HOLOMORPHY OF EISENSTEIN SERIES – A NEW METHOD AND APPLICATIONS IN THE CASE OF THE GENERAL LINEAR GROUP

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Abstract. The paper proposes a new method for proving holomorphy of Eisenstein series on the adèlic points of a reductive connected linear algebraic group defined over a number field. It relies on the description of the Franke filtration of the spaces of automorphic forms with a fixed cuspidal support in terms of main values of derivatives of Eisenstein series. The method is applied in the case of the arbitrary degenerate Eisenstein series on the general linear group. It completely determines the regions of holomorphy in that case.

Contents

Introduction

The purpose of this paper is to present a new method for proving the holomorphy of Eisenstein series. The proposed method is general in principle, although its applicability may depend on the setting. It is based on the Franke filtration of the spaces of automorphic forms [3] and its description in terms of (derivatives of) Eisenstein series, and their residues. The method is appropriate for Eisenstein series associated with both cuspidal and residual automorphic representations of the Levi factor of a parabolic subgroup. The latter are referred to as degenerate Eisenstein series in this paper. The

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limitation of the method is that it can only prove holomorphy of Eisenstein series at the values of its complex parameter in a certain range. It cannot detect poles, not to mention their order or residues.

The Franke filtration, introduced in [3] and refined in [4] and [6], is a finite decreasing filtration of the space of automorphic forms on the adèlic points of a connected reductive linear algebraic group defined over a number field. The quotients of the filtration are described in terms of parabolically induced representations. The explicit description of the quotients is very useful in applications in which the internal structure of the space of automorphic forms plays a decisive role.

Already in Franke's paper [3], the filtration is applied to prove that every non-cuspidal automorphic form is obtained as a derivative of an Eisenstein series or its residue, which has the consequence that the cohomology of congruence arithmetic subgroups can be calculated from the automorphic cohomology of the ambient group. Refinements of the Franke filtration and further applications, especially in the cohomology of arithmetic groups, can be found among others in the papers [4], [8], [10], [11], [20], [22], [23]. We refer to the introduction of [7] for more details.

However, the application of the Franke filtration in this paper is of entirely different flavor. Instead of using the information about analytic behavior of Eisenstein series to determine the Franke filtration and then apply it to different problems, we do exactly the opposite. We use, for the first time, the information about the Franke filtration to deduce analytic properties of Eisenstein series.

We consider the Franke filtration of the spaces of automorphic forms with the fixed cuspidal support, as in [6], and use the very construction of isomorphisms between parabolically induced representations and the quotients of the filtration. These isomorphisms are defined using the main values of the (derivatives of) Eisenstein series. For our purposes, the key observation is that in the case of non-holomorphic Eisenstein series, these main values are well-defined only as elements of the quotient of the filtration.

In other words, if an Eisenstein series had a pole at some fixed value of its complex parameter, then the isomorphism arising from that Eisenstein series to the Franke filtration would be defined in terms of the main values of its derivatives. These main values would be well-defined only up to the span of coefficients in the principal part of its Laurent expansion, which should thus belong to a deeper filtration step. Hence, if we can show, from the definition of the quotients of the filtration for the given fixed cuspidal support, that such (non-trivial) deeper filtration step does not exist, then the Eisenstein series could not have had the pole in the first place, thus implying its holomorphy. This process will be formally structured in three steps in Sect. 3.

All the existing methods for studying the analytic properties of the Eisenstein series, such as the Langlands–Shahidi method [21], consider the Eisenstein series as an isolated object, and examine its constant term or other types of objects associated with it. In our method the possible deeper step of the filtration would arise from another (degenerate) Eisenstein series. Hence, the main new idea underlying the proposed method is that we should look at all the Eisenstein series with the given cuspidal support at the same time in order to prove that the considered Eisenstein series is holomorphic.

Although the above description of the method seems quite simple, it relies on the deep result of Franke and its refinements. In addition, showing that there are no deeper filtration steps than the one of the considered Eisenstein series may be difficult to achieve in some settings. That is the reason why we mention above that our method is general in principle, but the application depends on the setting.

However, as we elaborate in the body of the paper, there are many cases in which our method is applicable, and provides an easy proof of the holomorphy of certain Eisenstein series which may be difficult to prove, or even out of reach, by other methods. In fact, our motivation to write this paper arose from the recent preprint by Ginzburg and Soudry [5], generalizing the previous work of Hanzer and Muić [12]. However, our results go far beyond their work.

These papers determine the poles of the degenerate Eisenstein series on the general linear group associated with the tensor product of two Speh representations viewed as a residual representation of the Levi factor of a maximal proper parabolic subgroup, evaluated at the values of their complex parameter in the closed right half-plane. The difference is that Hanzer and Muić assume that the Speh representations are characters, that is, with the cuspidal support in the Borel subgroup. The methods of the two papers are different. Hanzer and Muić carefully study the constant term along the Borel subgroup, in which only the Hecke L-functions appear, and using a highly technical combinatorics of the summands succeed in the complete description of the poles. On the other hand, to determine the poles in the more general situation, Ginzburg and Soudry use a careful study of certain Fourier coefficient analogous to the Bernstein–Zelevinsky derivative [1], [24], that plays the role of the method of descent, which is again very sophisticated and highly technical.

In the paper, we explain how the proposed method, applied to the case of the general linear group, can determine precisely the regions of holomorphy of the Eisenstein series studied by Ginzburg and Soudry in [5]. The remaining points in which our method cannot decide are precisely those in which the considered Eisenstein series has a pole according to [5]. See Sect. 5 for more details.

Our main result is much more general than the one in [5] and [12]. Namely, it determines the holomorphy regions for (degenerate) Eisenstein series on the general linear group associated with a discrete spectrum representation of the Levi factor of a non-maximal parabolic subgroup. It turns out again that the remaining points are precisely the poles of these Eisenstein series, so that we completely determine the analytic behavior (except the order of poles) of the most general Eisenstein series on $GL_n(\mathbb{A})$. This is far beyond the results of [5] and [12]. See Sect. 6 for more details.

The paper is organized as follows. Section 1 provides the notation and basic notions required in the paper. In particular, it introduces the notion of automorphic forms and representations. In Section 2, the Franke filtration of the space of automorphic forms with a fixed cuspidal support is defined, and the isomorphisms between parabolically induced representations and the quotients of the filtration are explicitly described in terms of Eisenstein series. The new method for proving holomorphy of Eisenstein series is described and proved in Section 3. In Section 4, the notation is specialized to the case of the general linear group, and the description of the discrete spectrum of the general linear group and the formalism of segments are recalled. Finally, Section 5 and Section 6 provide evidence for the applicability of the proposed method by considering the case of Eisenstein series on the general linear group.

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1. Preliminaries and notation

Let F be an algebraic number field, $\mathbb{A}$ the ring of adèles, and $\mathbb{I}$ the group of idèles of F. The subring of finite adèles is denoted by $\mathbb{A}_f$, and the archimedean part by F_∞, so that $\mathbb{A} \cong F_\infty \times \mathbb{A}_f$.

Let G be a connected reductive linear algebraic group defined over F. For any F-algebra $\mathcal{R}$, let $G(\mathcal{R})$ be the group of $\mathcal{R}$-rational points of G. Observe that $G(\mathbb{A}) \cong G(F_\infty) \times G(\mathbb{A}_f)$. Denote by $\mathfrak{g}_\infty$ the real Lie algebra of the archimedean factor $G(F_\infty)$ of $G(\mathbb{A})$.

Fix a maximal compact subgroup K of G, which is a product over all places of maximal compact subgroups of local components of $G(\mathbb{A})$, such that it is hyperspecial at all but finitely many finite places of F. Write $K = K_\infty \times K_f$, where K_∞ is the archimedean, and K_f the finite part of K.

Fix once for all a minimal parabolic subgroup P_0 of G defined over F, which is in good position with respect to K in the sense of [19, Sect. I.1.4]. A parabolic subgroup of G is called standard if it contains P_0. All parabolic subgroups in the paper are assumed to be standard. Given a parabolic subgroup R of G, let $R = L_R N_R$ be its Levi decomposition, where L_R is the Levi factor, and N_R the unipotent radical. Let A_R denote a maximal split torus in the center of L_R.

Let W be the Weyl group of G with respect to P_0. The Weyl group of the Levi factor L_R of a parabolic subgroup R is denoted by W_{L_R}. Two parabolic subgroups of G are called associate if their Levi factors are conjugate by an element of the Weyl group W.

