EISENSTEIN COHOMOLOGY AND AUTOMORPHIC L-FUNCTIONS
A SURVEY OF A COLLABORATION WITH JOACHIM SCHWERMER

NEVEN GRBAC

To Joachim Schwermer, with gratitude and admiration, for the occasion of his 66th birthday

Abstract. During the past ten years of the most inspiring and very fruitful collaboration with Joachim Schwermer, we have carefully studied the non-vanishing conditions for certain summands in the decomposition along the cuspidal support of the (square-integrable) Eisenstein cohomology of a reductive group over a totally real number field. These conditions form a subtle combination of geometric conditions, arising from cohomological considerations, and arithmetic conditions, arising from the analytic properties of Eisenstein series and given in terms of automorphic L-functions. This paper is a survey of the most important results of our long-lasting collaboration.

1. Introduction

The cohomology of an arithmetic subgroup of a reductive group over a number field is closely related to automorphic forms with respect to that arithmetic subgroup. On the other hand, it has another interpretation in terms of geometric objects. In that way, it provides a direct link between arithmetic algebraic geometry and number theory. The flow of information may go in both directions. Using geometric constructions one may hope to show the existence of automorphic forms, satisfying certain properties, with respect to some arithmetic subgroups. In the other direction, one may use the decomposition and structure of the spaces of automorphic forms to get information about cohomology. The latter direction is exactly the main theme of my collaboration with Joachim Schwermer, and the subject of this survey paper. Schwermer has written several excellent overview papers about the cohomology of arithmetic groups, emphasising both, the geometric and the number theoretic aspects [23], [29], [30]. Hence, we omit here the preliminaries and details regarding the setting, wider scope and applications of cohomology of arithmetic groups, and focus on the results obtained in our collaboration.

The main object of concern, when expressing the cohomology of arithmetic groups in terms of automorphic forms, is the so-called automorphic cohomology of a reductive group. It captures, in the adèlic setting, the information about cohomology of congruence arithmetic subgroups. It is defined as the relative Lie algebra cohomology of the space of automorphic forms on the adèlic group.

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In the collaboration with Joachim Schwermer, we study the automorphic cohomology of a reductive group using information about automorphic forms on the adèlic points of that group. However, complete explicit information regarding the structure of spaces of automorphic forms is often not known. Hence, we are exploiting the interconnection of the two interpretations, geometric and automorphic, to show that only the automorphic forms with certain properties may possibly contribute to cohomology. These arguments exclude many difficult situations in the theory of automorphic forms from consideration. For the remaining possibilities, we use the information from the theory of automorphic forms to show the existence of certain cohomology classes represented by (non-cuspidal) square-integrable automorphic forms and study the internal structure of cohomology.

The paper is organized as follows. In Section 2 we define the main objects of concern and provide just the preliminaries required to state the results. Section 3 introduces the necessary conditions for non-vanishing of certain summands in cohomology and explains their consequences. In Section 4 the focus is on the subtle interplay of geometric and arithmetic necessary non-vanishing conditions for summands in square-integrable cohomology. Finally, Section 5 deals with internal structure of the full automorphic cohomology, in particular, with the existence of non-trivial cohomology classes in the summands of square-integrable cohomology.

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When I first met Joachim Schwermer at the Erwin Schrödinger Institute in Vienna in 2006, I was a young mathematician, just finishing my PhD, and looking for new problems. My thesis was about L^2 spaces of automorphic forms and their spectral decomposition, the Eisenstein series and automorphic L-functions and their analytic behavior. My adviser was Goran Muić at the University of Zagreb. At that time, I did not know anything about cohomology. In our discussions back in 2006, I learned that the subject of my thesis is closely related to the cohomology of arithmetic groups, and we were soon working together on many problems in cohomology of arithmetic groups, looking at the possible applications of the structural information about spaces of automorphic forms.

I will always be grateful to Schwermer, who introduced me to the subject and taught me everything I know about cohomology. His kindness and patience, especially at the beginning of our collaboration, gave me the courage and confidence to continue the quest. I hope that he has enjoyed our collaboration as much as I have, and that it will continue for many years to come. *Happy birthday Joachim!*

2. Automorphic cohomology and cohomology of arithmetic groups

In this section we define the automorphic cohomology and its decomposition along the cuspidal support. The individual summands in that decomposition are the main objects of our concern in this paper. For simplicity of exposition and to avoid some technical issues, we work with a semi-simple group, instead of a reductive group, and over the field $\mathbb{Q}$ of rational numbers, instead of any totally real number field.

Hence, let G be a semi-simple connected linear algebraic group defined over the field of rational numbers $\mathbb{Q}$. For a finite prime p, let $\mathbb{Q}_p$ be the field of p-adic numbers. For $p = \infty$, we have $\mathbb{Q}_\infty = \mathbb{R}$. Let $\mathbb{A}$ denote the ring of adèles of $\mathbb{Q}$, and $\mathbb{A}_f$ the subring of finite adèles. Let $\mathfrak{g}_\infty$ be the real Lie algebra of the Lie group $G(\mathbb{R})$. We fix, once for all, a minimal parabolic $\mathbb{Q}$-subgroup P_0 of G, and a maximal compact subgroup K of $G(\mathbb{A})$. We may assume that $K = \prod_p K_p$, where K_p is a maximal compact subgroup of $G(\mathbb{Q}_p)$, hyperspecial for almost all p, and that K is in good position with respect to P_0 as in [25, Sect. I.1.4].

