DEGENERATE EISENSTEIN COHOMOLOGY OF $SL_n(\mathbb{Z})$ BEYOND THE STABLE RANGE

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ABSTRACT. This paper investigates the cohomology of $SL_n(\mathbb{Z})$, $n \geq 2$, "right outside" what one calls the "stable range". More precisely, a qualitative non-vanishing result for the cohomology $H^q(SL_n(\mathbb{Z}))$ in degrees $q = n-1$ and $q = n$ is shown, whose major novelty is to include the existence of non-trivial cohomology classes, which are representable by everywhere unramified degenerate Eisenstein series. In particular, these classes cannot be detected by the work of Borel and Franke on the Borel map. In the last section, we describe non-constant automorphic representatives of non-zero classes for $SL_6(\mathbb{Z})$ and $SL_8(\mathbb{Z})$, whose degree lies right below the "cuspidal range".

CONTENTS

INTRODUCTION

In order to describe the context and the results of this paper, let $G/\mathbb{Q}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and fix a choice of a maximal compact subgroup K of the real Lie group $G(\mathbb{R})$, i.e., of the group of $\mathbb{R}$-points of G. We denote by $X = G(\mathbb{R})/K$ the associated symmetric space. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$.

Half a century ago, cf. [Bor74], A. Borel showed that the cohomology $H^q(\Gamma, \mathbb{C})$ of Γ is – below a certain degree $q(G)$ – entirely spanned by classes, which are represented by $G(\mathbb{R})$-invariant differential forms on X. Although Borel's bound is not sharp in general, his result implies that below degree $q(G)$, the cohomology $H^q(\Gamma, \mathbb{C})$ falls into what one calls ever since the "stable range", i.e., the maximal range of degrees of cohomology, in which $H^q(\Gamma, \mathbb{C})$ does not change, even if the rank of

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G in its Cartan-type classification is allowed to grow to infinity (and Γ varies among the arithmetic subgroups of G).

If Γ is a congruence subgroup, then the above can be rephrased in the more modern language of adèles $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ (over $\mathbb{Q}$) and automorphic forms: It can be expressed by saying that in a certain maximal range of degrees $0 \leq q \leq st(G)$, all classes in the cohomology $H^q(\Gamma, \mathbb{C})$ are obtained from $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$, i.e., from the $(\mathfrak{g}, K)$-cohomology of the global trivial automorphic representation $\mathbf{1}_{G(\mathbb{A})}$ of $G(\mathbb{A})$, realized as a square-integrable automorphic representation on the space of constant functions $G(\mathbb{A}) \to \mathbb{C}$. In other words, given the Lie group $G(\mathbb{R})$, it is enough to study the Poincaré-polynomial of $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{R})})$, which is usually well-understood in terms of differential geometry, in order to understand $H^q(\Gamma, \mathbb{C})$ for all congruence subgroups Γ of $G(\mathbb{Q})$ and degrees $q \leq st(G)$. See also [Spe83a] for further, general results on the contribution of $H^q(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{R})})$ to $H^q(\Gamma, \mathbb{C})$.

In this paper we explore new phenomena of non-vanishing for the (automorphic) cohomology of $\Gamma = SL_n(\mathbb{Z})$, $n \geq 4$, *right beyond* the "stable range" $st(SL_n) = n - 2$.

To put ourselves *in medias res*, we recall that according to the work of Franke [Fra98] and Franke–Schwermer [Fra-Schw98], $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ affords a description as Hecke-module as a direct sum

$$H^q(SL_n(\mathbb{Z}), \mathbb{C}) \cong \bigoplus_{\{P\}} \bigoplus_{\varphi(\pi)} H^q(\mathfrak{g}, K, \mathcal{A}_{\{P\},\varphi(\pi)}(G))^{SL_n(\hat{\mathbb{Z}})}.$$

Here, the first sum ranges over all associate classes $\{P\}$ of standard parabolic $\mathbb{Q}$-subgroups P of SL_n and the second sum ranges over all associate classes $\varphi(\pi)$ of cuspidal automorphic representations of the Levi subgroup of P. The spaces $\mathcal{A}_{\{P\},\varphi(\pi)}(G)$ then denote the module of all possible partial derivatives of regularized Eisenstein series attached to $\varphi(\pi)$, cf. §2.2.2, and the exponent $SL_n(\hat{\mathbb{Z}})$ stands for the invariants under the natural action of $SL_n(\hat{\mathbb{Z}})$, i.e., the everywhere unramified vectors, where $\hat{\mathbb{Z}}$ is the Prüfer ring, i.e., the profinite completion of $\mathbb{Z}$.

In particular, the summand $H^q(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ attached to the Borel subgroup $P = B$ and the cuspidal support represented by the Hecke character

$$\chi = e^{\langle \rho_B, H_B(\cdot) \rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the adèlic points of the maximal torus T of SL_n shows up in this direct sum. In fact, the latter summand comprises all of $H^q(SL_n(\mathbb{Z}), \mathbb{C})$, if $n \leq 11$, cf. our Thm. 2.2, which is based on fundamental work of Chenevier-Lannes [Che-Lan19].

Our main result in degree $q = n-1$ now reads as follows: Given an order partition $\underline{n} = (n_1, \ldots, n_k)$ of n into positive integers, $P_{\underline{n}}$ denotes the corresponding standard parabolic subgroup of SL_n, cf. §1.2. Let $a(q)$ be the number of ways to write an integer q as the sum of different integers of the form $4\ell + 1$, $\ell \geq 1$. See also Lem. 3.2. Then we obtain

Theorem A. *Let $n \geq 4$. Then, if n is odd, the cohomology space $H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ is isomorphic to the $G(\mathbb{A}_f)$-module $H^{n-1}(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$, whereas if n is even, it contains $H^{n-1}(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})})$ as a submodule with the quotient given as the kernel of the natural connecting morphism ("Bockstein* *homomorphism")*

$$\mathrm{Eis}^n := \ker\left(\bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \to \mathbf{1}^{a(n)}_{G(\mathbb{A}_f)}\right).$$

The unramified classes in Eis^n *are represented by degenerate Eisenstein series associated to the trivial representation* $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ *of the Levi factor of* $P_{\underline{n}}$*, evaluated at the evaluation point* $\lambda = \rho_{P_{\underline{n}}}$*, where* $\underline{n} \in \{(n-1,1),(1,n-1)\}$*.*

Consequently, $\dim_\mathbb{C} H^{n-1}(SL_n(\mathbb{Z}),\mathbb{C})$ *grows at least as* $a(n-1)$*, if* $n \geq 5$ *is odd, and as* $a(n-1) + \dim_\mathbb{C}\left((\mathrm{Eis}^n)^{SL_n(\hat{\mathbb{Z}})}\right)$*, if* $n \geq 4$ *is even.*

As a corollary, if $n \geq 5$ *is odd, the free part of the* $\mathbb{Z}$*-module* $H^{n-1}(SL_n(\mathbb{Z}))$ *is non-zero, if either* $n \geq 43$*, or* $n \in \{15, 19, 23, 27, 31, 35, 39\}$ *and vanishes if* $n \in \{5, 7, 9, 11\}$*.*

Moreover, if $n \geq 4$ *is even, then the subspace of* $SL_n(\hat{\mathbb{Z}})$*-invariant, i.e., unramified, vectors in* Eis^n *does not vanish, if* $n \in \{4, 6, 8, 10, 12, 16, 20, 24, 28, 32, 36, 40\}$*. As a corollary, the free part of the* $\mathbb{Z}$*-module* $H^{n-1}(SL_n(\mathbb{Z}))$ *is non-zero for all even* $n \geq 4$*.*

We emphasize that this is the first time that explicit automorphic representatives of non-trivial cohomology classes for $SL_n(\mathbb{Z})$ are constructed which are neither holomorphic values, nor square-integrable residues of Eisenstein series. We refer to our Thm. 4.6 and Thm. 4.10 for a proof of Thm. A and all details left out here.

It should be noted that Thm. A recovers Franke's main result on the kernel (resp. image) of the "Borel map" in degree $q = n - 1$ in [Fra08], see in particular pp. 58 – 62 *ibidem*, but clearly goes beyond it (in degree $q = n - 1$), as we establish and describe several new non-zero classes in $H^{n-1}(SL_n(\mathbb{Z}),\mathbb{C})$, which are not representable by constant automorphic forms, but are represented by everywhere unramified degenerate Eisenstein series: These are the classes in $(\mathrm{Eis}^n)^{SL_n(\hat{\mathbb{Z}})}$, for whose automorphic description the reader is referred to Thm. 4.10.