For a parabolic subgroup R, denote by $X^*(L_R)$ the $\mathbb{Z}$-module of rational characters of the Levi factor L_R. Let $\check{\mathfrak{a}}_R = X^*(L_R) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\check{\mathfrak{a}}_{R,\mathbb{C}}$ be its complexification. If R and Q are two parabolic subgroups of G such that $Q \subset R$, then the restriction of characters gives rise to an inclusion

$$\check{\mathfrak{a}}_R \to \check{\mathfrak{a}}_Q,$$

and the same for the complexifications. In the special case of $Q = P_0$, we denote all the inclusions of various $\check{\mathfrak{a}}_R$ into $\check{\mathfrak{a}}_{P_0}$ by ι.

Since G is reductive, and not necessarily semisimple, the space $\check{\mathfrak{a}}_G$ may be non-trivial. Then, for any proper parabolic subgroup $R \neq G$ of G, the natural complement of the space $\check{\mathfrak{a}}_G$ in $\check{\mathfrak{a}}_R$ is denoted by $\check{\mathfrak{a}}_R^G$. This space corresponds to the characters trivial on the connected component of the center. Note that in the case of a semisimple group G, we have $\check{\mathfrak{a}}_R^G = \check{\mathfrak{a}}_R$. The inclusions ι into $\check{\mathfrak{a}}_{P_0}$, introduced above, are still denoted by ι when restricted to spaces $\check{\mathfrak{a}}_R^G$ and viewed as inclusions into $\check{\mathfrak{a}}_{P_0}^G$.

Similarly as for characters, let $X_*(L_R)$ be the $\mathbb{Z}$-module of rational cocharacters of L_R, the space $\mathfrak{a}_R = X_*(L_R) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathfrak{a}_{R,\mathbb{C}}$ its complexification. These spaces are dual to the corresponding spaces of characters in the standard pairing denoted $\langle\, ,\, \rangle$. Let $H_R : G(\mathbb{A}) \to \mathfrak{a}_R$ denote the map defined by

$$\exp\langle \lambda, H_R(g) \rangle = |\lambda(l)|,$$

where $\lambda \in \check{\mathfrak{a}}_R$, and $g = nlk$, with $n \in N_R(\mathbb{A})$, $l \in L_R(\mathbb{A})$, $k \in K$, is any Iwasawa decomposition of $g \in G(\mathbb{A})$. Here $|\cdot|$ is the absolute value on $\mathbb{I}$, and exp at the finite places of F stands for the exponential function with cardinality of the residue field as the basis. The definition is independent of the choice of the Iwasawa decomposition of g.

Let $\mathcal{A}$ denote the space of all automorphic forms on $G(\mathbb{A})$, in the sense of [2], trivial on the connected component of $A_G(F_\infty)$. It carries a structure of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. An automorphic form is called cuspidal, if its constant term along all proper parabolic subgroups of G vanishes.

The discrete spectrum $L^2_{\text{disc}}(G(\mathbb{A}))$ of $G(\mathbb{A})$ is the span of all irreducible subrepresentations of the right regular representation of $G(\mathbb{A})$ on the L^2 space of classes of functions on $G(\mathbb{A})$ with respect to a fixed Haar measure, left invariant under $G(F)$ and the connected component of $A_G(F_\infty)$. It decomposes in a direct sum of irreducible subrepresentations. The space of smooth K-finite functions in an irreducible summand of $L^2_{\text{disc}}(G(\mathbb{A}))$ is referred to as a unitary discrete spectrum representation of $G(\mathbb{A})$, although it is only a $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module. A unitary discrete spectrum representation of $G(\mathbb{A})$ that consists of cuspidal automorphic forms is called a unitary cuspidal automorphic representation of $G(\mathbb{A})$.

2. Franke filtration and Eisenstein series

Let π be a (unitary irreducible) cuspidal automorphic representation of the Levi factor $L_P(\mathbb{A})$ of a parabolic subgroup P, and let ν_0 in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P^G$ be fixed. Let ϕ denote the associate class of cuspidal automorphic representations of the Levi factors of the parabolic subgroups in the associate class $\{P\}$ of P, as in [16, Sect. 1.3], which is represented by the representation $\pi \otimes \exp\langle \nu_0, H_P(\cdot) \rangle$ of $L_P(\mathbb{A})$. The space $\mathcal{A}_{\{P\},\phi}$ of automorphic forms with the cuspidal support ϕ can be defined using Eisentein series as in [4, Sect. 1.3]. This definition is in accordance with the more classical definition of [19, Sect. III.2.6], as elaborated in [4, Thm. 1.4].

The Franke filtration is described here following the approach first applied in [6]. The point is to fix the full cuspidal support $\{P\}$ and ϕ, in the notation as above, and consider the space of automorphic forms $\mathcal{A}_{\{P\},\phi}$ with the fixed cuspidal support. The quotients of the filtration can then be described in terms of explicit parabolically induced representations.

The Franke filtration of $\mathcal{A}_{\{P\},\phi}$ is defined in terms of triples (R, Π, λ_0), where

- R is a (standard) parabolic subgroup of G which contains an element of the associate class $\{P\}$,
- Π is a (unitary) discrete spectrum representation of the Levi factor $L_R(\mathbb{A})$ of R,
- λ_0 is an element in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^G$,

such that the representation $\Pi \otimes \exp\langle\lambda_0, H_R(\cdot)\rangle$ has the cuspidal support in the associate class represented by $\pi \otimes \exp\langle\nu_0, H_P(\cdot)\rangle$. By the construction of the residual spectrum, this means that $\Pi \otimes \exp\langle\lambda_0, H_R(\cdot)\rangle$ is obtained as the span of square-integrable iterated residues of the Eisenstein series associated with an element of the associate class of π at the appropriate value of ν.

The set of all such triples is denoted by $\mathcal{M} = \mathcal{M}_{\{P\},\phi}$, where we omit the reference to the cuspidal support from the notation, because it is fixed. The set $\mathcal{M}$ is considered as the set of objects of a groupoid in which the isomorphisms between triples are defined in terms of conjugation by Weyl group elements. More precisely, an isomorphism from (R, Π, λ_0) to (R', Π', λ_0') is any Weyl group element $w \in W(L_R)$ such that the conjugation by w gives $wL_Rw^{-1} = L_{R'}$, $\Pi^w = \Pi'$ and $(\exp\langle\lambda_0, H_R(\cdot)\rangle)^w = \exp\langle\lambda_0', H_{R'}(\cdot)\rangle$. Here $W(L_R)$ stands for the set of minimal length representatives w of the left cosets of the Weyl group W_{L_R} of L_R in the full Weyl group W such that the conjugation wL_Rw^{-1} is the Levi factor of a standard parabolic subgroup in the associate class $\{R\}$.

There is a functor, denoted by $M = M_{\{P\},\phi}$, from the groupoid $\mathcal{M}$ to the category of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules, defined on objects by

$$M\big((R, \Pi, \lambda_0)\big) = \mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})}\big(\Pi \otimes \exp\langle\lambda_0 \otimes H_R(\cdot)\rangle\big) \otimes S(\check{\mathfrak{a}}_{R,\mathbb{C}}^G),$$

where $S(\check{\mathfrak{a}}_{R,\mathbb{C}}^G)$ is the symmetric algebra on $\check{\mathfrak{a}}_{R,\mathbb{C}}^G$. The $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-module structure on the right-hand side is defined as in [3, p. 218]. We omit the definition of M on isomorphisms of the groupoid $\mathcal{M}$, and only refer to [3, p. 234], because it is not required in the rest of the paper.

Consider the set $\mathcal{S} = \mathcal{S}_{\{P\},\phi}$ of inclusions $\iota(\lambda_0)$ into $\check{\mathfrak{a}}_{P_0}^G$ of the third entries λ_0 of all the triples in $\mathcal{M}$. It is a finite set, as explained in [7, Sect. 4.2]. A partial order on $\mathcal{S}$ is defined by

$$\iota(\lambda_0) \succ \iota(\lambda_0')$$

if and only if $\iota(\lambda_0) \neq \iota(\lambda_0')$ and $\iota(\lambda_0) - \iota(\lambda_0')$ belongs to the closure of the negative obtuse Weyl chamber in $\check{\mathfrak{a}}_{P_0}^G$. We make the choice of an integer-valued function $T = T_{\{P\},\phi}$ on $\mathcal{S}$ which obeys the partial order, that is,

$$T(\iota(\lambda_0)) > T(\iota(\lambda_0')) \quad \text{whenever} \quad \iota(\lambda_0) \succ \iota(\lambda_0').$$

The quotients of the Franke filtration are determined by the triples in $\mathcal{M}$ and the choice of function T. The following theorem is essentially [3, Thm. 14] of Franke, refined as in [6].