Let $\mathcal{A} = \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the space of automorphic forms on $G(\mathbb{A})$ as defined in [4]. It carries the structure of a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module arising from right translation. Let E be a finite-dimensional irreducible algebraic representation of G. We define the automorphic cohomology of G with respect to E as the relative Lie algebra cohomology of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}$ of automorphic forms on $G(\mathbb{A})$ with respect to E, that is,

$$H^*(G, E) = H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A} \otimes_{\mathbb{C}} E).$$

It carries the structure of a $G(\mathbb{A}_f)$-module. This object captures the information about the cohomology of congruence arithemetic subgroups of G. This fact is proved in [5], [2] and [8].

More precisely, according to [5], the Eilenberg–MacLane cohomology of an arithmetic subgroup Γ of G is isomorphic to the de Rham cohomology of the corresponding locally symmetric space $\Gamma\backslash X$, where $X = G(\mathbb{R})/K_\infty$. This, in turn, is isomorphic to the relative Lie algebra cohomology of the $(\mathfrak{g}_\infty, K_\infty)$-module of smooth functions on $\Gamma\backslash G(\mathbb{R})$. The regularization theorem of Borel, proved in [2], shows that in this last cohomology space the same space is obtained if smooth functions are replaced with smooth functions of uniform moderate growth. For a congruence subgroup Γ, Franke proved in [8] that instead of smooth functions of moderate growth, the same cohomology space is obtained as the relative Lie algebra cohomology of the $(\mathfrak{g}_\infty, K_\infty)$-module of automorphic forms on $G(\mathbb{R})$ with respect to Γ. Writing this in the adèlic setting, and passing to the direct limit with respect to the inclusion of open compact subgroups of $G(\mathbb{A}_f)$, one obtains the automorphic cohomology $H^*(G, E)$ as defined above. Conversely, given a compact open subgroup C of $G(\mathbb{A}_f)$, one can recover the cohomology of the corresponding congruence arithmetic subgroup Γ by taking the C-invariants of the $G(\mathbb{A}_f)$ action on the automorhpic cohomology $H^*(G, E)$.

The first step in the study of automorphic cohomology is to apply Wigner's lemma [5, Sect. I.4], which says that only automorphic forms whose infinitesimal character is compatible with E may possibly contribute to $H^*(G, E)$. Thus, we let $\mathcal{J}$ be the ideal of finite codimension in the center $\mathcal{Z}$ of the universal enveloping algebra of the complexification of $\mathfrak{g}_\infty$ which annihilates the conjugate dual of E. Then

$$H^*(G, E) \cong H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J}} \otimes_{\mathbb{C}} E),$$

where $\mathcal{A}_{\mathcal{J}}$ consists of automorphic forms that are annihilated by a power of $\mathcal{J}$.

The space $\mathcal{A}_{\mathcal{J}}$ admits a decomposition along the cuspidal support. Let $\{P\}$ be the associate class of parabolic $\mathbb{Q}$-subgroups of G, represented by a standard parabolic $\mathbb{Q}$-subgroup P with a Levi decomposition $P = M_P N_P$. Let ϕ_π be an associate class of cuspidal automorphic representations of the Levi factors of parabolic subgroups in $\{P\}$, represented by a cuspidal automorphic representation π of $M_P(\mathbb{A})$. More precisely, $\phi_\pi = (\phi_Q)_{Q\in\{P\}}$, where ϕ_Q is the finite set of cuspidal automorphic representations of the Levi factor $M_Q(\mathbb{A})$ which are $G(\mathbb{Q})$-conjugate to π. Note that we do not assume that π is unitary.

Let $\check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X^*(P)$ is the $\mathbb{Z}$-module of $\mathbb{Q}$-rational characters of P, and let $\check{\mathfrak{a}}_{P,\mathbb{C}}$ be its complexification. It is well known, see for example [25, Sect. I], that the elements of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ give rise to characters of $M_P(\mathbb{A})$. Abusing the notation, for $\nu \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, we denote the corresponding character of $M_P(\mathbb{A})$ by the same letter ν. Since we may replace the representatives π and P with their conjugates, we will always assume, without loss of generality, that $\pi \cong \pi_0 \otimes \nu_0$, where π_0 is a unitary cuspidal automorphic representation of $M_P(\mathbb{A})$, and ν_0 is a character of $M_P(\mathbb{A})$ corresponding to an element $\nu_0 \in \check{\mathfrak{a}}_P$ which belongs to the closure of the positive Weyl chamber determined by P.