Our next theorem says, that Franke's result, [Fra08], pp. 58 – 62, is sharp, insofar as it describes the summand $H^q(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ in degree $q = n$, i.e., there are no other classes in $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ – and hence in all of $H^n(SL_n(\mathbb{Z}),\mathbb{C})$, if $5 \leq n \leq 11$ – than the ones obtained by the Borel map:

Theorem B. *Let* $n \geq 5$*. Then, the cohomology space* $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ *is isomorphic to the image of the natural map of* $G(\mathbb{A}_f)$*-modules* $H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g}, K, \mathcal{A}(G))$*. As a corollary,*

$$\dim_\mathbb{C} H^n(SL_n(\mathbb{Z}),\mathbb{C}) \geq \begin{cases} a(n) - 2 & \text{if } n \text{ is even} \\ a(n) & \text{if } n \text{ is odd} \end{cases}$$

In particular, the free part of the $\mathbb{Z}$*-module* $H^n(SL_n(\mathbb{Z}))$ *is non-zero, in the following cases:*

- *for odd* n*, if either* $n \geq 25$*, or* $n \in \{5, 9, 13, 17, 21\}$*;*
- *for even* n*, if either* $n \geq 50$*, or* $n \in \{22, 26, 30, 34, 38, 42, 46\}$*.*

We refer to Thm. 4.3 for all explanations and a proof.

We would like to compare our results to the vibrant recent literature on the subject: Thm. 1.1 of the recent preprint [Bro23] also implies a growth-condition on the dimension of the cohomology of $SL_n(\mathbb{Z})$ by studying the kernel of the Borel map by means of Hopf algebras and a new approach to the Borel-Serre compactification. Though our methods here are automorphic and hence totally different, it is interesting to notice that for odd $n \geq 5$ the dimension of the space of n-forms of "non-compact type" (as they are used and called in [Bro23], Thm. 1.1) is the same as our constant $a(n)$. Our formulas, however, differ from Brown's in the case of even $n \geq 4$, as here we get non-trivial Eisenstein cohomology classes, which are not representable by constant automorphic functions, i.e., are not in the image of the Borel map.

There are several other complementary (and sometimes partly overlapping) results in the recent literature: We would like to mention [AMP24, BHP24, BCGP24, Ash24, KMP21, PSS20, Chu-Put17, CFP14] as a chronologically decreasing selection of interesting recent sources and refer to Rem. 5.1 and §5.3 for more comments on the three most recent of these preprints.

In the very last section we consider questions related to the non-vanishing of the Eisenstein cohomology of $SL_{2m}(\mathbb{Z})$ in degree $q = m^2 - 1$, i.e., right below the range, in which cuspidal cohomological representations of $SL_{2m}(\mathbb{A})$ could contribute non-trivially to cohomology. Again, the non-trivial classes detected and considered here are not in the image of the Borel map, and they are represented by degenerate Eisenstein series:

In §5.1, we reestablish the non-vanishing of $H^8(SL_6(\mathbb{Z}))$, as originally shown by Elbaz-Vincent–Gangl–Soulé, [EVGS13], but also determine, which (non-constant) degenerate Eisenstein series of $SL_6(\mathbb{A})$ represent the non-trivial classes in $H^8(SL_6(\mathbb{Z}), \mathbb{C})$. See §5.1 for details.

Similarly, as communicated to the second named author by Brown, A. Ash has asked for a description of the cohomology of $SL_8(\mathbb{Z})$. Among others, degree $q = 15$ was of particular interest. Here we show that $H^{15}(SL_8(\mathbb{Z}), \mathbb{C})$ is two-dimensional by automorphic methods, and we describe, which (non-constant) degenerate Eisenstein series of $SL_8(\mathbb{A})$ represent the non-trivial classes in $H^{15}(SL_8(\mathbb{Z}), \mathbb{C})$. We refer to §5.2 for this result.

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1. Preliminaries and notation

1.1. **Groups.** The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have their usual meaning. The ring of adèles of $\mathbb{Q}$ will be denoted by $\mathbb{A}$, its subring of non-archimedean elements by $\mathbb{A}_f$.

For $n \geq 1$, let GL_n be the general linear group defined over $\mathbb{Q}$. If H is any $\mathbb{Q}$-subgroup of GL_n, then $S(H)$ will denote its elements of determinant equal to 1. In particular, we will write $G := S(GL_n) = SL_n$ for the ($\mathbb{Q}$-split) special linear group defined over $\mathbb{Q}$. If, however, H is a real Lie group, we will use $\mathfrak{h}$ to denote its Lie algebra and $\mathfrak{h}_{\mathbb{C}}$ for its complexification.

1.2. **Parabolic data.** We fix once and for all the Borel subgroup B of G, consisting of upper-triangular matrices in G. Let $B = TU$ be the Levi decomposition of B, where T is a maximal split torus in B, and U the unipotent radical. Then,

$$T(R) = \left\{ \mathrm{diag}(t_1, \dots, t_n) \,:\, t_i \in R^{\times}, \prod_i t_i = 1 \right\}$$

for any abelian $\mathbb{Q}$-algebra R. More generally, let $P \supseteq B$ be a standard parabolic $\mathbb{Q}$-subgroup of G, cf. [Bor-Wal00], 0.3.4. They are parameterized by the tuples $(n_1, \dots, n_k)$, $k \geq 1$, $n_i \in \mathbb{N}$, $\sum n_i = n$, according to the block-sizes of the corresponding Levi subgroup $L \cong S(GL_{n_1} \times \cdots \times GL_{n_k}) \subset P$. Its group of real points $L(\mathbb{R})$ admits a unique maximal semisimple direct factor, denoted by M. Its Lie algebra is naturally complemented by the real Lie algebra $\mathfrak{a}_P$ of the split component A_P of L. Its (complexified) dual is as usual denoted by $\check{\mathfrak{a}}_P$ (resp. by $\check{\mathfrak{a}}_{P,\mathbb{C}}$). We write $S(\check{\mathfrak{a}}_{P,\mathbb{C}})$ for the attached symmetric (i.e., universal enveloping) tensor algebra. Moreover, we recall the set W^P of *Kostant representatives* from [Bor-Wal00], III.1.4: It is a uniquely determined set of right coset representatives of the quotient $W_L \backslash W$, where W (resp. W_L) denotes the Weyl group of G (resp. L) with respect to T. (Here we used the fact that G is $\mathbb{Q}$-split.)

1.3. **Compact subgroups.** We assume to have fixed a maximal compact subgroup K of $G(\mathbb{R})$ and K_f of $G(\mathbb{A}_f)$ in good position with respect to B and T, in the sense of [Mœ-Wal95], I.1.4 or [Gro23], §9.2. More explicitly, $K = SO(n)$, the compact special orthogonal group of $n \times n$-matrices and $K_f = SL_n(\hat{\mathbb{Z}}) = \prod_p SL_n(\mathbb{Z}_p)$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the Prüfer ring, i.e., the profinite completion of $\mathbb{Z}$.

1.4. **Certain characters.** We denote by $\mathrm{sgn} : \mathbb{R}^* \to \{\pm 1\}$ the sign-character of the multiplicative group $\mathbb{R}^*$ of non-zero real numbers. If $\lambda \in \check{\mathfrak{a}}_P$, then $\mathbb{C}_\lambda$ denotes the one-dimensional module of $L(\mathbb{R})$ of highest weight λ, i.e., if $L(\mathbb{R}) \cong S(GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_k}(\mathbb{R}))$ and $\lambda = (\lambda_1, \dots\dots, \lambda_k)$, then $\mathbb{C}_\lambda = \det_{n_1}^{\lambda_1} \otimes \cdots \otimes \det_{n_k}^{\lambda_k}$, where $\det_{n_i}$ denotes the determinant on $GL_{n_i}(\mathbb{R})$. Going adelic, if $\lambda \in \check{\mathfrak{a}}_P$, then $e^{\langle \lambda, H_P(\cdot) \rangle}$ denotes the one-dimensional representation of $L(\mathbb{A})$ constructed from λ and the Harish-Chandra height function $H_P(\cdot)$, cf. [Fra98], p. 185. If H is any subgroup of $G(\mathbb{A})$, then $\mathbf{1}_H$ denotes the trivial representation of H.

2. A sufficient condition for the non-vanishing of $H^q(SL_n(\mathbb{Z}))$

2.1. **Recap: The cohomology of $SL_n(\mathbb{Z})$ via automorphic forms.** For the sake of later reference, we shall shortly recall some facts about the cohomology of $SL_n(\mathbb{Z})$ and its interconnection to the cohomology of the space of automorphic forms of $SL_n(\mathbb{A})$.

In order to do so, we need to take a "transcendental" point of view, i.e., work with coefficient modules over $\mathbb{C}$. Just in this section, let us abbreviate $\Gamma = SL_n(\mathbb{Z})$ and let us also view $\mathbb{Z}$ and $\mathbb{C}$ as trivial modules under Γ. It is well-known that the group homology $H_*(\Gamma) := H_*(\Gamma, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. Indeed, this follows easily from the fact that Γ has a subgroup of finite index, which is torsion free, whence Γ itself is an arithmetic group of finite type, cf. [Ser79], §1.3. The universal coefficient theorem for group homology hence shows that as $\mathbb{C}$-vector spaces

$$H_q(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_q(\Gamma, \mathbb{C}).$$

Using duality between singular homology and cohomology, we get again an isomorphism of group cohomology as $\mathbb{C}$-vector spaces

$$H^q(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^q(\Gamma, \mathbb{C}).$$

It follows that the free part of the $\mathbb{Z}$-module $H^q(SL_n(\mathbb{Z}))$ must be non-zero, if $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ is.