Theorem 2.1. *The* $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-module* $\mathcal{A}_{\{P\},\phi}$ *of automorphic forms on* $G(\mathbb{A})$*, with cuspidal support in the associate class* $\{P\}$ *of parabolic subgroups and the associate class* ϕ *of cuspidal automorphic representations of their Levi factors, admits a descending filtration*

$$\mathcal{A}_{\{P\},\phi} \supseteq \cdots \supseteq \mathcal{A}^i_{\{P\},\phi} \supseteq \mathcal{A}^{i+1}_{\{P\},\phi} \supseteq \dots,$$

indexed by integers, where the quotients of the filtration are isomorphic to

$$\mathcal{A}^i_{\{P\},\phi}/\mathcal{A}^{i+1}_{\{P\},\phi} \cong \operatorname*{colim}_{\substack{(R,\Pi,\lambda_0)\in\mathcal{M} \\ T(\iota(\lambda_0))=i}} \operatorname{Ind}^{G(\mathbb{A})}_{R(\mathbb{A})}\big(\Pi \otimes \exp\langle\lambda_0, H_R(\cdot)\rangle\big) \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$$

as $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$*-modules, and only finitely many quotients are non-trivial. The colimit is taken with respect to the restriction of the functor* M*, see* [17, Sect. III.2].

In the next section, we also require the precise definition of the isomorphisms of Thm. 2.1 between the quotients of the filtration and the parabolically induced representations. They are given in terms of main values of derivatives of Eisenstein series. Hence, we first recall the definition of the Eisenstein series associated with a discrete spectrum representation Π of $L_R(\mathbb{A})$.

Let Π denote a (unitary irreducible) discrete spectrum representation of the Levi factor $L_R(\mathbb{A})$ of a (standard) parabolic subgroup R of G. Let $\mathcal{W}_\Pi$ denote the space of smooth K-finite functions on

$$L_R(F)N_R(\mathbb{A})A_R(F_\infty)^\circ\backslash G(\mathbb{A}),$$

where $A_R(F_\infty)^\circ$ is the connected component of $A_R(F_\infty)$, such that the functions on $L_R(\mathbb{A})$ given by the assignment

$$l \mapsto f(lg), \qquad l \in L_R(\mathbb{A}),$$

belong to the Π-isotypic component V_Π of the discrete spectrum of $L_R(\mathbb{A})$, for all $g \in G(\mathbb{A})$. For a given $f \in \mathcal{W}_\Pi$, and a complex parameter $\lambda \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}$, let

$$f_\lambda(g) = f(g)\exp\langle\lambda + \rho_R, H_R(g)\rangle, \qquad g \in G(\mathbb{A}).$$

Then, f_λ with λ ranging through $\check{\mathfrak{a}}^G_{R,\mathbb{C}}$ form a section of the induced representations

$$I_R(\lambda, V_\Pi) = \operatorname{Ind}^{G(\mathbb{A})}_{R(\mathbb{A})}\big(V_\Pi \otimes \exp\langle\lambda, H_R(\cdot)\rangle\big).$$

We may formally define the Eisenstein series associated with Π as the family of series given by

$$E(\lambda, g, f, \Pi) = \sum_{\gamma\in R(F)\backslash G(F)} f_\lambda(\gamma g), \qquad \lambda \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}, g \in G(\mathbb{A}), f \in \mathcal{W}_\Pi.$$

The series converges absolutely and locally uniformly for $Re(\lambda)$ in a sufficiently positive cone in the positive Weyl chamber defined by R, and admits the analytic continuation to a meromorphic function of $\lambda \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}$. The standard reference for these and other properties of Eisenstein series is [19], see also [15].

In order to describe the isomorphisms of Thm. 2.1, we now follow the exposition in [4]. Given a triple $(R, \Pi, \lambda_0) \in \mathcal{M}$, we define the Eisenstein series $E(\lambda, g, f, \Pi)$, where $\lambda \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}$, $g \in G(\mathbb{A})$, and f ranges through the space $\mathcal{W}_\Pi$, as above. The symmetric algebra $S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$ may be viewed as the space of partial derivatives in Cartesian coordinates on $\check{\mathfrak{a}}^G_{R,\mathbb{C}}$. For a multi-index α, let $\frac{\partial^\alpha}{\partial\lambda^\alpha}$ denote the corresponding derivative viewed as an element of $S(\check{\mathfrak{a}}^G_{R,\mathbb{C}})$. Then, the main value

$$\mathrm{MV}_{\lambda=\lambda_0}\left(\frac{\partial^\alpha}{\partial\lambda^\alpha} E(\lambda, g, f, \Pi)\right)$$

of the derivative of Eisenstein series is defined as follows.

According to [15, Sect. 7], see also [19, Chap. IV], the possible singularities of $E(\lambda, g, f, \Pi)$ at $\lambda = \lambda_0$ are along a finite set of root hyperplanes. Hence, we may choose a non-zero vector $\vartheta \in \check{\mathfrak{a}}^G_R$ that is not contained in the tangent space of any of the singular hyperplanes. Consider the Laurent expansion

$$\frac{\partial^\alpha}{\partial\lambda^\alpha} E(\lambda_0 + z\vartheta, g, f, \Pi) = \sum_{j=-m}^{\infty} a_j(g) z^j$$

of the derivative of the Eisenstein series in the direction of ϑ, where $z \in \mathbb{C}$. We would like to define the main value as the constant term $a_0(g)$ in this expansion. It is an automorphic form, but not well-defined because it depends on the choice of direction ϑ. However, it is well defined as an element of the quotient of the space of automorphic forms by the space spanned by the coefficients $a_j(g)$, $j < 0$, in the principal part of the Laurent expansions. The key observation of Franke, verified in the proof of [3, Thm. 14], is that all the coefficients in the principal part of the Laurent expansion always belong to some deeper filtration step than the considered Eisenstein series. Hence, if $T(\iota(\lambda_0)) = i$, then

$$\mathrm{MV}_{\lambda=\lambda_0}\left(\frac{\partial^\alpha}{\partial\lambda^\alpha} E(\lambda, g, f, \Pi)\right) = a_0(g) \in \mathcal{A}^i_{\{P\},\phi}/\mathcal{A}^{i+1}_{\{P\},\phi}$$

is well-defined viewed as an element of the quotient of the filtration.

Therefore, given a triple (R, Π, λ_0) such that $\iota(\lambda_0) = i$, the map

$$\mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})}\big(V_\Pi \otimes \exp\langle\lambda_0, H_R(\cdot)\rangle\big) \otimes S(\check{\mathfrak{a}}^G_{R,\mathbb{C}}) \to \mathcal{A}^i_{\{P\},\phi}/\mathcal{A}^{i+1}_{\{P\},\phi}$$

defined by the assignment

$$f_{\lambda_0} \otimes \frac{\partial^\alpha}{\partial\lambda^\alpha} \mapsto \mathrm{MV}_{\lambda=\lambda_0}\left(\frac{\partial^\alpha}{\partial\lambda^\alpha} E(\lambda, g, f, \Pi)\right)$$

is an intertwining of $(\mathfrak{g}_\infty, K_\infty; G(\mathbb{A}_f))$-modules. Taking into account all the triples in $\mathcal{M}$ such that the value of T is i, the surjectivity follows from the definition of the filtration. The colimit is required to deal with the kernel arising from the relations given by the functional equations of Eisenstein series.

3. The method for proving holomorphy

Our main goal is to describe the method for finding regions in $\check{\mathfrak{a}}_{R,\mathbb{C}}^G$ in which the degenerate Eisenstein series $E(\lambda, g, f, \Pi)$ introduced above is holomorphic. The idea is to use the Franke filtration, and in particular the position in the filtration of the automorphic forms arising from that Eisentein series. Without loss of generality, we may and will assume an appropriate normalization of Π, as in [14, p. 121], such that all the poles of the Eisenstein series are at λ in the real space $\check{\mathfrak{a}}_R^G$.