We now define, following [9, Sect. 1.3], the space $\mathcal{A}_{\{P\},\phi_\pi}$ of automorphic forms supported in the associate class ϕ_π of cuspidal automorphic representations of the Levi factors of parabolic subgroups in the associate class $\{P\}$. This definition is equivalent to the definition given in [25, Sect. III.2.6] according to [9, Thm. 1.4]. Let $\pi \cong \pi_0 \otimes \nu_0$ as above. For simplicity of exposition, we suppose that π_0 is of multiplicity one in the space of cuspidal automorphic forms on $M_P(\mathbb{A})$. Let W_{π_0} be the space of smooth right K-finite functions f on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$ such that the function $f_g(m) = f(mg)$ on $M_P(\mathbb{A})$ belongs to the space of π_0 for all $g \in G(\mathbb{A})$. Given $f \in W_{\pi_0}$, we define the Eisenstein series, at least formally, by the series

$$E(f,\nu)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} (\nu + \rho_P)(\gamma g) f(\gamma g)$$

where $\nu \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, $g \in G(\mathbb{A})$, and $\rho_P \in \check{\mathfrak{a}}_P$ is the half-sum of positive roots in N_P. We view $\nu + \rho_P$ as a character of $G(\mathbb{A})$ extended from a character of $M_P(\mathbb{A})$ via Iwasawa decomposition trivially on $N_P(\mathbb{A})$ and K. The defining series of $E(f,\nu)(g)$ converges absolutely and locally uniformly in a positive cone deep enough in the positive Weyl chamber of $\check{\mathfrak{a}}_{P,\mathbb{C}}$ determined by P. It has the analytic continuation to a meromorphic function of ν on the whole space $\check{\mathfrak{a}}_{P,\mathbb{C}}$. The poles in the closure of the positive Weyl chamber are along the singular hyperplanes which form a locally finite family. For these properties of Eisenstein series see [25, Sect. IV.1] or [22]. We refer to the Eisenstein series $E(f,\nu)$, with $f \in W_{\pi_0}$, as the Eisenstein series associated to π_0.

We are interested in the analytic behavior of the Eisenstein series $E(f,\nu)$ at $\nu = \nu_0$. Since ν_0 is in the closure of the positive Weyl chamber and the family of singular hyperplanes is locally finite around ν_0, there is a (possibly empty) finite set of singular hyperplanes passing through ν_0. Hence, there is a polynomial $F(\nu)$ such that $F(\nu)E(f,\nu)$ is holomorphic around $\nu = \nu_0$. Then, the space $\mathcal{A}_{\{P\},\phi_\pi}$ is defined as the span of all the coefficients in the Taylor expansions of $F(\nu)E(f,\nu)$ around $\nu = \nu_0$, with f ranging over W_{π_0}. Although $F(\nu)$ is not unique, this definition does not depend on that choice.

The automorphic forms in $\mathcal{A}_{\{P\},\phi_\pi}$ are compatible with E, that is, belong to the space $\mathcal{A}_{\mathcal{J}}$, if and only if the cuspidal support ϕ_π is compatible with E as in [9, Sect. 1.3]. We denote by $\Phi_{\mathcal{J},\{P\}}$ the set of all associate classes ϕ_π of cuspidal automorphic representations of the Levi factors of parabolic subgroups in $\{P\}$ which are compatible with E. If $\phi_\pi \in \Phi_{\mathcal{J},\{P\}}$ we write $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ for $\mathcal{A}_{\{P\},\phi_\pi}$. Then, the space $\mathcal{A}_{\mathcal{J}}$ admits the decomposition along the cuspidal support

$$\mathcal{A}_{\mathcal{J}} = \bigoplus_{\{P\}} \bigoplus_{\phi_\pi \in \Phi_{\mathcal{J},\{P\}}} \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$$

as a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. The summands indexed by the full group $\{G\}$ consist of cuspidal automorphic forms.

The decomposition of $\mathcal{A}_{\mathcal{J}}$ gives rise to the corresponding decomposition in cohomology. Thus, the automorphic cohomology admits the decomposition along the cuspidal support

$$H^*(G,E)=\bigoplus_{\{P\}}\bigoplus_{\phi_\pi\in\Phi_{\mathcal{J},\{P\}}} H^*(\mathfrak{g}_\infty,K_\infty;\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E).$$

Since the summands indexed by the full group $\{G\}$ come from cuspidal automorphic forms, their sum is called cuspidal cohomology. The natural complement of cuspidal cohomology consists of summands indexed by $\{P\}\neq\{G\}$. Their sum is called Eisenstein cohomology, because the cohomology classes can be represented by derivatives of Eisenstein series and their residues.

The main object of our concern is the non-vanishing and structural description of the individual summands of Eisenstein cohomology in the decomposition along the cuspidal support. In other words, the problem is to determine for which cuspidal supports ϕ_π the summand

$$H^*(\mathfrak{g}_\infty,K_\infty;\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E),\quad \{P\}\neq\{G\} \tag{A}$$

is not trivial, and in that case determine the structure of that cohomology space.

Let $\mathcal{L}$ denote the $(\mathfrak{g}_\infty,K_\infty;G(\mathbb{A}_f))$-submodule of $\mathcal{A}$ consisting of square-integrable automorphic forms. The space $\mathcal{L}$ admits the decomposition along the cuspidal support. We denote by $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ the summand in that decomposition supported in ϕ_π. It is the space of square-integrable automorphic forms in $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$. The Langlands spectral decomposition (cf. [25], [22]) implies that $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ is spanned by all square-integrable iterated residues at $\nu=\nu_0$ of the Eisenstein series $E(f,\nu)$ associated to π_0. Recall that we write here $\pi\cong\pi_0\otimes\nu_0$ as before.