Let now $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$, cf. [Bor-Jac79, Gro23], on which the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$ acts trivially. Then, it is well-known, that $H^q(\Gamma, \mathbb{C})$ allows a description as the space of K_f-invariant vectors in the $(\mathfrak{g}, K)$-cohomology of $\mathcal{A}(G)$, cf. [Bor-Wal00], Thm. VII.2.2 in combination with Strong Approximation for G, cf. [Pla-Rap94], Thm. 7.12, and [Fra98], Thm. 18:

$$H^q(SL_n(\mathbb{Z}), \mathbb{C}) \cong H^q(\mathfrak{g}, K, \mathcal{A}(G))^{K_f}. \tag{2.1}$$

Therefore, each cohomology class in $H^q(SL_n(\mathbb{Z}), \mathbb{C})$ may be represented by everywhere unramified, i.e., K_f-right invariant, automorphic forms in $\mathcal{A}(G)$.

2.2. **Automorphic background à la Franke and a first consequence for the cohomology of $SL_n(\mathbb{Z})$.**

2.2.1. *Parabolic supports.* Let $\{P\}$ be the associate class of the parabolic $\mathbb{Q}$-subgroup $P = L_P N_P$ of $G = SL_n$: It consists by definition of all parabolic $\mathbb{Q}$-subgroups $Q = L_Q N_Q$ of G, for which L_Q and L_P are conjugate by an element in $SL_n(\mathbb{Q})$. We denote by $\mathcal{A}_{\{P\}}(G)$ the space of all $f \in \mathcal{A}(G)$, which are negligible along every parabolic $\mathbb{Q}$-subgroup $Q \notin \{P\}$: This means that for all $g \in SL_n(\mathbb{A})$, the function $L_Q(\mathbb{A}) \to \mathbb{C}$, which is given by $\ell \mapsto f_Q(\ell g)$, where f_Q denotes the constant term of f along Q, is orthogonal (with respect to the Petersson inner product) to the space of all cuspidal automorphic forms on $L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})$. Having set up these notations, Langlands obtained the following decomposition of $\mathcal{A}(G)$ as a $(\mathfrak{g}, K, G(\mathbb{A}_f))$-module, cf. [BLS96] Thm. 2.4:

$$\mathcal{A}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\{P\}}(G).$$

2.2.2. *Cuspidal supports.* We recall now, cf. [Fra-Schw98], 1.2, and [Gro23], §15.2, the notion of an *associate class* $\varphi(\pi)$ of cuspidal automorphic representations of the Levi subgroups of the elements in the class $\{P\}$. Therefore, let $\{P\}$ be represented by $P = LN$. Then, an associate class $\varphi(\pi)$ may be parameterized by $\pi = \tilde{\pi} \cdot e^{\langle \lambda_\pi, H_P(\cdot) \rangle}$, where

(1) $\tilde{\pi}$ is a unitary cuspidal automorphic representation of $L(\mathbb{A})$, whose central character vanishes on the identity component $A_P(\mathbb{R})^\circ$ of $A_P(\mathbb{R})$,
(2) $\lambda_\pi \in \check{\mathfrak{a}}_{P,\mathbb{C}}$, which is compatible with the infinitesimal character $\chi_{\tilde{\pi}_\infty}$ of $\tilde{\pi}_\infty$ (cf. [Fra-Schw98], 1.2, or [Gro23], §15.2, in particular (15.13)).

We let $\mathcal{W}_{P,\widetilde{\pi}}$ be the space of all smooth, K-finite functions

$$f : L(\mathbb{Q})N(\mathbb{A})A_P(\mathbb{R})^\circ\backslash G(\mathbb{A}) \to \mathbb{C},$$

such that for every $g \in G(\mathbb{A})$ the function $\ell \mapsto f(\ell g)$ on $L(\mathbb{A})$ is contained in the $\widetilde{\pi}$-isotypic component of the cuspidal spectrum $L^2_{cusp}(L(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash L(\mathbb{A}))$ of $L(\mathbb{A})$. For a function $f \in \mathcal{W}_{P,\widetilde{\pi}}$, $\lambda \in \check{\mathfrak{a}}_{P,\mathbb{C}}$ and $g \in G(\mathbb{A})$ an *Eisenstein series* is formally defined as

$$E_P(f,\lambda)(g) := \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g)e^{\langle\lambda+\rho_P,H_P(\gamma g)\rangle}.$$

It is known to converge absolutely and uniformly on compact subsets of $G(\mathbb{A}) \times \check{\mathfrak{a}}_{P,\mathbb{C}}$, if the real part of λ is sufficiently positive. In that case, $E_P(f,\lambda)$ is an automorphic form and the map $\lambda \mapsto E_P(f,\lambda)(g)$ can be analytically continued to a meromorphic function on all of $\check{\mathfrak{a}}_{P,\mathbb{C}}$, cf. [Mœ-Wal95], II.1.5, IV.1.8, IV.1.9, [Lan76], §7, or, most concretely, the main result of [Ber-Lap23]. Given $\varphi(\pi)$, represented by a cuspidal representation π of the above form, a $(\mathfrak{g},K,G(\mathbb{A}_f))$-submodule

$$\mathcal{A}_{\{P\},\varphi(\pi)}(G)$$

of $\mathcal{A}_{\{P\}}(G)$ was defined in [Fra-Schw98], 1.3 as follows: It is the span of all possible partial derivatives of holomorphic values or residues of all Eisenstein series attached to $\tilde{\pi}$, evaluated at the point $\lambda = \lambda_\pi$. This definition is independent of the choice of the representatives P and π, due to the functional equations satisfied by the Eisenstein series considered. For details, we refer the reader to [Fra-Schw98], 1.2–1.4, as the original source, or to [Gro23], §15.2–15.3. The following is a theorem of Franke–Schwermer, see, [Fra-Schw98], Thm. 1.4, or [Gro23], Thm. 15.21,

Theorem 2.1 (Franke–Schwermer). *There is an isomorphism of $(\mathfrak{g},K,G(\mathbb{A}_f))$-modules*

$$\mathcal{A}_{\{P\}}(G) \cong \bigoplus_{\varphi(\pi)} \mathcal{A}_{\{P\},\varphi(\pi)}(G).$$

Using Thm. 2.1, the next result refines the above description of $H^q(SL_n(\mathbb{Z}),\mathbb{C})$ in terms of automorphic forms and reveals that the cohomology of $SL_n(\mathbb{Z})$ is in fact quite simply structured, if $n \leq 11$. Namely, we will show that in the latter case it is strictly supported by the trivial character of the Borel subgroup $B = TU$ of G. For this recall that an irreducible cuspidal automorphic representation π of $L(\mathbb{A})$ is called *of level 1*, if its non-archimedean component π_f, as a representation of $L(\mathbb{A}_f)$, satisfies $\pi_f^{K_f\cap L(\mathbb{A}_f)} \neq \{0\}$, i.e., if π is unramified at all non-archimedean places. Moreover, recall from §1.2 that we denoted by M the maximal semisimple real Lie-subgroup of $L(\mathbb{R})$.

Theorem 2.2. *For all $n \geq 2$ and all degrees q of cohomology, there is an isomorphism of modules of the Hecke algebra of $SL_n(\mathbb{Z})$ (or, equivalently of the maximal open compact subgroup $K_f = SL_n(\hat{\mathbb{Z}})$)*

$$H^q(SL_n(\mathbb{Z}),\mathbb{C}) \cong \bigoplus_{\{P\}} \bigoplus_{\substack{\varphi(\pi):\\ \chi_{\tilde{\pi}_\infty}=-w(\rho)|_M\\ \pi_f \text{ is of level 1}}} H^q(\mathfrak{g},K,\mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f},$$

where $w \in W^P$ runs through the Kostant representatives for P.

If $n \leq 11$, then the following much simpler description holds:

$$H^q(SL_n(\mathbb{Z}),\mathbb{C}) \cong H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)}(G))^{K_f}, \tag{2.2}$$

where $\varphi(\chi)$ is the cuspidal support represented by the Hecke character

$$\chi = e^{\langle \rho_B, H_B(\cdot)\rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$.