Fix $\lambda_0 \in \check{\mathfrak{a}}_R^G$ in the closure of the positive Weyl chamber of $\check{\mathfrak{a}}_R^G$. In order to study the analytic properties of $E(\lambda, g, f, \Pi)$ at $\lambda = \lambda_0$, we consider the space of automorphic forms on $G(\mathbb{A})$ with the same cuspidal support as the Eisenstein series evaluated at $\lambda = \lambda_0$. Suppose that this cuspidal support is represented by a (unitary irreducible) cuspidal automorphic representation π of the Levi factor $L_P(\mathbb{A})$ of a parabolic subgroup P and the evaluation point ν_0 in the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_P^G$. Therefore, we consider the space of automorphic forms $\mathcal{A}_{\{P\},\phi}$, where $\{P\}$ is the associate class of P, and ϕ the associate class of $\pi \otimes \exp\langle \nu_0, H_P(\cdot)\rangle$.

Theorem 3.1. *Let $E(\lambda, g, f, \Pi)$ be the Eisenstein series associated with the discrete spectrum representation Π of the Levi factor $L_R(\mathbb{A})$ of a parabolic subgroup R of G, where $\lambda \in \check{\mathfrak{a}}_{R,\mathbb{C}}^G$, $g \in G(\mathbb{A})$, and f ranges through the space $\mathcal{W}_\Pi$, as defined in Sect. 2. Let $\lambda_0 \in \check{\mathfrak{a}}_R^G$ be fixed. Let $\{P\}$ and ϕ denote the associate class of the full cuspidal support of the given Eisenstein series evaluated at $\lambda = \lambda_0$, so that the triple (R, Π, λ_0) belongs to the set $\mathcal{M}_{\{P\},\phi}$.*

Suppose that the triple (R, Π, λ_0) is such that $\iota(\lambda_0)$ is a maximal element in the partial order on the set $\mathcal{S}_{\{P\},\phi}$. Then, the Eisenstein series $E(\lambda, g, f, \Pi)$ is holomorphic at $\lambda = \lambda_0$ for every $f \in \mathcal{W}_\Pi$.

Proof. Since $\iota(\lambda_0)$ is a maximal element in the partial order of $\mathcal{S}_{\{P\},\phi}$, the function $T_{\{P\},\phi}$ in the definition of the Franke filtration of $\mathcal{A}_{\{P\},\phi}$ can be chosen in such a way that $T_{\{P\},\phi}(\iota(\lambda_0)) = i_{\max}$ is the maximum of T. Hence, according to Thm. 2.1, the main values at $\lambda = \lambda_0$ of the derivatives of the Eisenstein series $E(\lambda, g, f, \Pi)$ contribute to the deepest non-trivial quotient of the filtration $\mathcal{A}_{\{P\},\phi}^{i_{\max}}$.

However, the construction of the isomorphisms of Thm. 2.1, in particular the main values, reveals that if the Eisenstein series $E(\lambda, g, f, \Pi)$ is not holomorphic at $\lambda = \lambda_0$, then the non-trivial coefficients in the principal part of the Laurent expansion should belong to a deeper filtration step. Thus, since there is no deeper filtration step, the Eisenstein series $E(\lambda, g, f, \Pi)$ must be holomorphic at $\lambda = \lambda_0$, as claimed. □

As a consequence of this theorem, we have a new method for proving holomorphy of Eisenstein series, already sketched in the introduction. Having all the notation in place, we now provide the step-by-step procedure for applying the method.

Problem: Let $E(\lambda, g, f, \Pi)$ be the Eisenstein series associated with the discrete spectrum representation Π of the Levi factor $L_R(\mathbb{A})$ of a parabolic subgroup R of G, where $\lambda \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}$, $g \in G(\mathbb{A})$, and f ranges through the space $\mathcal{W}_\Pi$, as defined in Sect. 2. Let $\lambda_0 \in \check{\mathfrak{a}}^G_{R,\mathbb{C}}$ be the evaluation point in the closure of the positive Weyl chamber. The problem is to prove that the Eisenstein series $E(\lambda, g, f, \Pi)$ is holomorphic at $\lambda = \lambda_0$.

Step 1: Determine the full cuspidal support of the Eisenstein series $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$. In other words, find the associate class of parabolic subgroups $\{P\}$, a unitary cuspidal automorphic representation π of the Levi factor $L_P(\mathbb{A})$ of a representative P of $\{P\}$, and an evaluation point $\nu_0 \in \check{\mathfrak{a}}^G_{P,\mathbb{C}}$ in the closure of the positive Weyl chamber, such that $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$ belongs to the space $\mathcal{A}_{\{P\},\phi}$ of automorphic forms on $G(\mathbb{A})$ supported in the associate class $\{P\}$ and ϕ represented by $\pi \otimes \exp\langle \nu_0, H_P(\cdot)\rangle$.

Step 2: Explore the Franke filtration of the space of automophic forms $\mathcal{A}_{\{P\},\phi}$ determined in Step 1, that is, study the corresponding set of triples $\mathcal{M}_{\{P\},\phi}$ and the partial order on the associated set $\mathcal{S}_{\{P\},\phi}$. More precisely, determine in the partial order the position of $\iota(\lambda_0)$ for the triple $(R, \Pi, \lambda_0) \in \mathcal{M}_{\{P\},\phi}$ arising from the Eisenstein series $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$.

Step 3: If it is determined in Step 2 that $\iota(\lambda_0)$ is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$, then the Eisenstein series $E(\lambda, g, f, \Pi)$, is holomorphic at $\lambda = \lambda_0$, according to Thm. 3.1. Otherwise, our method cannot decide if the Eisenstein series $E(\lambda, g, f, \Pi)$, is holomorphic at $\lambda = \lambda_0$ or not.

In retrospect, looking at the steps of the proposed method for proving holomorphy, it seems as we are in the famous cartoon of Sidney Harris [13], in which two mathematicians discuss the proof on the blackboard containing the step saying "Then a miracle occurs", and one of them complains: "I think you should be more explicit here in step two". One could indeed argue that our Step 2 in the proposed method is comparable to the miracle of that cartoon. In fact, the applicability of the method depends heavily on the question whether we can be "more explicit here in step two", and the rest of the paper is devoted to providing evidence that we can.

4. Discrete spectrum of GL_n and the formalism of segments

This section is a preparation for Sect. 5 and Sect. 6 in which the proposed method is applied to the case of degenerate Eisenstein series on the general linear group. We first specialize the notation of Sect. 1 to the case of the general linear group $G = GL_n$. See [9, Sect. 1] for more details.

The fixed choice of a minimal parabolic subgroup P_0 in GL_n is the Borel subgroup of upper triangular matrices, and its Levi factor is the maximal split torus of diagonal matrices. Let e_j, $j = 1, \ldots, n$, denote the projections of the torus to the j-th diagonal entry. We fix $(e_1, \ldots, e_n)$ as an ordered basis of $\check{\mathfrak{a}}_{P_0}$ and $\check{\mathfrak{a}}_{P_0,\mathbb{C}}$, and write their elements as n-tuples in that basis. The spaces $\check{\mathfrak{a}}^{GL_n}_{P_0}$ and $\check{\mathfrak{a}}^{GL_n}_{P_0,\mathbb{C}}$ consist of n-tuples with the sum of coordinates zero.

The standard parabolic subgroups in GL_n consist of block upper triangular matrices, and their Levi factors of block diagonal matrices. Given a standard parabolic subgroup with the blocks of sizes $n_1 \times n_1, n_2 \times n_2, \ldots, n_k \times n_k$ along the diagonal, the spaces $\check{\mathfrak{a}}_R$ and $\check{\mathfrak{a}}_{R,\mathbb{C}}$ have the basis consisting of the determinant on each diagonal block. In these coordinates, $(s_1, \ldots, s_k) \in \check{\mathfrak{a}}_R^{GL_n}$ if and only if $n_1 s_1 + \cdots + n_k s_k = 0$. The inclusion ι of $\check{\mathfrak{a}}_R^{GL_n}$ into $\check{\mathfrak{a}}_{P_0}^{GL_n}$ is given by

$$\iota\big((s_1, \ldots, s_k)\big) = \big(s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_k, \ldots, s_k\big),$$

where s_j appears n_j times on the right-hand side.

The closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$ consists of all elements $(s_1, \ldots, s_k) \in \check{\mathfrak{a}}_R^{GL_n}$ such that $s_1 \geq s_2 \geq \cdots \geq s_k$. The closure of the negative obtuse Weyl chamber in $\check{\mathfrak{a}}_{P_0}^{GL_n}$ consists of all $(\zeta_1, \ldots, \zeta_n) \in \check{\mathfrak{a}}_{P_0}^{GL_n}$ such that

$$\zeta_1 + \cdots + \zeta_j \leq 0,$$

for $j = 1, \ldots, n-1$. Hence, given two different elements $(\zeta_1, \ldots, \zeta_n)$, $(\zeta_1', \ldots, \zeta_n') \in \check{\mathfrak{a}}_{P_0}^{GL_n}$, the partial order required for the Franke filtration is defined by

$$(\zeta_1, \ldots, \zeta_n) \succ (\zeta_1', \ldots, \zeta_n')$$

if and only if

$$\zeta_1 + \cdots + \zeta_j \leq \zeta_1' + \cdots + \zeta_j'$$

for $j = 1, \ldots, n-1$.