The inclusion of $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ into $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ gives rise to a map in cohomology

$$H^*(\mathfrak{g}_\infty,K_\infty;\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E)\longrightarrow H^*(\mathfrak{g}_\infty,K_\infty;\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E).$$

However, this map is not necessarily injective. Its image is the summand supported in ϕ_π in the so-called square-integrable cohomology, and denoted by

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty,K_\infty;\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E).$$

The square-integrable cohomology consists of cohomology classes that can be represented by square-integrable automorphic forms. In the case $\{P\}=\{G\}$, the summand in square-integrable cohomology is the same as the corresponding summand in full cohomology, because the unitary cuspidal automorphic forms are square-integrable. Thus, the interesting part of square-integrable cohomology lies in Eisenstein cohomology. We are interested in describing the square-integrable cohomology

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty,K_\infty;\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}\otimes_{\mathbb{C}} E),\quad \{P\}\neq\{G\} \tag{L}$$

in the summand (A) in the decomposition of Eisenstein cohomology along the cuspidal support.

3. Non-vanishing conditions

In this section we review the necessary conditions for non-vanishing of the summand (A) in the decomposition of Eisenstein cohomology along the cuspidal support. These conditions arise from the representation theoretic consideration of the space $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ of automorphic forms supported in π. We omit the details and refer to [27] and [24].

The crucial point is that there is a finite filtration of the space $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$, defined through the analytic behavior at $\nu = \nu_0$ of the Eisenstein series $E(f,\nu)$ associated to π_0, such that the successive quotients of the filtration are parabolically induced representations. It is the so-called Franke filtration introduced in [8, Sect. 6]. If the Eisenstein series $E(f,\nu)$ associated to π_0 is holomorphic at $\nu = \nu_0$, then $E(f,\nu_0)$, together with its derivatives at $\nu = \nu_0$, defines an intertwining map of the representation parabolically induced from $\pi_0 \otimes \nu_0$ to $G(\mathbb{A})$, tensored by the symmetric algebra $S(\check{\mathfrak{a}}_{P,\mathbb{C}})$ of derivatives with respect to ν, into the space of automorphic forms. Otherwise, if the Eisenstein series has a pole at $\nu = \nu_0$, one should consider the residues of Eisenstein series, and use the degenerate Eisenstein series on $G(\mathbb{A})$, supported in π_0, to construct intertwining maps between certain induced representations and the filtration quotients of the Franke filtration.

Hence, roughly speaking, the relative Lie algebra cohomology of $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ vanishes, if the relative Lie algebra cohomology of certain induced representation vanishes. Applying Frobenius reciprocity reduces this latter cohomology to the cohomology of $\pi_0 \otimes \nu_0$ with respect to the coefficient system given by the Lie algebra cohomology $H^*(\mathfrak{n}_P, E)$, where $\mathfrak{n}_P$ is the Lie algebra of the unipotent radical $N_P(\mathbb{R})$. Then, the non-vanishing conditions arise from the non-vanishing of the cuspidal cohomology at the level of Levi factors. To make these conditions precise we need some more notation.

Given a standard parabolic $\mathbb{Q}$-subgroup P of G, let A_P denote the maximal $\mathbb{Q}$-split torus in the center of the Levi factor M_P of P. The Lie algebra $\mathfrak{a}_P$ of $A_P(\mathbb{R})$ is isomorphic to $\mathfrak{a}_P \cong X_*(A_P) \otimes \mathbb{R}$, where $X_*(A_P)$ denotes the $\mathbb{Z}$-module of $\mathbb{Q}$-rational cocharacters of A_P. In the case of the minimal parabolic $\mathbb{Q}$-subgroup P_0, we write simply A_0 and $\mathfrak{a}_0$, instead of A_{P_0} and $\mathfrak{a}_{P_0}$, respectively. We retain the notation $\check{\mathfrak{a}}_P = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$, and write $\check{\mathfrak{a}}_0$ in the case $P = P_0$. There is a natural pairing of $\mathfrak{a}_P$ and $\check{\mathfrak{a}}_P$. The inclusion of A_P into A_0 gives rise to a map of $\mathfrak{a}_P$ into $\mathfrak{a}_0$. On the other hand, the restriction of characters from P to P_0 gives rise to a map of $\check{\mathfrak{a}}_P$ to $\check{\mathfrak{a}}_0$. These two maps provide natural decompositions

$$\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P \quad \text{and} \quad \check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P.$$

The space $\check{\mathfrak{a}}_0^P$ may be viewed as the space of infinitesimal characters of representations of $M_P(\mathbb{R})$. The projection of an element $\lambda \in \check{\mathfrak{a}}_0$ to $\check{\mathfrak{a}}_P$ and $\check{\mathfrak{a}}_0^P$ is obtained by restriction to $\mathfrak{a}_P$ and $\mathfrak{a}_0^P$ and is thus denoted by $\lambda\big|_{\mathfrak{a}_P}$ and $\lambda\big|_{\mathfrak{a}_0^P}$, respectively.