Proof. From Franke–Schwermer's theorem, cf. Thm. 2.1, we get

$$\begin{aligned} H^q(SL_n(\mathbb{Z}),\mathbb{C}) &\cong H^q(\mathfrak{g},K,\mathcal{A}(G))^{K_f} \\ &\cong \bigoplus_{\{P\}} \bigoplus_{\varphi(\pi)} H^q(\mathfrak{g},K,\mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f}. \end{aligned}$$

For any representative $\pi = \tilde{\pi}\cdot e^{\langle \lambda_\pi, H_P(\cdot)\rangle}$ of an associate class $\varphi(\pi)$, the natural $(\mathfrak{g},K,G(\mathbb{A}_f))$-homomorphism,

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi\otimes S(\check{\mathfrak{a}}_{P,\mathbb{C}})) \longrightarrow \mathcal{A}_{\{P\},\varphi(\pi)}(G)$$

given by summation of locally regularized Eisenstein series around λ_π is surjective, cf. [Fra-Schw98], 3.3.(4). Hence, in order to obtain a non-zero space

$$H^q(\mathfrak{g},K,\mathcal{A}_{\{P\},\varphi(\pi)}(G))^{K_f},$$

it is necessary that $\tilde{\pi}_\infty$ has the same infinitesimal character as the contragredient of a representation of M of highest weight $(w(\rho)-\rho)|_M$, w being a Kostant representative for P, see [Fra-Schw98], 1.2.c. For a given representative $\pi = \tilde{\pi}\cdot e^{\langle \lambda_\pi, H_P(\cdot)\rangle}$ of a class $\varphi(\pi)$, this Kostant representative w is indeed unique, as we must have $\lambda_\pi = -w(\rho)|_{A_P}$ as well, cf. [Bor-Wal00], Thm. III.3.3. This implies that $\chi_{\tilde{\pi}_\infty} = -w(\rho)|_M$, see also [Bor-Wal00], Thm. III.3.3.(i).(2). Moreover, invoking Frobenious reciprocity for non-archimedean parabolic induction, it is clear that π_f must be unramified at every place, i.e., of level 1. Collecting all that together, this shows the first assertion.

Let now be $n \leq 11$. Then, by [Che-Lan19], Thm. F on p. 13 (see also [Che-Tai20], Thm. 3 and §2.4.6), there is no level 1 irreducible unitary cuspidal automorphic representation of $L(\mathbb{A})$, whose infinitesimal character matches the one of the contragredient of $(w(\rho)-\rho)|_M$, w being a Kostant representative for P, if $L = S(GL_{n_1}\times ... \times GL_{n_r})$ contains a general linear group of rank $n_i > 1$. It therefore follows that $\varphi(\pi)$ must be represented by an irreducible cuspidal automorphic representation π with $P = B$ and $\tilde{\pi} = \mathbf{1}_{T(\mathbb{A})}$. Moreover, for $\varphi(\pi)$ to give rise to a non-zero space $H^*(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\pi)}(G))$, we must have $\lambda_\pi = -w(\rho)|_{A_B}$ and, in fact, this element has to be in the closed positive Weyl chamber of $\check{\mathfrak{a}}_{B,\mathbb{C}} = \check{\mathfrak{t}}_{\mathbb{C}}$, cf. [Fra-Schw98], 5.5. together with p. 772 *ibidem*. The latter condition, however, is only satisfied by the longest element w_G of $W^B = W$, which gives $\lambda_\pi = -w_G(\rho)|_{A_B} = \rho_B$. This shows the claim. □

Remark 2.3. As indicated in the introduction, this simple description of $H^q(SL_n(\mathbb{Z}),\mathbb{C})$ as in (2.2) will generally fail, if $n \geq 12$, because of the existence of an irreducible unitary cuspidal automorphic representation $\tilde{\tau}$ of $GL_2(\mathbb{A})$ of level 1 and of infinitesimal character $\chi_{\tilde{\tau}_\infty} = (\frac{11}{2}, -\frac{11}{2})$ (namely the one constructed out of a non-zero cuspidal modular form of weight 12 and full level, i.e., out of a non-zero element in $S_{12}(SL_2(\mathbb{Z}))$, e.g., the Ramanujan Delta-function). Indeed, suitably extended by 10 Hecke characters, one obtains an irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A}) \times \prod_{i=1}^{10} GL_1(\mathbb{A})$ of level 1 which satisfies $\chi_{\tilde{\pi}_\infty} = (\frac{11}{2}, -\frac{11}{2}, \frac{9}{2}, \frac{7}{2}, ..., -\frac{9}{2}) = -w(\rho_{GL_{12}})$ for a suitable Kostant representative w.

3. An examination of Franke's filtration and consequences for automorphic cohomology

3.1. Franke's filtration of the cuspidal support of the trivial automorphic representation. We recall that in [Fra98], §6, a certain, technically involved, finite-step filtration was defined, which can be refined to apply to the individual summands $\mathcal{A}_{\{P\},\varphi(\pi)}(G)$, cf. [Grb12], §3, [Gro13], §3.1, [Grb-Gro13], §3, or [Grb23], Chap. 4. The reader, who prefers to read a presentation of this subject, which is taylored to the (special) linear group, is invited to consult [Grb-Gro24], §2, for all relevant details. Our next result makes this filtration explicit for the datum $(\{B\},\varphi(\chi))$, $\chi = e^{\langle \rho_B, H_B(\cdot)\rangle}$.

Theorem 3.1. *Let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character*

$$\chi = e^{\langle \rho_B, H_B(\cdot)\rangle} = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$. Then, Franke's filtration of the space $\mathcal{A}_{\{B\},\varphi(\chi)}$ of automorphic forms with cuspidal support in the associate class $\varphi(\chi)$ can be defined as the filtration

$$\mathcal{A}_{\{B\},\varphi(\chi)} = \mathcal{A}^0_{\{B\},\varphi(\chi)} \supsetneqq \mathcal{A}^1_{\{B\},\varphi(\chi)} \supsetneqq \cdots \supsetneqq \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \supsetneqq \{0\}$$

of length n, where the quotients of the filtration for $i = 0, 1, \ldots, n-1$ are isomorphic to

$$\begin{aligned}\mathcal{A}^i_{\{B\},\varphi(\chi)}/\mathcal{A}^{i+1}_{\{B\},\varphi(\chi)} &\cong \bigoplus_{\substack{\underline{n}=(n_1,\ldots,n_r)\\ \text{with } r=n-i}} \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}) \\ &\cong \bigoplus_{\substack{\underline{n}=(n_1,\ldots,n_r)\\ \text{with } r=n-i}} \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(\bigotimes_{j=1}^{r} |\det{}_{n_j}|^{\frac{n_{j+1}+\cdots+n_r-(n_1+\cdots+n_{j-1})}{2}}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\end{aligned}$$

as $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules, where the direct sum is over the set of all ordered partitions $\underline{n} = (n_1, \ldots, n_r)$ of n into positive integers with $r = n - i$, i.e., over all parabolic subgroups of rank i. In particular,

$$\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \cong \mathbf{1}_{G(\mathbb{A})},$$

where $\mathbf{1}_{G(\mathbb{A})}$ is the trivial representation of $G(\mathbb{A})$, realized as the residual automorphic representation on the space of constant functions on $G(\mathbb{A})$.

Proof. It follows from Theorem 4.1 in [Grb-Gro24], that Franke's filtration of the space of automorphic forms with cuspidal support in $\varphi(\chi)$ can be arranged in such a way that the contributions to the quotients of the filtration are determined by the rank of the parabolic subgroup on which the degenerate Eisenstein series are supported, i.e., by the rank of the parabolic subgroup from which the contribution is parabolically induced. The result then follows from the decomposition of the sequence of exponents of the cuspidal support into segments.

The exponents in the induced representation from the parabolic subgroup $P_{\underline{n}}$ may be easily obtained by a direct calculation, or, can be found e.g., in (1.10) of [Gro-Lin21]. □

3.2. The isomorphisms in Franke's filtration. In the construction of explicit representatives of non-trivial automorphic cohomology classes, an explicit description of the isomorphism between the parabolically induced representations and the quotients of the filtration in Thm. 3.1 is required. This isomorphism is obtained using the main values of the derivatives of Eisenstein series, which we now discuss, as in [Fra98], see also [Fra-Schw98].

Consider the parabolically induced representation

$$\mathrm{Ind}_{P_{\underline{n}}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})$$

which appears as a direct summand of a quotient of the filtration in Thm. 3.1, and all the summands are of that form.

From the data in the induced representation, we construct a degenerate Eisenstein series associated to the trivial representation $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ of the Levi factor of $P_{\underline{n}}$, defined in the same way as in Sect. 2.2.2. We denote it by $E_{P_{\underline{n}}}(f,\lambda)$, where $\lambda \in \check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}$ is the complex parameter, and f ranges through the space $\mathcal{W}_{P_{\underline{n}},\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$. The evaluation point of interest is at $\lambda = \rho_{\underline{n}}$, which is a singularity of $E_{P_{\underline{n}}}(f,\lambda)$.

The symmetric algebra $S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})$ is identified with the partial derivatives in (our fixed choice of) Cartesian coordinates on $\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}}$. Given a multi-index α, the corresponding derivative $\frac{\partial^\alpha}{\partial\lambda^\alpha}$ is thus viewed as an element of the symmetric algebra.