The Weyl group of GL_n is isomorphic to the symmetric group of n letters, which acts on $\check{\mathfrak{a}}_{P_0}$ by permutations of coordinates. Associate parabolic subgroups of GL_n are those obtained from each other by a permutation of diagonal blocks.

Recall that the discrete spectrum of the general linear group is determined by Mœglin–Waldspurger in [18] and can be described in terms of segments. A segment is a tensor product of the form

$$\Delta(\sigma, [a, b]) = \sigma|\det|^b \otimes \sigma|\det|^{b-1} \otimes \cdots \otimes \sigma|\det|^a,$$

where σ is a cuspidal automorphic representation of $GL_m(\mathbb{A})$, and a, b are real numbers such that

$$k = k(\Delta(\sigma, [a, b])) = b - a + 1$$

is a positive integer called the length of the segment. The length is equal to the number of factors in the tensor product. The numbers $a, a+1, \ldots, b$ are called the exponents of the segment, and we often regard $[a, b]$ as the set of exponents of $\Delta(\sigma, [a, b])$. We denote by

$$s = s(\Delta(\sigma, [a, b])) = \frac{a+b}{2}$$

the mean value of all the exponents in $[a, b]$.

The segment $\Delta(\sigma, [a, b])$ is a representation of the product of k copies of $GL_m(\mathbb{A})$. Thus, it may be viewed as a representation of the Levi factor $L_P(\mathbb{A})$ of a standard parabolic subgroup P of GL_{km} consisting of block upper triangular matrices with k blocks of size $m \times m$ along the diagonal. The parabolically induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{GL_{km}(\mathbb{A})}\left(\sigma|\det|^{b}\otimes\sigma|\det|^{b-1}\otimes\cdots\otimes\sigma|\det|^{a}\right)$$

admits a unique irreducible quotient, denoted by

$$J(\Delta(\sigma,[a,b])).$$

According to [18], it is isomorphic to

$$J(\sigma,k)|\det|^{s},$$

where k and s are as above, and

$$J(\sigma,k)=J\left(\Delta\left(\sigma,\left[-\frac{k-1}{2},\frac{k-1}{2}\right]\right)\right)$$

is obtained as the unique irreducible quotient of the analogous induced representation constructed from the segment of length k symmetric around zero, which is isomorphic to a summand in the decomposition of the discrete spectrum of $GL_{km}(\mathbb{A})$. Conversely, all the summands in the decomposition of the discrete spectrum of $GL_n(\mathbb{A})$ are obtained in this way for varying factorizations $n = km$ in positive integers, and cuspidal automorphic representations of $GL_m(\mathbb{A})$. These are the main results of [18].

Note that the representations of the form $J(\sigma, k)$ are referred to as Speh representations by Ginzburg and Soudry in [5], while Hanzer and Muić in [12] consider the special case in which σ is a unitary Hecke character of the group of idèles $\mathbb{I} = GL_1(\mathbb{A})$.

Recall that Bernstein–Zelevinsky [1], [24], developed the now standard formalism regarding segments for the purpose of studying the representation theory of p-adic general linear groups. However, the same formalism is appropriate in our case of discrete spectrum representations of $GL_n(\mathbb{A})$.

By definition, two given segments are linked if the tensor product constructed using all the elements of the union of their factors is again a segment, which is not equal to any of the given segments. More precisely, if the given segments are

$$\Delta_1=(\sigma_1,[a_1,b_1]) \qquad \text{and} \qquad \Delta_2=(\sigma_2,[a_2,b_2])\,,$$

they are linked if and only if $\sigma_1 \cong \sigma_2$ and the union of the sets of exponents $[a_1, b_1]$ and $[a_2, b_2]$ forms a set $[a, b]$ of exponents of some segment, but different from $[a_1, b_1]$ and $[a_2, b_2]$.

In Sect. 6, we repeatedly use the following fact. If segments Δ_1 and Δ_2 as above with $\sigma_1 \cong \sigma_2$ are not linked, then their sets of exponents are either disjoint or one of them is contained in the other. In the latter case, abusing the language, we sometimes say that one of the segments in contained in the other.

5. Degenerate Eisenstein series arising from a maximal parabolic subgroup of GL_n

In this section, we apply the method proposed in Sect. 3 to the case of the general linear group. In a recent paper [9], we studied the Franke filtration in that case. The main point is that the understanding of the Franke filtration allows us to obtain, using the proposed method, a complete description of the regions of holomorphy for the degenerate Eisenstein series constructed from a tensor product of two Speh representations as in [5].

As in [5], we consider the degenerate Eisenstein series associated with the discrete spectrum representation Π of the Levi factor $L_R(\mathbb{A})$ of the maximal proper parabolic subgroup R of GL_n consisting of the block upper triangular matrices with blocks of size $n_1 \times n_1$ and $n_2 \times n_2$ along the diagonal. Although we consider the same Eisenstein series, our notation differs from the notation in [5]. The reason is that we use the results and ideas of [9], so it is more convenient to use notation of that paper. The representation Π is given as the tensor product

$$\Pi \cong J(\sigma_1, k_1) \otimes J(\sigma_2, k_2)$$

of two Speh representations, where σ_1 and σ_2 are cuspidal automorphic representations of $GL_{m_1}(\mathbb{A})$ and $GL_{m_2}(\mathbb{A})$, respectively, and k_1, k_2 are positive integers such that $n_1 = k_1 m_1$ and $n_2 = k_2 m_2$.

The complex parameter for the Eisenstein series is taken from the one-dimensional space $\check{\mathfrak{a}}_{R,\mathbb{C}}^{GL_n}$, which is identified with $\mathbb{C}$ in such a way that $s \in \mathbb{C}$ corresponds to the character

$$|\det|^{\frac{n_2}{n_1+n_2}s} \otimes |\det|^{-\frac{n_1}{n_1+n_2}s}$$

of $L_R(\mathbb{A})$. This identification looks different than the characters used in [5], because we prefer to stay in the space $\check{\mathfrak{a}}_{R,\mathbb{C}}^{GL_n}$, which is convenient from the Franke filtration point of view. The choice of character in [5] is not in $\check{\mathfrak{a}}_{R,\mathbb{C}}^{GL_n}$, but since the analytic properties of Eisenstein series depend only on the difference of the two exponents, we can easily translate the result between two identifications. The poles in [5] should be multiplied by two to get our result.

Given the value $s_0 \geq 0$ of the complex parameter, we are interested in the analytic behavior at $s = s_0$ of the Eisenstein series $E(s, g, f, \Pi)$ associated with Π as in Sect. 2, where $\lambda \in \check{\mathfrak{a}}_{R,\mathbb{C}}^{GL_n}$, $g \in GL_n(\mathbb{A})$, and f ranges over the space $\mathcal{W}_\Pi$. The following theorem determines the regions of holomorphy that are within reach of our method. It turns out that the remaining points are exactly the poles determined by Ginzburg and Soudry in [5]. Although finding poles is out of our scope, this shows that the method is sharp in this special case.

Theorem 5.1. *Let $E(s, g, f, \Pi)$ be the Eisenstein series associated with the discrete spectrum representation Π of the Levi factor L_R of the parabolic subgroup R of GL_n, as above. Let $s_0 \geq 0$ be a fixed element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$. Then, $E(s, g, f, \Pi)$ is holomorphic at $s = s_0$ for every $f \in \mathcal{W}_\Pi$ in the following cases:*

(1) *if $\sigma_1 \not\cong \sigma_2$ and $s_0 \geq 0$ is arbitrary,*
(2) *if $\sigma_1 \cong \sigma_2$ and $s_0 \geq 0$ is such that $s_0 \neq \frac{k_1+k_2}{2} - i$ for $i = 0, 1, \ldots$ $\ldots, \min(k_1, k_2) - 1$.*

Proof. We apply our method explained in Sect. 3. In the proof below, we follow the steps listed there.