Let ρ_0 denote the half-sum of positive roots in the absolute root system for G, viewed as an element of $\check{\mathfrak{a}}_0$. Let Λ denote the highest weight of the finite-dimensional representation E of $G(\mathbb{C})$, viewed as an element of $\check{\mathfrak{a}}_0$.

Let W be the absolute Weyl group of G. For a standard parabolic $\mathbb{Q}$-subgroup P, let W_P be the absolute Weyl group of its Levi factor, viewed as a subgroup of W. We denote by W^P the set of coset representatives for $W_P \backslash W$, which are of minimal length in their coset.

According to [21], the Lie algebra cohomology of the unipotent radical is

$$H^q(\mathfrak{n}_P, E) = \bigoplus_{\substack{w \in W^P \\ \ell(w) = q}} F_{\mu_w},$$

where $\ell(w)$ is the length of w, and F_{μ_w} is the irreducible finite-dimensional representation of $M_P(\mathbb{C})$ of highest weight

$$\mu_w = w(\Lambda + \rho_0) - \rho_0.$$

Then, the non-vanishing conditions arise from the non-vanishing conditions for the cuspidal cohomology of the Levi factor with respect to coefficient systems F_{μ_w}. The results are summarized in the following theorem.

Theorem 3.1 (Necessary non-vanishing conditions). *Let $P = M_P N_P$ be a standard proper parabolic $\mathbb{Q}$-subgroup of G. Let $\pi = \pi_0 \otimes \nu_0$ be a cuspidal automorphic representation of $M_P(\mathbb{A})$, where π_0 is unitary, and ν_0 an element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P$ viewed as a character of $M_P(\mathbb{A})$.*

Then, the summand (A)*, that is,*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E)$$

in the decomposition of Eisenstein cohomology is trivial except possibly if the following conditions

(1) $\nu_0 = -w(\Lambda + \rho_0)\big|_{\mathfrak{a}_P}$,

(2) $-w(\Lambda + \rho_0)\big|_{\mathfrak{a}_0^P}$ *is the infinitesimal character of* π_0,

(3) $-w_{l,P}\left(\mu_w\big|_{\mathfrak{a}_0^P}\right) = \mu_w\big|_{\mathfrak{a}_0^P}$, *where $w_{l,P}$ is the longest element of W_P,*

(4) $\pi_{0,\infty}$ *is cohomological,*

are all satisfied with the same minimal coset representative $w \in W^P$.

Regarding the proof of these facts, the first two conditions come from the compatibility with $\mathcal{J}$, and were proved in [27, Cor. 3.5] and [27, page 55], respectively. The third condition follows from [3]. The last condition is clear from the definition of cohomological representations. Recall that a representation of $G(\mathbb{R})$ is called cohomological, if it has non-trivial cohomology with respect to some coefficient system [34].

As an example of the strength of these necessary conditions for non-vanishing, we have the following theorem. We state here the theorem only for $\mathbb{Q}$-split classical groups, although an analogous result holds for the general linear group as well [16], [14]. The case of non-split classical groups can be handled in the same way, because the non-vanishing conditions are given in terms of absolute root systems and absolute Weyl groups. The precise statements for non-split classical groups can thus be deduced from the split case. For example, the case of unitary groups is considered in our work in progress [18].

Theorem 3.2 ([16], [14]). *Let G be one of the $\mathbb{Q}$-split classical groups Sp_n, SO_{2n+1}, SO_{2n} of $\mathbb{Q}$-rank n. Let $P = M_P N_P$ be the standard parabolic subgroup with the Levi factor $M_P \cong GL_{n_1} \times \cdots \times GL_{n_k} \times G'$, where G' is a (possibly trivial) smaller group of the same type. Let $\pi_0 \cong \tau_1 \otimes \cdots \otimes \tau_k \otimes \sigma$ be a unitary cuspidal automorphic representation of $M_P(\mathbb{A})$, where τ_i, resp. σ, is a unitary cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$, resp. $G'(\mathbb{A})$. Let $\nu_0 \in \check{\mathfrak{a}}_P$ correspond to the character $|\det|^{s_1} \otimes \cdots \otimes |\det|^{s_k}$ of $M_P(\mathbb{A})$, where $s_i \in \mathbb{R}$. Let*

$$\pi \cong \pi_0 \otimes \nu_0 \cong \tau_1|\det|^{s_1} \otimes \cdots \otimes \tau_k|\det|^{s_k} \otimes \sigma.$$

Then, the summand (A) *in the decomposition of Eisenstein cohomology along the cuspidal support, that is,*