Given a function $f \in \mathcal{W}_{P_{\underline{n}},\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ and a derivative $\frac{\partial^\alpha}{\partial\lambda^\alpha} \in S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})$, we would like to evaluate the derivative of the Eisenstein series

$$\frac{\partial^\alpha}{\partial\lambda^\alpha} E_{P_{\underline{n}}}(f,\lambda)$$

at the evaluation point $\lambda = \rho_{P_{\underline{n}}}$. However, as already mentioned above, this Eisenstein series and its derivative are not holomorphic at $\lambda = \rho_{P_{\underline{n}}}$. Therefore, we must use the notion of its main value

$$\mathrm{MV}_{\lambda=\rho_{P_{\underline{n}}}}\left(\frac{\partial^\alpha}{\partial\lambda^\alpha} E_{P_{\underline{n}}}(f,\lambda)\right)$$

at $\lambda = \rho_{P_{\underline{n}}}$, as defined in [Fra98, p. 235], see also [Fra-Schw98, p. 775].

Although the main value is not well-defined as an automorphic form, one of the crucial observations of Franke is that it defines a unique element of the quotient of the Franke filtration. Hence, the map from the parabolically induced representation

$$\mathrm{Ind}_{P_{\underline{n}}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})$$

to the quotient of the filtration, given by the assignment

$$f \otimes \frac{\partial^\alpha}{\partial\lambda^\alpha} \mapsto \mathrm{MV}_{\lambda=\rho_{P_{\underline{n}}}}\left(\frac{\partial^\alpha}{\partial\lambda^\alpha} E_{P_{\underline{n}}}(f,\lambda)\right), \tag{3.1}$$

for all $f \in \mathcal{W}_{P_{\underline{n}},\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ and $\frac{\partial^\alpha}{\partial\lambda^\alpha} \in S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})$, is a well-defined injective intertwining of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules. (Here, we silently identified the normalized global induction $\mathrm{Ind}_{P_{\underline{n}}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right)$ with $\mathcal{W}_{P_{\underline{n}},\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ through the $e^{\langle -\rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}$-twisted evaluation of functions at $id \in G(\mathbb{A})$, cf. [Gro23], (15.23).) Since the Eisenstein series of different summands in Thm. 3.1 are not related by functional equations, the above construction for all summands gives rise to the isomorphisms between the direct sums of parabolically induced representations and the quotients of the filtration in the theorem.

3.3. **Cohomology of the trivial representation of $SL_n(\mathbb{R})$.**

Lemma 3.2. *Let $n \geq 1$. The Poincaré polynomial of the cohomology $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ of the trivial reprepsentation $\mathbf{1}_{SL_n(\mathbb{R})} = \mathbb{C}$ of $SL_n(\mathbb{R})$ is given by*

$$P_n(t) = \begin{cases} \prod_{i=1}^{k-1}(1+t^{4i+1}) \cdot (1+t^n) & \text{if } n = 2k \\ \prod_{i=1}^{k}(1+t^{4i+1}) & \text{if } n = 2k+1 \end{cases}$$

Consequently, $\dim_{\mathbb{C}} H^0(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) = 1$, *whereas for the degrees $1 \leq q \leq \dim_{\mathbb{R}} X$ the complex dimension of the cohomology $H^q(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ is given as follows: Let $a(q)$ be the number of ways to write an integer q as the sum of different integers of the form $4\ell+1$, $\ell \geq 1$. (Here, we formally set $a(q) = 0$, if $q \leq 0$.) Then,*

$$\dim_{\mathbb{C}} H^q(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) = \begin{cases} a(q) + a(q-n) & \text{if } n = 2k \\ a(q) & \text{if } n = 2k+1 \end{cases}$$

Proof. As $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C}) \cong H^*_{\mathrm{dR}}(SU(n)/SO(n), \mathbb{C})$, the Poincaré polynomial of the cohomology space $H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ can be read off [GHV76], Table 1, p. 493. The claim on the complex dimension of $\dim_{\mathbb{C}} H^*(\mathfrak{sl}_n(\mathbb{R}), SO(n), \mathbb{C})$ hence follows immediately. □

3.4. **The Kostant representatives.** Our next task is to determine the Kostant representatives producing the correct exponents for the induced representations appearing in our Thm. 3.1. To this end, recall the (very) well-known fact that the Weyl group W of G with respect to the fixed maximal split torus T is isomorphic to the symmetric group of permutations $\mathfrak{S}_n$ of n letters, and, via this isomorphism, the action of $w \in W \cong \mathfrak{S}_n$ on the character of the torus given by the sequence of exponents $(s_1, \ldots, s_n)$ is by permutation of these exponents. Here, recall that $(s_1, \ldots, s_n) \in \mathbb{C}^n$ corresponds to the character given by the assignment

$$(t_1, \ldots, t_n) \mapsto |t_1|^{s_1} \ldots |t_n|^{s_n},$$

where $(t_1, \ldots, t_n) \in T(\mathbb{A})$. The Weyl group is generated by the simple reflections w_i, $i = 1, \ldots, n-1$, corresponding to the simple roots of G. The length $\ell(w)$ of an element $w \in W$ is the number of simple reflections in any reduced decomposition of w into a product of simple reflections.

Clearly, if a standard parabolic $\mathbb{Q}$-subgroup $P = P_{(n_1,\ldots,n_k)}$ of G corresponds to the ordered partition $(n_1, \ldots, n_k)$ of n into positive integers, then

$$W_L \cong \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$$

where $\mathfrak{S}_{n_j}$ is the symmetric group of permutations of n_j letters.

Proposition 3.3. *Let $P_{\underline{n}}$ be the standard parabolic subgroup of G corresponding to the ordered partition $\underline{n} = (n_1, \ldots, n_k)$ of n into positive integers, and let $L_{\underline{n}}$ be its Levi factor. Let $w_{\underline{n}}$ be the Kostant representative in $W^{P_{\underline{n}}}$ such that*

$$-w_{\underline{n}}(\rho)\Big|_{\check{\mathfrak{a}}_{P_{\underline{n}}}}$$

equals the exponents that appear in the induced representation from $P_{\underline{n}}$ of Theorem 3.1. Then the length of $w_{\underline{n}}$ is given by

$$\ell(w_{\underline{n}}) = \sum_{1 \leq i < j \leq k} n_i n_j. \tag{3.2}$$

In particular, $w_{\underline{n}}$ is the longest element in $W^{P_{\underline{n}}}$.

Proof. Since the exponents in ρ are all different, there is a unique representative $w_{\underline{n}}$ in $W^{P_{\underline{n}}}$ producing the required exponents for each $\underline{n}$. It is the Weyl group element which acts as the longest permutation of blocks of sizes $n_k, \dots, n_1$. More precisely, the block of last n_1 exponents should be sent to the beginning of the sequence, the next to the last n_2 exponents should be sent to the second block of n_2 exponents, and so on, without changing the order inside the blocks. The first step of moving the block of last n_1 exponents can be made in $n_1(n-n_1)$ simple reflections, obtained as interchange of position of all n_1 exponents in the last block with all $n-n_1$ exponents outside the last block. The second step of moving the next to the last block of n_2 exponents to become the second block can be made in $n_2(n-n_1-n_2)$ simple reflections. And so on, we obtain

$$\begin{aligned}\ell(w_{\underline{n}}) &\leq n_1(n-n_1)+n_2(n-n_1-n_2)+\cdots+n_{k-1}(n-n_1-n_2-\cdots-n_{k-1})\\ &= n_1(n_2+\cdots+n_k)+n_2(n_3+\cdots+n_k)+\cdots+n_{k-1}n_k\\ &= \sum_{1\leq i<j\leq k} n_i n_j.\end{aligned}$$

The other inequality follows from the fact that the steps of the above procedure are independent, and each step cannot be made using less simple reflections. From the fact that $w_{\underline{n}}$ is the longest permutation of blocks of the parabolic, it is clear that it is the longest element in $W^{P_{\underline{n}}}$. □

3.5. **Automorphic cohomology in low degrees.**

Proposition 3.4. *Let $n \geq 4$ and let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character*

$$\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$$

of the torus $T(\mathbb{A})$. Then, the natural map $\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)} \hookrightarrow \mathcal{A}_{\{B\},\varphi(\chi)}$ induces an isomorphism of $G(\mathbb{A}_f)$-modules

$$H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)}) \cong H^q(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$$

for all degrees $0 \leq q \leq n$.

Proof. Let $n \geq 4$ as in the statement of the proposition and let $k \geq 3$. The short exact sequence of $(\mathfrak{g},K,G(\mathbb{A}_f))$-modules

$$\{0\} \to \mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-k}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)} \to \{0\}$$

gives rise to a long exact sequence of $G(\mathbb{A}_f)$-modules

$$\begin{aligned}\cdots \to &H^q(\mathfrak{g},K,\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}) \to H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}) \to H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}) \to\\ \to &H^{q+1}(\mathfrak{g},K,\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}) \to \dots\end{aligned} \tag{3.3}$$

It is hence enough to show that

$$H^q(\mathfrak{g},K,\mathcal{A}^{n-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-k+1}_{\{B\},\varphi(\chi)}) = \{0\} \tag{3.4}$$

for all $k \geq 3$ and all $q \leq n$.