Step 1 amounts to finding the cuspidal support of the given Eisenstein series $E(s,g,f,\Pi)$ evaluated at $s=s_0$. Since the two Speh representations have their segments as the cuspidal support, we only have to twist them by the character associated with s_0. Hence, the cuspidal support is in the associate class ϕ represented by the representation

$$\sigma_1|\det|^{\frac{k_1-1}{2}+\frac{n_2}{n_1+n_2}s_0}\otimes\sigma_1|\det|^{\frac{k_1-3}{2}+\frac{n_2}{n_1+n_2}s_0}\otimes\cdots\otimes\sigma_1|\det|^{-\frac{k_1-1}{2}+\frac{n_2}{n_1+n_2}s_0}$$

$$\otimes\sigma_2|\det|^{\frac{k_2-1}{2}-\frac{n_1}{n_1+n_2}s_0}\otimes\sigma_2|\det|^{\frac{k_2-3}{2}-\frac{n_1}{n_1+n_2}s_0}\otimes\cdots\otimes\sigma_2|\det|^{-\frac{k_2-1}{2}-\frac{n_1}{n_1+n_2}s_0}.$$

of the Levi factor of the standard parabolic subgroup P of GL_n consisting of block upper triangular matrices with k_1 blocks of size $m_1\times m_1$, followed by k_2 blocks of size $m_2\times m_2$, along the diagonal.

In Step 2, we should explore the Franke filtration of the space of automorphic forms $\mathcal{A}_{\{P\},\phi}$, with the cuspidal support as above, in terms of triples in $\mathcal{M}_{\{P\},\phi}$. This is the difficult step in the general setting, as explained at the end of Sect. 3, but in the case of GL_n, it reduces to the matter of combinatorics of segments according to [9, Lemma 3.1]. More precisely, Lemma 3.1 of [9] says that the set $\mathcal{M}_{\{P\},\phi}$ of triples is in finite-to-one correspondence with the partitions of the cuspidal support in segments, and the correspondence is explicitly described.

The triple $(R,\Pi,s_0)\in\mathcal{M}_{\{P\},\phi}$ arising from the considered Eisenstein series $E(s,g,f,\Pi)$ evaluated at $s=s_0$, corresponds to the partition of the cuspidal support in two segments

$$\begin{aligned}\Delta_1 =&\Delta\left(\sigma_1,\left[-\frac{k_1-1}{2}+\frac{n_2}{n_1+n_2}s_0,\frac{k_1-1}{2}+\frac{n_2}{n_1+n_2}s_0\right]\right),\\ \Delta_2 =&\Delta\left(\sigma_2,\left[-\frac{k_2-1}{2}-\frac{n_1}{n_1+n_2}s_0,\frac{k_2-1}{2}-\frac{n_1}{n_1+n_2}s_0\right]\right).\end{aligned}$$

The rest of the proof is divided in two cases, depending on whether there exist two factors in the tensor products, one from segment Δ_1, the other from Δ_2, which form a new segment.

Case 1. No two factors form a segment: We first determine the conditions on Π and s_0 for which we are in this case. First of all, if $\sigma_1\not\cong\sigma_2$ it is clear that we cannot form a segment from factors in Δ_1 and Δ_2, because in a segment all factors have the same cuspidal automorphic representation. Secondly, if $\sigma_1\cong\sigma_2$, the segment cannot be formed if the exponents in segments Δ_1 and Δ_2 are not in the same class of $\mathbb{R}/\mathbb{Z}$, because then their difference is not an integer. This happens if and only if

$$s_0\notin\frac{k_1+k_2}{2}+\mathbb{Z}.$$

And thirdly, even if $\sigma_1\cong\sigma_2$ and $s_0\in\frac{k_1+k_2}{2}+\mathbb{Z}$, the segment still cannot be formed if all the exponents of Δ_1 are more than one apart from all the exponents of Δ_2. A simple calculation shows that this happens if and only if

$$s_0 > \frac{k_1+k_2}{2}.$$

We now show that, under any of the above conditions on Π and s_0, the Eisenstein series $E(s,g,f,\Pi)$ is holomorphic at $s=s_0$. In the formalism of segments recalled in Sect. 4, observe that in this case the segments Δ_1 and Δ_2 are not linked, but also one of them is not contained in the other.

An important observation at this point is the following. Since in this case we cannot combine two factors, one from Δ_1, the other from Δ_2, to form a segment, all triples in $\mathcal{M}_{\{P\},\phi}$ are obtained by making partitions of Δ_1 and Δ_2 in smaller segments. On the other hand, [9, Lemma 3.2] deals with such situations. If a triple in $\mathcal{M}_{\{P\},\phi}$ is obtained from another triple in $\mathcal{M}_{\{P\},\phi}$ by making a partition of its segments, then the corresponding element in $\mathcal{S}_{\{P\},\phi}$ of the former triple is less than the corresponding element in $\mathcal{S}_{\{P\},\phi}$ of the latter, in the partial order defining the filtration. In other words, $\iota(s_0)$ for the triple (R,Π,s_0) is a maximal element of $\mathcal{S}_{\{P\},\phi}$. Therefore, according to Step 3 of our method, $E(s,g,f,\Pi)$ is holomorphic at $s=s_0$, as claimed.

Case 2. There are two factors which form a segment: From the first paragraph in Case 1, it follows that in Case 2 we have $\sigma_1\cong\sigma_2$ and

$$s_0 \in \left\{\frac{k_1+k_2}{2} - i \,:\, i\in\mathbb{Z},\, 0\le i\le \frac{k_1+k_2}{2}\right\},$$

but we cannot apply our method for all these s_0. That is, of course, as expected, because the method does not determine poles. Hence, we now show that the Eisenstein series $E(s,g,f,\Pi)$ is holomorphic at $s=s_0$ for

$$s_0 \in \left\{\frac{k_1+k_2}{2} - i \,:\, i\in\mathbb{Z},\, \min(k_1,k_2)\le i\le \frac{k_1+k_2}{2}\right\},$$

even if $\sigma_1\cong\sigma_2$.

Since $\sigma_1\cong\sigma_2$, we write σ for both σ_1 and σ_2, and set $m=m_1=m_2$. Inserting $n_1=k_1m$ and $n_2=k_2m$ in the considered cuspidal support, we obtain that the cuspidal support is

$$\sigma|\det|^{\frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0}\otimes\sigma|\det|^{\frac{k_1-3}{2}+\frac{k_2}{k_1+k_2}s_0}\otimes\cdots\otimes\sigma|\det|^{-\frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0}$$

$$\otimes\sigma|\det|^{\frac{k_2-1}{2}-\frac{k_1}{k_1+k_2}s_0}\otimes\sigma|\det|^{\frac{k_2-3}{2}-\frac{k_1}{k_1+k_2}s_0}\otimes\cdots\otimes\sigma|\det|^{-\frac{k_2-1}{2}-\frac{k_1}{k_1+k_2}s_0},$$

and the Eisenstein series $E(s,g,f,\Pi)$ evaluated at $s=s_0$ is associated with the triple $(R,\Pi,s_0)\in\mathcal{M}_{\{P\},\phi}$ corresponding to the partition of the cuspidal support in two segments

$$\begin{aligned}\Delta_1 =&\Delta\left(\sigma, \left[-\frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0, \frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0\right]\right),\\ \Delta_2 =&\Delta\left(\sigma, \left[-\frac{k_2-1}{2}-\frac{k_1}{k_1+k_2}s_0, \frac{k_2-1}{2}-\frac{k_1}{k_1+k_2}s_0\right]\right).\end{aligned}$$

It can be easily checked that the considered values of s_0 are precisely those for which the set of exponents of one of the segments is a subset of the other, so that the segments Δ_1 and Δ_2 are not linked. Suppose, without loss of generality, that Δ_1 contains Δ_2, that is, $k_1 \geq k_2$. Then, the minimum $\min(k_1, k_2) = k_2$, and the point of evaluation is in the set

$$s_0 \in \left\{\frac{k_1+k_2}{2} - i \,:\, i \in \mathbb{Z},\, k_2 \leq i \leq \frac{k_1+k_2}{2}\right\}.$$

In particular, $s_0 \leq \frac{k_1-k_2}{2}$.