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E),$$

is trivial except possibly if all $s_i \in \frac{1}{2}\mathbb{Z}$*.*

We call this property the half-integrality condition. Its importance lies in the fact that further study of the summand (A) requires understanding of the analytic properties of the Eisenstein series $E(f,\nu)$ at $\nu = \nu_0$, and this is simplified by the half-integrality restriction on the possible s_i. For example, if σ is globally generic in the theorem above, the Langlands–Shahidi method [22], [25], [31], [32], tells us that the poles of the Eisenstein series at $\nu = \nu_0$ are determined by the poles of the complete automorphic L-functions in its constant term. In the cases considered in the theorem, the automorphic L-functions in the constant term of the Eisenstein series are the Rankin–Selberg automorphic L-functions of pairs for the general linear group times the classical group, and the symmetric and exterior square automorphic L-functions attached to τ_i at the value $2s_i$ of their complex parameter. For the generic representations, by the global functorial lifting from classical groups to the general linear group [6], [7], the former are related to the Rankin–Selberg automorphic L-functions of pairs for the general linear group, which are well understood. For the latter, according to the above theorem, only $2s_i \in \mathbb{Z}$, that is, the analytic behavior at integral values of the complex parameter, matters for cohomology. This excludes the critical strip $0 < Re(s) < 1$ from the consideration. The holomorphy of the symmetric and exterior square (complete) automorphic L-functions for the values of its complex parameter in the critical strip is only recently proved [11], [19], using Arthur's endoscopic classification of automorphic representations in the discrete spectrum for G as in the theorem [1]. At the time of our study, Arthur's classification was still conjectural, and we used the above theorem to get unconditional results on automorphic cohomology.

4. Square-integrable cohomology

The first step towards complete understanding of the internal structure of the summand (A) in the decomposition of Eisenstein cohomology along the cuspidal support is understanding the summand (L) in the square-integrable cohomology, that is,

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E), \quad \{P\} \neq \{G\}.$$

This is due to the fact that the (possibly trivial) space $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ is a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-submodule of $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$, which always forms the lowest filtration step in the Franke filtration of the latter.

The summand (L) in square-integrable cohomology is, of course, trivial if the summand (A) in the full Eisenstein cohomology is trivial. Hence, all necessary conditions for non-vanishing stated in Theorem 3.1 should be satisfied in order that (L) is possibly non-trivial. Since these conditions are obtained from cohomological considerations, we refer to them as geometric conditions.

On the other hand, the summand (L) is certainly trivial if the space $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ is trivial. The space $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ is spanned by the square-integrable iterated residues at $\nu = \nu_0$ of the Eisenstein series associated to π_0. The existence of (non-zero) such residues is determined by the analytic behavior of the Eisenstein series at $\nu = \nu_0$, which is closely related to the analytic behavior of the automorphic L-functions in its constant term, according to the Langlands–Shahidi method [22], [25], [31], [32]. Hence, the necessary conditions for non-vanishing of the summand (L) arising from non-triviality of the space $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ can be written in terms of the analytic properties of automorphic L-functions. We refer to them as arithmetic non-vanishing conditions.

The geometric and arithmetic necessary conditions for non-vanishing of the summand (L) of square-integrable cohomology form a subtle combination, which provides a strong restriction on possible contributions to square-integrable cohomology. As an example, we state here a theorem for summands (L) supported in the Siegel maximal proper parabolic $\mathbb{Q}$-subgroup of the symplectic group over $\mathbb{Q}$. This is a special case of the results obtained in [15], which are dealing with arbitrary maximal proper parabolic $\mathbb{Q}$-subgroup of the symplectic group. The analogous results for the case of odd special orthogonal group are obtained by Gotsbacher and Grobner in [10].

Theorem 4.1 ([15])**.** *Let $G = Sp_n$ be the symplectic group over $\mathbb{Q}$ of $\mathbb{Q}$-rank n. Let P be the Siegel standard maximal proper parabolic $\mathbb{Q}$-subgroup, that is, the Levi factor is $M_P \cong GL_n$. Let π_0 be a unitary cuspidal automorphic representation of $M_P(\mathbb{A}) \cong GL_n(\mathbb{A})$. Let $\nu_0 \in \check{\mathfrak{a}}_P$ correspond to the character $|\det|^{s_0}$, with $s_0 \geq 0$, of $M_P(\mathbb{A}) \cong GL_n(\mathbb{A})$. Let $\pi \cong \pi_0 \otimes |\det|^{s_0}$. Let $\Lambda = \sum_{i=1}^{n} \lambda_i e_i$, with $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, be the highest weight of the finite-dimensional representation E of G, where e_i is the projection of a fixed maximal split torus of G onto its i^{th} component.*

Then, the summand (L) in square-integrable cohomology supported in π, that is,

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E),$$

is trivial, except possibly if the following conditions are satisfied.

1. *$s_0 = 1/2$,*
2. *the exterior square automorphic L-function $L(s, \pi_0, \wedge^2)$ has a pole at $s = 1$,*
3. *the principal automorphic L-function $L(s, \pi_0)$ is non-zero at $s = 1/2$,*
4. *the rank n is even,*
5. *the highest weight Λ satisfies $\lambda_{2j-1} = \lambda_{2j}$ for $j = 1, \ldots, n/2$,*
6. *the infinite component $\pi_{0,\infty}$ of π_0 is the tempered representation of $GL_n(\mathbb{R})$ which is isomorphic to the fully induced representation*

$$\pi_{0,\infty} \cong \mathrm{Ind}_{Q(\mathbb{R})}^{GL_n(\mathbb{R})} \left(\bigotimes_{j=1}^{n/2} D(2\mu_j + 2n - 4j + 4) \right),$$

where $\mu_j = \lambda_{2j-1} = \lambda_{2j}$, $D(k)$ with $k \geq 2$ is the discrete series representation of $GL_2(\mathbb{R})$ of lowest $O(2)$-type k, and Q is the standard parabolic subgroup of GL_n with the Levi factor isomorphic to a direct product of $n/2$ copies of GL_2.