Recalling Frobenius reciprocity (as it was used in the proof of [Bor-Wal00], Thm. III.3.3 or in Eq. (5) on p. 257 in [Fra98]) and the fact that for each parabolic subgroup $P_{\underline{n}}$ of G,

$$H^*(\mathfrak{a}_{P_{\underline{n}}}, S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})) = H^0(\mathfrak{a}_{P_{\underline{n}}}, S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})) \cong \mathbb{C},$$

see, [Fra98], p. 256, the $G(\mathbb{A}_f)$-module $H^q\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{\underline{n}}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right)$ is isomorphic to

$$H^{q-\ell(w_{\underline{n}})}(\mathfrak{m}_{\underline{n}}, K\cap M_{\underline{n}}, e^{\langle 2\rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_\infty\rangle}\otimes \mathbb{C}_{-2\rho_{P_{\underline{n}}}})\otimes \mathrm{Ind}_{P_{\underline{n}}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right),$$

where $\ell(w_{\underline{n}}) = \sum_{1\leq i<j\leq k} n_i n_j$ is the length of the uniquely determined Kostant representative $w_{\underline{n}} \in W^{P_{\underline{n}}}$, given by Prop. 3.3. Hence, our Thm. 3.1 implies that it is enough to prove that for all $k \geq 3$

$$\min_{\underline{n}=(n_1,\dots,n_k)} \sum_{1\leq i<j\leq k} n_i n_j \geq n+1, \tag{3.5}$$

in order to show (3.4) for all $k\geq 3$ and all $q \leq n$. To this end, we rewrite

$$\begin{aligned}\sum_{1\leq i<j\leq k} n_i n_j &= \sum_{1\leq i<j\leq k-1} n_i n_j + \sum_{i=1}^{k-1} n_i(n - n_1 - n_2 - \cdots - n_{k-1}) \\ &= -\sum_{i=1}^{k-1} n_i^2 + n\cdot \sum_{i=1}^{k-1} n_i - \sum_{1\leq i<j\leq k-1} n_i n_j,\end{aligned}$$

revealing $\ell(w_{\underline{n}})$ as a quadratic polynomial in the variables n_i, $1\leq i\leq k-1$. Since the coefficient of n_i^2 is always negative, the minimum over all ordered partitions $\underline{n} = (n_1,\dots,n_k)$ is attained at the boundary of the domain of possible values, which, in the present case, is (all boundary values lead to the same outcome) at $n_1 = n_2 = \cdots = n_{k-1} = 1$. Hence, by inserting, we get that for all $k\geq 3$,

$$\min_{\underline{n}=(n_1,\dots,n_k)} \sum_{1\leq i<j\leq k} n_i n_j = n(k-1) - \frac{k(k-1)}{2}.$$

Checking, when this expression satisfies (3.5), hence leads by a simple calculation to checking when

$$n \geq \frac{k^2-k+2}{2(k-2)}. \tag{3.6}$$

Viewing the right-hand side of (3.6) as a function $\phi(k)$ of a real variable k, it a matter of basic calculus to show that the only local extreme in the domain $3\leq k\leq n$ is the local minimum at $k=4$. Hence, the maximum of $\phi(k)$ is attained at the boundary of the domain $3\leq k\leq n$, and it remains to check that the inequalities

$$\begin{aligned} n &\geq \phi(3) = 4, \\ n &\geq \phi(n) = \frac{n^2-n+2}{2(n-2)}, \end{aligned}$$

hold for $n\geq 4$. The former inequality is obvious, and the latter follows by writing it as a quadratic inequality in n. Thus, the result follows. □

3.6. **Quotient-cohomology.** Let $n\geq 4$ and let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}}\otimes|\cdot|^{\frac{n-3}{2}}\otimes|\cdot|^{\frac{n-5}{2}}\otimes\cdots\otimes|\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$. We start with an analysis of the $G(\mathbb{A}_f)$-modules $H^q(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$ in degrees $q=n-1$ and $q=n$. Recall from Thm. 3.1 that $\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \cong \mathbf{1}_{G(\mathbb{A})}$. We will first prove a simple lemma:

Lemma 3.5. *For $n \geq 4$, the map* (3.1) *induces an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong H^{n-1}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(1,n-1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right) \oplus H^{n-1}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right).$$

For $n \geq 5$, the map (3.1) *induces an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^{n}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong H^{n}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(1,n-1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right) \oplus H^{n}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right),$$

whereas for $n = 4$

$$H^{4}(\mathfrak{g}, K, \mathcal{A}^{2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{3}_{\{B\},\varphi(\chi)}) \cong H^{4}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(1,3)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,3)}}, H_{P_{(1,3)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,3)},\mathbb{C}})\right) \oplus H^{4}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(2,2)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(2,2)}}, H_{P_{(2,2)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(2,2)},\mathbb{C}})\right) \oplus H^{4}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{(3,1)}(\mathbb{A})}\left(e^{\langle \rho_{P_{(3,1)}}, H_{P_{(3,1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(3,1)},\mathbb{C}})\right).$$

Proof. Thm. 3.1 implies that for all degrees q,

$$H^{q}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong \bigoplus_{\underline{n}=(n_1,n_2)} H^{q}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right),$$

with $n = n_1 + n_2$, $n_1, n_2 \in \mathbb{Z}_{>0}$. Literally the same strategy, as in the proof of Prop. 3.4, shows that the lowest degree of cohomology, in which $H^{q}\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right)$ may be non-zero is bounded from below by $\ell(w_{\underline{n}}) = \sum_{1\leq i<j\leq 2} n_i n_j = n_1 \cdot n_2$. Since $n_1 + n_2 = n \geq 4$, it is a very simple exercise to prove that $n_1 \cdot n_2 > n-1$ if $2 \leq n_1 \leq n-2$. This shows the claim in degree $q = n-1$. Similarly, if $n \geq 5$, then $n_1 \cdot n_2 > n$ if $2 \leq n_1 \leq n-2$, which shows the assertion for degree $q = n \geq 5$. The case $n = 4$ is treated analogously, keeping in mind that the partition $(2, 2)$ also contributes. □

We continue by refining Lem. 3.5. We first treat the case of $q = n$.

Proposition 3.6. *Under the assumptions of Sect. 3.6, for $n \geq 5$,*

$$H^{n}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) = \{0\},$$

whereas for $n = 4$

$$H^{4}(\mathfrak{g}, K, \mathcal{A}^{2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{3}_{\{B\},\varphi(\chi)}) \cong \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(2,2)}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,2)}}, H_{P_{(2,2)}}(\cdot)_f\rangle}\right).$$

Proof. Let $n \geq 4$. We first consider $H^n\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)$. This cohomology space is obviously isomorphic to

$$H^n\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right) \otimes \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right),$$

so, this space being zero is equivalent to the vanishing of

$$H^n\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right).$$

Recall that by our Prop. 3.3, $\ell(w_{(n-1,1)}) = n-1$. Hence, invoking [Bor-Wal00], Thm. III.3.3 and [Fra98], p. 256,

$$\begin{aligned} &H^n\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right) \\ &\cong H^1(\mathfrak{m}_{(n-1,1)}, K \cap M_{(n-1,1)}, e^{\langle 2\rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_\infty\rangle} \otimes \mathbb{C}_{-2\rho_{P_{(n-1,1)}}}) \\ &\cong H^1(\mathfrak{m}_{(n-1,1)}, K \cap M_{(n-1,1)}, \mathrm{sgn}(\det_{n-1}) \otimes \mathrm{sgn}^{n-1}). \end{aligned}$$

Using [Bor-Wal00], I.1.3.(2) and I.5.1.(4), the latter space is isomorphic to the vector space of $S(O(n-1) \times O(1))/SO(n-1) \times SO(1)$-invariant elemets in the direct sum

$$H^1(\mathfrak{sl}_{n-1}(\mathbb{R}), SO(n-1), \mathrm{sgn}(\det_{n-1})) \otimes H^0(\mathfrak{sl}_1(\mathbb{R}), SO(1), \mathrm{sgn}) \oplus H^0(\mathfrak{sl}_{n-1}(\mathbb{R}), SO(n-1), \mathrm{sgn}(\det_{n-1})) \otimes H^1(\mathfrak{sl}_1(\mathbb{R}), SO(1), \mathrm{sgn}).$$

