\lfloor \frac{\alpha-1}{2} \right\rfloor, 0, \frac{\alpha-1}{2} - \left\lfloor \frac{\alpha-1}{2} \right\rfloor - 1, \dots, -\frac{\alpha-1}{2}\right)$

where the position of the diagonal block of size $2m-\alpha$ and the factor $\chi \circ \det$ is determined by the position of the coordinate 0 in $\underline{z}$. The notation $\lfloor x \rfloor$ stands for the greatest integer not greater than x.

We now show that $\iota(\underline{z})$ contributes to a deeper filtration step than $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$, i.e.,

$$\iota(\underline{z}) \succ \iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right).$$

Writing in coordinates

$$\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right) = \left(\frac{m-\alpha}{2}, \dots, \frac{m-\alpha}{2}, -\frac{m-\alpha}{2}, \dots, -\frac{m-\alpha}{2},\right)$$

where $\frac{m-\alpha}{2}$ and $-\frac{m-\alpha}{2}$ occur m times each, and

$$\iota(\underline{z}) = \left(\frac{\alpha-1}{2}, \frac{\alpha-3}{2}, \dots, \frac{\alpha-1}{2} - \left\lfloor \frac{\alpha-1}{2} \right\rfloor, 0, 0, \dots, 0, \frac{\alpha-1}{2} - \left\lfloor \frac{\alpha-1}{2} \right\rfloor - 1, \dots, -\frac{\alpha-1}{2}\right),$$

where the zero occurs $n-\alpha$ times in this sequence, not counting the zero possibly obtained form the expression $\frac{\alpha-1}{2} - \left\lfloor \frac{\alpha-1}{2} \right\rfloor$, which is zero if α is odd.

Under the assumption $\alpha \leq \frac{m+1}{2}$ of the theorem, we have for the first partial sum that

$$\frac{m-\alpha}{2} \geq \frac{\alpha-1}{2},$$

and for the second partial sum the strict inequality

$$2 \cdot \frac{m-\alpha}{2} = m-\alpha > \frac{\alpha-1}{2} + \frac{\alpha-3}{2} = \alpha - 2.$$

Further on, all non-negative entries of $\iota(\underline{z})$, except possibly the first one, are strictly less than $\frac{m-\alpha}{2}$. Hence, the first m partial sums of $\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)$ are greater or equal than the first m partial sums of $\iota(\underline{z})$, and the inequality is strict except for the first partial sums in the case $\alpha = \frac{m+1}{2}$. For the remaining partial sum the same conclusion holds, using the fact that $(\zeta_1, \dots, \zeta_n) \in \check{\mathfrak{a}}^G_{B,\mathbb{C}}$ satisfies $\zeta_1 + \cdots + \zeta_n = 0$. Thus, we proved that

$$\iota(\underline{z}) \succ \iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right).$$

holds.

Our considerations imply that we may choose the function $T_{\{B\},\varphi(\pi)}$ for triples $(R, \Pi, \underline{z})$ with R containing the diagonal block of size $2m-\alpha$ as

$$T_{\{B\},\varphi(\pi)}(\iota(\underline{z})) = \ell - k + 1,$$

where k is the relative rank of R, and choose the value

$$T_{\{B\},\varphi(\pi)}\left(\iota\left(\frac{m-\alpha}{2}, -\frac{m-\alpha}{2}\right)\right) = \ell - \alpha$$

for the triple with $R = P_{(m,m)}$. For all the remaining triples we may define $T_{\{B\},\varphi(\pi)}$ to be less than $\ell - \alpha$.

It remains to prove the description of the quotients of the filtration. This follows from the description of the Franke filtration in Section 2. The result for

$$\mathcal{A}^{\ell-\alpha}_{\{B\},\varphi(\pi)} / \mathcal{A}^{\ell-\alpha+1}_{\{B\},\varphi(\pi)}$$

is clear, as the only triple which contributes to this filtration step is the one with $R = P_{(m,m)}$. The contribution to

$$\mathcal{A}^{\ell-\alpha+i}_{\{B\},\varphi(\pi)}/\mathcal{A}^{\ell-\alpha+i+1}_{\{B\},\varphi(\pi)}, \quad \text{for } i = 1, \dots, \alpha - 1,$$

is given by the triples $(R, \Pi, \underline{z})$ in $\mathcal{M}_{\{B\},\varphi(\pi)}$ with

$$T_{\{B\},\varphi(\pi)}(\iota(\underline{z})) = \ell - \alpha + i.$$

For our choice of $T_{\{B\},\varphi(\pi)}$, these are all triples with R of relative rank $\alpha - i + 1$ and containing the diagonal block of size $2m - \alpha$. Such triples were studied just above the statement of the theorem. Among such triples, the only morphisms are isomorphisms between pairs of triples corresponding to the same partition of the cuspidal support π into disjoint segments. Hence, in the colimit, only one member of each pair survives. Therefore, the direct sum is over the set $\mathcal{M}'_{\{B\},\varphi(\pi)}$ defined above, which contains exactly one triple corresponding to each partition of the cuspidal support π containing the segment Δ. □

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Neven Grbac, Juraj Dobrila University of Pula, Zagrebačka 30, HR-52100 Pula, Croatia
Email address: neven.grbac@unipu.hr

Harald Grobner, Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
Email address: harald.grobner@univie.ac.at