We now follow one of the ideas used in the proofs of Thm. 5.2 and Thm. 6.2 in [9]. The goal is to prove that

$$\iota(s_0) = \Bigg(\underbrace{\frac{k_2}{k_1+k_2}s_0, \dots, \frac{k_2}{k_1+k_2}s_0}_{n_1=k_1m \text{ times}}, \underbrace{-\frac{k_1}{k_1+k_2}s_0, \dots, -\frac{k_1}{k_1+k_2}s_0}_{n_2=k_2m \text{ times}}\Bigg) \in \check{\mathfrak{a}}_{P_0}^{GL_n},$$

arising from the triple $(R, \Pi, s_0) \in \mathcal{M}_{\{P\},\phi}$, is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$. Let (R', Π', λ_0') be any other triple in $\mathcal{M}_{\{P\},\phi}$. We show below that $\iota(\lambda_0')$ is either incomparable to or less than $\iota(s_0)$ in the partial order on $\mathcal{S}_{\{P\},\phi}$.

The triple (R', Π', λ_0') corresponds, as in [9, Lemma 3.1], to some partition of the cuspidal support in segments. Consider the segment Δ_1' in which the largest exponent of the cuspidal support belongs. Since for the considered s_0 we have that Δ_1 contains Δ_2, the segment Δ_1' is of the form

$$\Delta_1' = \Delta\left(\sigma, \left[-\frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0 + l, \frac{k_1-1}{2}+\frac{k_2}{k_1+k_2}s_0\right]\right),$$

where $0 \leq l < k_1$ is an integer. Observe that Δ_1' is the support of the representation

$$J(\sigma, k_1 - l)|\det|^{\frac{k_2}{k_1+k_2}s_0+\frac{l}{2}},$$

so that λ_0' contains the exponent $\frac{k_2}{k_1+k_2}s_0 + \frac{l}{2}$. Let

$$\iota(\lambda_0') = (\zeta_1', \dots, \zeta_n') \in \check{\mathfrak{a}}_{P_0}^{GL_n}$$

be written in coordinates. Since λ_0' is in the closure of the positive Weyl chamber, the coordinates ζ_j' are non-increasing, and thus

$$\zeta_1' \geq \frac{k_2}{k_1+k_2}s_0 + \frac{l}{2}.$$

However, if $l > 0$, it follows that ζ'_1 is strictly greater than the first coordinate of $\iota(s_0)$, which implies that either $\iota(\lambda'_0) \prec \iota(s_0)$, or they are incomparable, as required.

It remains to study the case of $l = 0$, so that $\Delta'_1 = \Delta_1$. But then, the rest of the partition corresponding to (R', Π', λ'_0) is just the partition of the segment Δ_2 in smaller segments. As in Case 1, invoking again [9, Lemma 3.2], it follows that $\iota(\lambda'_0) \prec \iota(s_0)$.

Having proved that $\iota(s_0)$ is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$, we apply Step 3 of our method to conclude that the Eisenstein series $E(s, g, f, \Pi)$ is holomorphic at $s = s_0$, even if $\sigma_1 \cong \sigma_2$ and $s_0 = \frac{k_1+k_2}{2} - i$ for $i = \min(k_1, k_2), \dots, \frac{k_1+k_2}{2}$, as claimed.

□

The previous theorem can be rephrased in the formalism of segments recalled in Sect. 4. This fact is also remarked on the very first page of [5]. Since it is required in more general applications of Sect. 6, we make it explicit in a separate corollary below.

Corollary 5.2. *Let $E(s, g, f, \Pi)$ be the Eisenstein series associated with the discrete spectrum representation Π of the Levi factor L_R of the parabolic subgroup R of GL_n, as above. Let $s_0 \geq 0$ be a fixed element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$. Then, $E(s, g, f, \Pi)$ is holomorphic at $s = s_0$ for every $f \in \mathcal{W}_\Pi$, if the two segments associated with Π twisted by s_0 are not linked.*

Proof. The fact that the two conditions for holomorphy in Thm. 5.1 are equivalent to the condition that the two corresponding segments are not linked is already observed in the proof of the theorem. More precisely, Case 1 in the proof deals with the case in which two segments are not linked and disjoint, while Case 2 deals with the case in which one of the segments is contained in the other. □

6. Degenerate Eisenstein series arising from arbitrary parabolic subgroup of GL_n

In this section, we apply our method to the general case of arbitrary (degenerate) Eisenstein series associated with a discrete spectrum representation of the Levi factor of a parabolic subgroup of GL_n. This is beyond the case of maximal parabolic subgroups considered in [5] and [12].

Our final result is a complete description of the poles in the closure of the positive Weyl chamber of arbitrary (degenerate) Eisenstein series on $GL_n(\mathbb{A})$. It is given in terms of the formalism of segments, recalled in Sect. 4. The holomorphy regions are determined by the proposed method. Our method determines the complete family of regions of holomorphy of the Eisenstein series. In proving that the remaining points are poles of Eisenstein series, relying on an inductive process, we use [5]. In other words, our method is sharp in the general case of Eisenstein series on $GL_n(\mathbb{A})$.

We begin by introducing the considered Eisenstein series. The notation regarding the general linear group and the formalism of segments of Sect. 4

is retained. Let R be a parabolic subgroup of GL_n consisting of block upper triangular matrices with blocks of size $n_1 \times n_1, n_2 \times n_2, \ldots, n_l \times n_l$ along the diagonal. Then, the Levi factor L_R is isomorphic to a direct product of $GL_{n_1} \times \cdots \times GL_{n_l}$. Let

$$\Pi \cong \Pi_1 \otimes \cdots \otimes \Pi_l$$

be a discrete spectrum representation of the Levi factor $L_R(\mathbb{A})$, where Π_i is a discrete spectrum representation of $GL_{n_i}(\mathbb{A})$.

Let $E(\lambda, g, f, \Pi)$ denote the degenerate Eisenstein series associated with Π as in Sect. 2, where $\lambda \in \check{\mathfrak{a}}_{R,\mathbb{C}}^{GL_n}$, $g \in GL_n(\mathbb{A})$, and f ranges over the space $\mathcal{W}_\Pi$. Let $\lambda_0 = (s_1, \ldots, s_l)$ be a given element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$. We are interested in the analytic behavior of the Eisenstein series $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$.

Let $\{P\}$ and ϕ denote the full cuspidal support of the Eisenstein series $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$, so that the triple (R, Π, λ_0), from which the Eisenstein series $E(\lambda, g, f, \Pi)$ and the evaluation point λ_0 arise, belongs to the set $\mathcal{M}_{\{P\},\phi}$. We omit the explicit description of P and a representative of ϕ, because it is not used in the considerations below. According to [9, Lemma 3.1], the triple (R, Π, λ_0) corresponds to a partition of the cuspidal support in segments. It is the partition associated with

$$\Pi \otimes \exp\langle \lambda_0, H_R(\cdot) \rangle \cong \Pi_1 |\det|^{s_1} \otimes \cdots \otimes \Pi_l |\det|^{s_l}.$$

If $\Pi_i = J(\sigma_i, k_i)$ in the notation of Sect. 4, where σ_i is a cuspidal automorphic representation of $GL_{m_i}(\mathbb{A})$, and k_i is a positive integer such that $k_i m_i = n_i$, then the corresponding segment is

$$\Delta_i = \Delta\left(\sigma_i, \left[-\frac{k_i - 1}{2} + s_i, \frac{k_i - 1}{2} + s_i\right]\right).$$

The main result below is stated in terms of segments Δ_i, $i = 1, \ldots, l$.

Theorem 6.1. *Let $E(\lambda, g, f, \Pi)$ be the Eisenstein series associated with the discrete spectrum representation Π of the Levi factor L_R of the parabolic subgroup R of GL_n, as above. Let $\lambda_0 = (s_1, \ldots, s_k)$ be a fixed element of the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_R^{GL_n}$. Then, $E(\lambda, g, f, \Pi)$ is holomorphic at $\lambda = \lambda_0$ for every $f \in \mathcal{W}_\Pi$ if and only if there is no pair of linked segments among the segments Δ_i, $i = 1, \ldots, l$, corresponding to the triple $(R, \Pi, \lambda_0) \in \mathcal{M}_{\{P\},\phi}$.*

Proof. One implication follows from [5]. If there are two linked segments among Δ_i, then the Eisenstein series on the Levi factor arising from these two segments has a pole, according to [5], and thus, the full Eisenstein series $E(\lambda, g, f, \Pi)$ on $G(\mathbb{A})$ has a pole at $\lambda = \lambda_0$. We prove by our method the other implication, that is, if there is no pair of linked segments among Δ_i, then the Eisenstein series $E(\lambda, g, f, \Pi)$ is holomorphic at $\lambda = \lambda_0$.

We skip Step 1 of the method, as the cuspidal support is already implicitly determined by the segments Δ_i. Let $\mathcal{P}$ denote the partition of the cuspidal support in segments Δ_i, $i = 1, \ldots, l$.