However, as vector spaces, the latter sum is isomorphic to

$$H^1(\mathfrak{sl}_{n-1}(\mathbb{R}), SO(n-1), \mathbb{C}) \otimes H^0(\mathfrak{sl}_1(\mathbb{R}), SO(1), \mathbb{C}) \oplus H^0(\mathfrak{sl}_{n-1}(\mathbb{R}), SO(n-1), \mathbb{C}) \otimes H^1(\mathfrak{sl}_1(\mathbb{R}), SO(1), \mathbb{C}),$$

which vanishes by Lem. 3.2. It is clear that the same argument implies the vanishing of the cohomology space $H^n\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(1,n-1)},\mathbb{C}})\right)$. Hence, by Lem. 3.5, we are only left to show that there is an isomorphism

$$H^4\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(2,2)}(\mathbb{A})}^{G(\mathbb{A})}\left(e^{\langle \rho_{P_{(2,2)}}, H_{P_{(2,2)}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(2,2)},\mathbb{C}})\right) \cong \mathrm{Ind}_{P_{(2,2)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,2)}}, H_{P_{(2,2)}}(\cdot)_f\rangle}\right).$$

i.e., that

$$H^4\left(\mathfrak{g}, K, \mathrm{Ind}_{P_{(2,2)}(\mathbb{R})}^{G(\mathbb{R})}\left(e^{\langle \rho_{P_{(2,2)}}, H_{P_{(2,2)}}(\cdot)_\infty\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{(2,2)},\mathbb{C}})\right) \cong \mathbb{C}$$

However, in view of the above argument, the latter cohomology space is isomorphic to

$$\left(H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbf{1}_{SL_2(\mathbb{R})}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbf{1}_{SL_2(\mathbb{R})})\right)^{S(O(2)\times O(2))/SO(2)\times SO(2)}. \tag{3.7}$$

The only non-trivial element of $S(O(2)\times O(2))/SO(2)\times SO(2)$ is represented by the pair of matrices $\mathrm{diag}(1,-1) \times \mathrm{diag}(1,-1)$ and $\mathrm{diag}(1,-1)$ acts trivially on $H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbf{1}_{SL_2(\mathbb{R})})$. Therefore, (3.7) is isomorphic to $H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C})$ and Lem. 3.2 for $n = 2$ finally implies the result. □

Next we treat the case of $q = n-1$. We get

Proposition 3.7. *Under the assumptions of Sect. 3.6,*

$$H^{n-1}(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})=\{0\},$$

if $n\geq 5$ *is odd, whereas*

$$\begin{aligned}H^{n-1}(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) &\cong \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(1,n-1)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(1,n-1)}},H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right)\\ &\oplus \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(n-1,1)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right)\end{aligned}$$

if $n\geq 4$ *is even.*

Proof. Let $n\geq 4$ be of arbitrary parity. Again, one gets that

$$\begin{aligned}&H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{A})}_{P_{(n-1,1)}(\mathbb{A})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\ &\cong H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{R})}_{P_{(n-1,1)}(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\ &\quad\otimes\mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(n-1,1)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right),\end{aligned}$$

and the same arguments as in the proof of Prop. 3.6 reveal that

$$\begin{aligned}&H^{n-1}\left(\mathfrak{g},K,\mathrm{Ind}^{G(\mathbb{R})}_{P_{(n-1,1)}(\mathbb{R})}\left(e^{\langle\rho_{P_{(n-1,1)}},H_{P_{(n-1,1)}}(\cdot)_\infty\rangle}\right)\otimes S(\check{\mathfrak{a}}_{P_{(n-1,1)},\mathbb{C}})\right)\\ &\cong H^0(\mathfrak{m}_{(n-1,1)},K\cap M_{(n-1,1)},\mathrm{sgn}(\det_{n-1})\otimes\mathrm{sgn}^{n-1}).\end{aligned}$$

$$\cong\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn}^{n-1})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}$$

We may represent the only non-trivial element of the quotient group $S(O(n-1)\times O(1))/SO(n-1)\times SO(1)$ by the pair $(\mathrm{diag}(id_{n-2},-1);-1)$, which obviously acts by multiplication by -1 on $H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))$ and by multiplication by $(-1)^\varepsilon$ by its adjoint action on $H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn}^\varepsilon)$. Hence,

$$\begin{aligned}&\left(H^0(\mathfrak{sl}_{n-1}(\mathbb{R}),SO(n-1),\mathrm{sgn}(\det_{n-1}))\otimes H^0(\mathfrak{sl}_1(\mathbb{R}),SO(1),\mathrm{sgn}^{n-1})\right)^{S(O(n-1)\times O(1))/SO(n-1)\times SO(1)}\\ &\cong\begin{cases}\mathbb{C} & \text{if } n \text{ is even}\\ \{0\} & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

Flipping factors in the Levi of the parabolic, the same argument applies to $P_{(1,n-1)}$ and so Lem. 3.5 finally implies the desired result. □

The above results on the cohomology of the quotient $\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}$ easily generalize to smaller degrees:

Lemma 3.8. *Under the assumptions of Sect. 3.6,*

$$H^q(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})=\{0\}$$

for all $0\leq q\leq n-2$.

Proof. The arguments presented in the proof of Prop. 3.4 show that $\ell(w_{\underline{n}})$, $\underline{n}=(n_1,n_2)$, is a lower bound for the degrees of cohomology q, in which $H^q(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$ may be non-zero. However, $\ell(w_{\underline{n}})$ is bounded from below by $n-1$, as we have seen in the proof of Lem. 3.5. □

4. Degenerate Eisenstein classes for $SL_n(\mathbb{Z})$

4.1. **Contribution of the constant automorphic forms.** Lem. 3.8, Prop. 3.7 and Prop. 3.6 allow us to determine non-trivial cohomology classes in $H^q(SL_n(\mathbb{Z}),\mathbb{C})$, $0 \leq q \leq n$, which are represented by constant automorphic forms on $G(\mathbb{A})$. This amounts to partly rewriting results of Borel and Franke in a more automorphic language: We refer to [Bor74], Thm. 7.5, and to [Fra08], pp. 58 – 62, where Franke gave a description of the kernel of the "Borel map", i.e., of the morphism in deRham-cohomology induced by the natural inclusion of $G(\mathbb{R})$-invariant differential forms in all differential forms, in terms of pull-backs of primitive classes and the Euler class of the canonical n-dimensional orientable real bundle on the compact dual X_u of $X = G(\mathbb{R})/K$.

Proposition 4.1. *Let $n \geq 2$. Then the natural inclusion $\mathbf{1}_{G(\mathbb{A})} \hookrightarrow \mathcal{A}(G)$ induces an embedding of $G(\mathbb{A}_f)$-modules $H^q(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})}) \hookrightarrow H^q(\mathfrak{g},K,\mathcal{A}(G))$ for all $0 \leq q \leq n-1$. If $n \geq 5$ is odd, then the assertion also holds in degree $q = n$.*

Proof. Since $H^q(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)})$ is a direct $G(\mathbb{A}_f)$-summand of $H^q(\mathfrak{g},K,\mathcal{A}(G))$, the first assertion follows from Prop. 3.4, Lem. 3.8, (3.3) and Thm. 3.1, whereas for the second one invokes the same references, but uses Prop. 3.7 instead of Lem. 3.8,. □

The case of degree $q = n$ for even n is treated in

Proposition 4.2. *Let $n \geq 4$. Then, the kernel of the natural homomorphism of $G(\mathbb{A}_f)$-modules $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g},K,\mathcal{A}(G))$, induced from the natural inclusion $\mathbf{1}_{G(\mathbb{A})} \hookrightarrow \mathcal{A}(G)$, has dimension less or equal to 2. Equivalently, the image of the $G(\mathbb{A}_f)$-module $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})})$ in $H^n(\mathfrak{g},K,\mathcal{A}(G))$ has dimension greater or equal to $\dim_{\mathbb{C}} H^n(\mathfrak{g},K,\mathbb{C}) - 2$.*

Proof. Let $\varphi(\chi)$ be the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$. Since $\mathcal{A}_{\{B\},\varphi(\chi)}$ is a direct $(\mathfrak{g},K,G(\mathbb{A}_f))$-summand of $\mathcal{A}(G)$, it is enough to show this for $H^n(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)})$. We consider the respective part of the long exact sequence in cohomology (3.3), which by Prop. 3.7 and Thm. 3.1 reads as

$$\begin{aligned} \cdots \to\ & \mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right) \oplus \mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right) \\ \to\ & H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to \ldots \end{aligned}$$

Since the trivial representation $\mathbf{1}_{G(\mathbb{A}_f)}$ of $G(\mathbb{A}_f)$ appears precisely once as a quotient of the induced representation $\mathrm{Ind}_{P_{(1,n-1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(1,n-1)}}, H_{P_{(1,n-1)}}(\cdot)_f\rangle}\right)$, respectively of the induced representation $\mathrm{Ind}_{P_{(n-1,1)}(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n-1,1)}}, H_{P_{(n-1,1)}}(\cdot)_f\rangle}\right)$, the connecting homomorphism above has at most two-dimensional image in $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})}) \cong H^n(\mathfrak{g},K,\mathbb{C}) \otimes \mathbf{1}_{G(\mathbb{A}_f)}$. Hence, the kernel of the natural map of $G(\mathbb{A}_f)$-modules $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$, has dimension less or equal to 2, or, equivalently, the image of the $G(\mathbb{A}_f)$-module $H^n(\mathfrak{g},K,\mathbf{1}_{G(\mathbb{A})})$ in $H^n(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$ has dimension greater or equal to $\dim_{\mathbb{C}} H^n(\mathfrak{g},K,\mathbb{C}) - 2$. However, by Prop. 3.4, $H^n(\mathfrak{g},K,\mathcal{A}^{n-2}_{\{B\},\varphi(\chi)})$ is nothing else than $H^n(\mathfrak{g},K,\mathcal{A}_{\{B\},\varphi(\chi)})$, whence the corollary follows. □

4.2. **A growth-result for $H^n(SL_n(\mathbb{Z}))$.** We recall the number $a(q)$ from Lem. 3.2, which denoted the number of ways to write a positive integer q as the sum of different integers of the form $4\ell + 1$,

$\ell \geq 1$. The following is our first main result: It says that Franke's description of the image of the Borel map in degree $q = n$ is exhausts the whole cohomology space $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$.