From the above theorem one may get the idea how difficult it is to determine the very existence of a cuspidal support which may possibly contribute non-trivially to square-integrable cohomology. In the special case treated in the theorem, the unitary part π_0 of the cuspidal support should have a precisely determined infinite component, while at the same time should satisfy both arithmetic conditions: the exterior square L-function attached to π_0 should have a pole at $s = 1$ and the principal L-function attached to π_0 should be non-zero at $s = 1/2$.

On the other hand, the theorem excludes from consideration many possibilities. It shows, for example, that if the rank of the symplectic group is odd, then there is no contribution to square-integrable cohomology from automorphic forms supported in the Siegel maximal proper parabolic subgroup. It also shows that for some coefficient systems, namely those not satisfying condition (5) of the theorem, there is no contribution to square-integrable cohomology from automorphic forms supported in the Siegel maximal proper parabolic subgroup.

To further illustrate the subtle combination of geometric and arithmetic necessary conditions for non-vanishing, we consider a simple low-rank example in which some of the more technical conditions of the above theorem are avoided.

Corollary 4.2. *Let $G = Sp_2$ be the symplectic group over $\mathbb{Q}$ of $\mathbb{Q}$-rank two. Let P be the Siegel standard maximal proper parabolic $\mathbb{Q}$-subgroup of G, that is, the Levi factor $M_P \cong GL_2$. Let π_0 be a unitary cuspidal automorphic representation of $M_P(\mathbb{A}) \cong GL_2(\mathbb{A})$. Let $\nu_0 \in \check{\mathfrak{a}}_P$ correspond to the character $|\det|^{s_0}$, with $s_0 \geq 0$, of $M_P(\mathbb{A}) \cong GL_2(\mathbb{A})$. Let $\pi \cong \pi_0 \otimes |\det|^{s_0}$. Let the coefficient system E be the trivial representation of G.*

Then, the summand (L) *in square-integrable cohomology supported in π, that is,*

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E),$$

is trivial, except possibly if the following conditions are satisfied.

(1) $s_0 = 1/2$,
(2) *the central character of π_0 is trivial,*
(3) *the principal automorphic L-function $L(s, \pi_0)$ is non-zero at $s = 1/2$,*
(4) *the infinite component $\pi_{0,\infty}$ of π_0 is the discrete series representation of $GL_2(\mathbb{R})$ of lowest $O(2)$-type* 4.

Hence, to show the existence of a cuspidal support π in the Siegel parabolic subgroup that could possibly contribute non-trivially to square-integrable cohomology of Sp_2, with respect to the trivial coefficient system, one should show the existence of a unitary cuspidal automorphic representation π_0 of $GL_2(\mathbb{A})$ with trivial central character, discrete series of lowest $O(2)$-type 4 as the infinite component, and such that $L(1/2, \pi_0) \neq 0$. In terms of classical automorphic forms, this may be rephrased as the existence problem for a holomorphic modular form of weight 4, trivial Nebentypus, arbitrary level, and such that $L(1/2, \pi_0) \neq 0$. This problem in wider generality was studied by Trotabas in [33]. A consequence of his work is the existence of a Hilbert modular form of any given even weight, trivial Nebentypus, arbitrary level, and such that $L(1/2, \pi_0) \neq 0$. In particular, it shows the existence of π_0 with the required properties.

In a recent preprint [17], we study the existence of non-trivial cohomology classes in square-integrable cohomology for the split symplectic and special orthogonal groups of rank two, as well as the exceptional group G_2, over a totally real number field. In that work, we encounter various arithmetic conditions, and to show the existence of the cuspidal support satisfying the necessary non-vanishing conditions, we use not only the result of Trotabas, but also a construction of monomial representations of $GL_2(\mathbb{A})$ via automorphic induction from the appropriate Hecke characters of the group of idèles.

However, the existence of the cuspidal support satisfying the necessary non-vanishing conditions is still not sufficient to imply the non-vanishing of the summand (L) in the decomposition of square-integrable cohomology along the cuspidal support. This requires further study of the internal structure of the summand (A) in the decomposition of full Eisenstein cohomology, which is the subject of the following section.

5. Internal structure of cohomology

As already mentioned above, once we establish the existence of a cuspidal support ϕ_π, satisfying all the necessary non-vanishing conditions for the summand (L) in the decomposition of square-integrable cohomology along the cuspidal support, that is,

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E), \quad \{P\} \neq \{G\},$$

the final step is to show that this summand is indeed non-trivial.

There exist two approaches to settle this problem. The first is due to Rohlfs–Speh [26, Thm. I.1=III.1]. They show that, given a cuspidal support ϕ_π satisfying the necessary non-vanishing conditions, the summand (L) of square-integrable cohomology is non-trivial in the lowest degree in which the relative Lie algebra cohomology

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E)$$

is non-trivial. In other words, they show that the map in cohomology induced by the inclusion of $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}$ into $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ is non-zero in that lowest degree. Their method of proving this fact is using the geometric expression of the automorphic cohomology in terms of differential forms.