In Step 2 of the method, we should study the position in the Franke filtration of the triple $(R, \Pi, \lambda_0) \in \mathcal{M}_{\{P\},\phi}$ associated with $E(\lambda, g, f, \Pi)$ evaluated at $\lambda = \lambda_0$. More precisely, we now prove that $\iota(\lambda_0)$ is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$ defining the filtration, under the assumption that there is no pair of linked segments among Δ_i. To that end, consider another triple (R', Π', λ_0') in $\mathcal{M}_{\{P\},\phi}$. Our goal is to prove that $\iota(\lambda_0')$ is either less than or incomparable to $\iota(\lambda_0)$.

The triple (R', Π', λ_0') corresponds, as in [9, Lemma 3.1], to a partition $\mathcal{P}'$ of the cuspidal support in segments Δ_j', $j = 1, \ldots, l'$. Since the correspondence in [9, Lemma 3.1] is finite-to-one, it is possible that (R, Π, λ_0) and (R', Π', λ_0') correspond to the same partition. But in that case $\iota(\lambda_0) = \iota(\lambda_0')$ so there is nothing to prove, and we may assume that partitions are different.

Since $\lambda_0 = (s_1, \ldots, s_k)$ is in the closure of the positive Weyl chamber, we have $s_1 \geq \cdots \geq s_k$. Hence, the segments Δ_i are ordered as in [9, Lemma 3.1]. Let i_0 be the minimal i such that Δ_i is not among segments Δ_j', or appears with higher multiplicity in the partition $\mathcal{P}$ than in $\mathcal{P}'$. Such i_0 certainly exists, because partitions are different. Replacing the triple (R, Π, λ_0) by another triple corresponding to the same partition $\mathcal{P}$, if necessary, we may rearrange the order of segments Δ_i in $\mathcal{P}$ with $s_i = s_{i_0}$ in such a way that the segments equal to Δ_{i_0} that appear in both partitions $\mathcal{P}$ and $\mathcal{P}'$ are in front of Δ_{i_0}. Thus, we may and will assume that $s_i \geq s_{i_0}$, for $i = 1, \ldots, i_0 - 1$, and the equality occurs only for those segments in $\mathcal{P}$ equal to Δ_{i_0} that remain unchanged in $\mathcal{P}'$.

Write in coordinates

$$\begin{aligned} \iota(\lambda_0) =& (\zeta_1, \ldots, \zeta_n), \\ \iota(\lambda_0') =& (\zeta_1', \ldots, \zeta_n'), \end{aligned}$$

and set

$$\begin{aligned} a_{i_0} =& -\frac{k_{i_0} - 1}{2} + s_{i_0}, \\ b_{i_0} =& \frac{k_{i_0} - 1}{2} + s_{i_0}, \end{aligned}$$

so that $\Delta_{i_0} = \Delta(\sigma_{i_0}, [a_{i_0}, b_{i_0}])$. Then,

$$\zeta_{n_1 + \cdots + n_{i_0 - 1} + 1} = s(\Delta_{i_0}) = \frac{a_{i_0} + b_{i_0}}{2} = s_{i_0}.$$

is the first coordinate obtained from Δ_{i_0} in $\iota(\lambda_0)$.

Consider the segments $\Delta_1, \Delta_2, \ldots, \Delta_{i_0 - 1}$ of partition $\mathcal{P}$, which are also in partition $\mathcal{P}'$. We first show that, in addition to $\Delta_1, \Delta_2, \ldots, \Delta_{i_0 - 1}$, there exists a segment Δ_j' in partition $\mathcal{P}'$, such that $s(\Delta_j') > s_{i_0}$.

Suppose the contrary, i.e., that such Δ_j' does not exist. In that case, the first $i_0 - 1$ segments of partition $\mathcal{P}'$ in the order of [9, Lemma 3.1] are $\Delta_1, \Delta_2, \ldots, \Delta_{i_0 - 1}$, so that $\zeta_j' = \zeta_j$ for $j = 1, \ldots, n_1 + \cdots + n_{i_0 - 1}$. Since $\Delta_{i_0} = \Delta(\sigma_{i_0}, [a_{i_0}, b_{i_0}])$ appears with higher multiplicity in $\mathcal{P}$ than in $\mathcal{P}'$, there must be a segment Δ_{j_0}' in $\mathcal{P}'$, different from $\Delta_1, \ldots, \Delta_{i_0}$, containing b_{i_0} and with σ_{i_0} as the cuspidal automorphic representation. Let $\Delta_{j_0}' = \Delta(\sigma_{i_0}, [c_{i_1}, d_{i_1}])$ be such a segment, so that b_{i_0} belongs to the set $[c_{i_1}, d_{i_1}]$ of its exponents.

If $c_{i_1} > a_{i_0}$, then

$$s(\Delta'_{j_0}) = \frac{c_{i_1} + d_{i_1}}{2} > \frac{a_{i_0} + b_{i_0}}{2} = s_{i_0},$$

so that Δ'_{j_0} contradicts the assumption that there are no such Δ'_{j_0} in $\mathcal{P}'$ in addition to $\Delta_1, \ldots, \Delta_{i_0-1}$. If $c_{i_1} = a_{i_0}$, then $d_{i_1} > b_{i_0}$ because $\Delta'_{j_0} \neq \Delta_{i_0}$, so that strict inequality $s(\Delta'_{j_0}) > s_{i_0}$ holds also in that case and yields the contradiction.

Hence, $c_{i_1} < a_{i_0}$ must hold, and in particular, c_{i_1} is not an exponent in Δ_{i_0}. We argue that c_{i_1} is an exponent in a segment Δ_{i_1} of partition $\mathcal{P}$ which contains Δ_{i_0}. This follows from the fact that exponents $c_{i_1}, c_{i_1}+1, \ldots, a_{i_0}-1$ of Δ'_{j_0} belong to the cuspidal support, so that they must belong to some segments in partition $\mathcal{P}$. If they are divided between several segments, then these segments would be linked, which contradicts the assumption on Δ_i. Otherwise, if they belong to a segment either disjoint to Δ_{i_0}, or intersecting Δ_{i_0} but not containing it, than that segment and Δ_{i_0} would be linked, which is again a contradiction.

Let $\Delta_{i_1} = \Delta(\sigma_{i_0}, [a_{i_1}, b_{i_1}])$ be the constructed segment in $\mathcal{P}$ which contains Δ_{i_0} and has c_{i_1} among its exponents. Then we may proceed as above by looking at the segment $\Delta'_{j_1} = \Delta(\sigma_{i_0}, [c_{i_2}, d_{i_2}])$ in $\mathcal{P}'$ containing b_{i_1}. The same argument implies that $c_{i_2} < a_{i_1}$ and there exists a segment Δ_{i_2} containing Δ_{i_1} and having c_{i_2} as exponent. We could continue this construction *ad infinitum*, but there are only finitely many segments in $\mathcal{P}$. Therefore, at some point, we obtain the segment Δ'_{j_x} for some $x = 0, 1, 2, \ldots$, such that $s(\Delta'_{j_x}) > s_{i_0}$, which is a contradiction to the assumption that no such Δ'_j exists in $\mathcal{P}'$.

In conclusion, we proved that, in addition to $\Delta_1, \Delta_2, \ldots, \Delta_{i_0-1}$, there exists a segment Δ'_j in partition $\mathcal{P}'$, such that $s(\Delta'_j) > s_{i_0}$. Then we have $\zeta'_j \geq \zeta_j$ for all $j = 1, \ldots, n_1 + \cdots + n_{i_0-1}$, and

$$\zeta'_{n_1+\cdots+n_{i_0-1}+1} > s_{i_0}.$$

This implies that for the partial sums the inequality

$$\zeta'_1 + \cdots + \zeta'_{n_1+\cdots+n_{i_0-1}+1} > \zeta_1 + \cdots + \zeta_{n_1+\cdots+n_{i_0-1}+1}$$

holds, so that $\iota(\lambda'_0)$ is either less than or incomparable to $\iota(\lambda_0)$, as required. Since the triple (R', Π', λ'_0) is arbitrary, $\iota(\lambda_0)$ is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$.

Now we proceed to Step 3 of our method to conclude that the Eisenstein series $E(\lambda, g, f, \Pi)$ is holomorphic at $\lambda = \lambda_0$, because $\iota(\lambda_0)$ is a maximal element in the partial order on $\mathcal{S}_{\{P\},\phi}$, under the assumption that the segments Δ_i are not linked. □

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