We use the result of Rohlfs–Speh in the preprint [17], already mentioned above, to show the actual non-vanishing of certain summands in the decomposition of square-integrable cohomology for the split symplectic and special orthogonal groups of rank two, and the exceptional group G_2, defined over a totally real number field. As an example, we present here the special case already considered in Corollary 4.2.

Theorem 5.1 ([17]). *Let $G = Sp_2$ be the symplectic group over $\mathbb{Q}$ of $\mathbb{Q}$-rank two. Let P be the Siegel standard maximal proper parabolic $\mathbb{Q}$-subgroup of G, that is, the Levi factor $M_P \cong GL_2$. Let the coefficient system E be the trivial representation of G.*

Then, there exists a unitary cuspidal automorphic representation π_0 of $M_P(\mathbb{A}) \cong GL_2(\mathbb{A})$ such that, for the cuspidal support $\pi \cong \pi_0 \otimes |\det|^{1/2}$, the summand (L) *in square-integrable cohomology supported in π, that is,*

$$H^*_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E),$$

does not vanish in degree $q = 2$. These classes, represented by square-integrable residues of Eisenstein series, contribute non-trivially in degree $q = 2$ to the full Eisenstein cohomology space $H^*_{\mathrm{Eis}}(Sp_2, \mathbb{C})$.

The second approach to the non-vanishing of summands (L) is using the Franke filtration. The Franke filtration, originally defined in [8, Sect. 6], and its refinement introduced in [9, Thm. 1.4], is a finite descending filtration of the $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module $\mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$. The main advantage of using this filtration is that the successive quotients of the filtration may be described as parabolically induced representations from certain Levi factors of G. This allows not only showing the non-vanishing of the summands (L) in square-integrable cohomology, but also the explicit calculation of the internal structure of the summands (A) in the full Eisenstein cohomology. The idea is to calculate first the cohomology of filtration quotients, and then use, step-by-step, the long exact sequences in cohomology.

The main disadvantage of the Franke filtration approach is that it may be very difficult to write the filtration in a form feasible for explicit calculation. The reason is that the definition of the filtration depends on the analytic behavior, not only of the Eisenstein series associated to π_0, but also all possible degenerate Eisenstein series with same cuspidal support, on the Levi factors of parabolic subgroups containing an element of the associate class $\{P\}$. The Franke filtration for the case of the cuspidal support in the associate class $\{P\}$ of a maximal proper parabolic subgroup P is described in [12].

The Franke filtration approach was pursued in [13] for the case of the symplectic group of rank two defined over a totally real number field. As an example of the Franke filtration approach to the calculation of cohomology, we present here again the same example as in Corollary 4.2 and Theorem 5.1. Observe that using the Franke filtration provides more information about the structure of cohomology. This result over $\mathbb{Q}$ was earlier obtained by Schwermer in [28] using a completely different method.

Theorem 5.2 ([13], [28])**.** *Let $G = Sp_2$ be the symplectic group over $\mathbb{Q}$ of $\mathbb{Q}$-rank two. Let P be the Siegel standard maximal proper parabolic $\mathbb{Q}$-subgroup of G, that is, the Levi factor $M_P \cong GL_2$. Let the coefficient system E be the trivial representation of G. Assume that π_0 is a unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$, such that the cuspidal support ϕ_π, represented by $\pi \cong \pi_0 \otimes |\det|^{1/2}$, satisfies the necessary conditions for non-vanishing of the summand* (L)*.*

Then, the summand (A) *in the decomposition of full Eisenstein cohomology along the cuspidal support is isomorphic to*

$$H^q(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}) \cong$$
$$\begin{cases} H^2_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}) = H^2(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}) \neq 0, & \text{for } q = 2, \\ \text{a submodule of } H^3(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi} / \mathcal{L}_{\mathcal{J},\{P\},\phi_\pi}), & \text{for } q = 3, \\ H^4_{(\mathrm{sq})}(\mathfrak{g}_\infty, K_\infty; \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}), \text{ possibly trivial}, & \text{for } q = 4, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, under the assumption that π_0 with the required properties exists, the square-integrable cohomology is non-trivial in degree $q = 2$.

The non-vanishing of the map in cohomology induced by $\mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\phi_\pi}$ in general was studied by Grobner in [20] using the Franke filtration. In that paper he shows that this map is injective in a certain range of cohomology degrees (all low degrees up to a certain bound). This, as a consequence, reproves the result of Rohlfs–Speh. However, in order to show actual non-vanishing, it is still necessary to check the existence of the cuspidal support ϕ_π such that the cohomology space

$$H^*(\mathfrak{g}_\infty, K_\infty; \mathcal{L}_{\mathcal{J},\{P\},\phi_\pi} \otimes_{\mathbb{C}} E)$$

is non-trivial in some of the degrees in which the map in cohomology is injective. This again boils down to the subtle combination of geometric and arithmetic conditions, as in Sect. 3 and Sect. 4, which were studied in [17] for aforementioned rank two cases. In particular, the existence of representations π_0 as in the theorem is obtained in [17].

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