Theorem 4.3. *Let $n \geq 5$. Then, the cohomology space $H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ is isomorphic to the image of the natural map of $G(\mathbb{A}_f)$-modules $H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g}, K, \mathcal{A}(G))$. Consequently,*

$$\dim_{\mathbb{C}} H^n(SL_n(\mathbb{Z}), \mathbb{C}) \geq \begin{cases} a(n) - 2 & \text{if } n \text{ is even} \\ a(n) & \text{if } n \text{ is odd} \end{cases}$$

In particular, the free part of the $\mathbb{Z}$-module $H^n(SL_n(\mathbb{Z}))$ is non-zero, in the following cases:

- *for odd n, if either $n \geq 25$, or $n \in \{5, 9, 13, 17, 21\}$;*
- *for even n, if either $n \geq 50$, or $n \in \{22, 26, 30, 34, 38, 42, 46\}$.*

Proof. We look at the following part of the long exact sequence of $G(\mathbb{A}_f)$-modules

$$\begin{aligned} \cdots \to & H^n(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to H^n(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to H^n(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \\ \to & H^{n+1}(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \cdots \end{aligned} \tag{4.1}$$

given by the short exact sequence of $(\mathfrak{g}, K, G(\mathbb{A}_f))$-modules

$$\{0\} \to \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)} \to \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \to \{0\}.$$

As $\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)} \cong \mathbf{1}_{G(\mathbb{A})}$, cf. Thm. 3.1, and $H^n(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$ is trivial, cf. Prop. 3.6, we get from Prop. 3.4 that

$$H^n(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)}) \cong H^n(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong \mathrm{Im}[H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g}, K, \mathcal{A}(G))].$$

The remaining assertions are a consequence of (2.1), our propositions Prop. 4.2 and Prop. 4.1 and Lem. 3.2. ☐

Corollary 4.4. *For $5 \leq n \leq 11$, the cohomology of a congruence subgroup Γ of $SL_n(\mathbb{Q})$ in degree $q = n$ is given as*

$$H^n(\Gamma, \mathbb{C}) = \mathrm{Im}[H^n(\mathfrak{g}, K, \mathbf{1}_{G(\mathbb{A})}) \to H^n(\mathfrak{g}, K, \mathcal{A}(G))].$$

Proof. This is clear from Thm. 4.3 and Thm. 2.2. ☐

Remark 4.5. Cor. 4.4 reestablishes the fact that the rank of the free part of the $\mathbb{Z}$-module $H^5(SL_5(\mathbb{Z}))$ is one, as already shown in [EVGS13], Thm. 7.3.

4.3. **Non-trivial degenerate Eisenstein classes in $H^{n-1}(SL_n(\mathbb{Z}), \mathbb{C})$.** In this section, we will go beyond our analysis of the contribution of the trivial automorphic representation $\mathbf{1}_{G(\mathbb{A})}$ to the cohomology of $SL_n(\mathbb{Z})$. In other words, we will prove the existence of several new non-zero classes in $H^{n-1}(SL_n(\mathbb{Z}))$, which *cannot be represented by constant automorphic functions* and moreover we will give a quite explicit description of the automorphic representatives of these classes per means of concrete (degenerate) Eisenstein series via the morphisms (3.1).

This aligns with the philosophy of Harder to construct non-trivial deRham cohomology classes in the cohomology of locally symmetric spaces by representing them by Eisenstein differential forms, i.e., by constructing a section to the natural restriction map to the cohomology of the boundary of the Borel-Serre compactification of the given locally symmetric space. We refer to [Har87] and to [Har90] for more details in the case of GL_n.

The next result is our second main theorem. Here, again, $\varphi(\chi)$ denotes the cuspidal support represented by the Hecke character $\chi = |\cdot|^{\frac{n-1}{2}} \otimes |\cdot|^{\frac{n-3}{2}} \otimes |\cdot|^{\frac{n-5}{2}} \otimes \cdots \otimes |\cdot|^{-\frac{n-1}{2}}$ of the torus $T(\mathbb{A})$.

Theorem 4.6. *Let $n \geq 4$. Then, if n is odd, the cohomology space $H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ is isomorphic to the $G(\mathbb{A}_f)$-module $\mathbf{1}^{a(n-1)}_{G(\mathbb{A}_f)}$, whereas, if n is even, $H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$ contains an isomorphic copy of $\mathbf{1}^{a(n-1)}_{G(\mathbb{A}_f)}$ as a submodule, with the quotient given as the kernel of the natural connecting morphism ("Bockstein homomorphism")*

$$\mathrm{Eis}^n := \ker\left(\bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \to \mathbf{1}^{a(n)}_{G(\mathbb{A}_f)}\right).$$

Consequently, $\dim_{\mathbb{C}} H^{n-1}(SL_n(\mathbb{Z}), \mathbb{C})$ *grows at least as $a(n-1)$, if $n \geq 5$ is odd, and as $a(n-1) +$* $\dim_{\mathbb{C}}\left((\mathrm{Eis}^n)^{SL_n(\hat{\mathbb{Z}})}\right)$*, if $n \geq 4$ is even.*

If $n \geq 5$ is odd, the free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$ is non-zero, if either $n \geq 43$, or $n \in \{15, 19, 23, 27, 31, 35, 39\}$ and vanishes if $n \in \{5, 7, 9, 11\}$.

Moreover, if $n \geq 4$ is even, then the subspace of K_f-unramified vectors in Eis^n *does not vanish, if $n \in \{4, 6, 8, 10, 12, 16, 20, 24, 28, 32, 36, 40\}$. Consequently, the free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$ is non-zero for all even $n \geq 4$.*

Proof. Let $n \geq 4$. To prove the first assertion on the cohomology $H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$, one considers the following part of the natural long exact sequence of $G(\mathbb{A}_f)$-modules

(4.2)
$$\cdots \to H^{n-2}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to$$
$$\to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to H^{n}(\mathfrak{g}, K, \mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \cdots.$$

Recalling Thm. 3.1 and Lem. 3.2, this exact sequence of $G(\mathbb{A}_f)$-modules becomes

$$\cdots \to H^{n-2}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \mathbf{1}^{a(n-1)}_{G(\mathbb{A}_f)} \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to$$
$$\to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \to \mathbf{1}^{a(n)}_{G(\mathbb{A}_f)} \to \cdots.$$

Moreover, using Lem. 3.8 and Prop. 3.7, it simplifies furthermore to

$$\{0\} \to \mathbf{1}^{a(n-1)}_{G(\mathbb{A}_f)} \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to \{0\},$$

if $n \geq 5$ is odd, whereas, if $n \geq 4$ is even, it becomes

$$\{0\} \to \mathbf{1}^{a(n-1)}_{G(\mathbb{A}_f)} \to H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \to \bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{\underline{n}}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \to \mathbf{1}^{a(n)}_{G(\mathbb{A}_f)} \to \cdots.$$

As by Prop. 3.4, $H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}) \cong H^{n-1}(\mathfrak{g}, K, \mathcal{A}_{\{B\},\varphi(\chi)})$, the first assertion follows. The implications for the free part of the $\mathbb{Z}$-module $H^{n-1}(SL_n(\mathbb{Z}))$, if $n \geq 5$ is odd, is then a consequence of the above and Thm. 2.2.

We now turn to the case when $n \geq 4$ is even and consider the subspace of K_f-invariant vectors in Eis^n. A direct check using Lem. 3.2 implies that for $n \in \{4, 6, 8, 10, 12, 16, 20, 24, 28, 32, 36, 40\}$ one has $a(n) \leq 1$. Since $\bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}_{P_{\underline{n}}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right)$ captures precisely two copies of $\mathbf{1}_{G(\mathbb{A}_f)}$ as quotients, the kernel Eis^n of the connecting morphism must contain one copy of $\mathbf{1}_{G(\mathbb{A}_f)}$. Hence, $(\mathrm{Eis}^n)^{K_f} \neq \{0\}$ as claimed. Since $a(n) \geq 1$ for $n \geq 26$ and for $n \in \{6, 10, 14, 18, 22\}$, we get $\dim_{\mathbb{C}}(H^{n-1}(SL_n(\mathbb{Z}), \mathbb{C})) \geq 1$ for *all* even $n \geq 4$ as desired. □

Remark 4.7. The non-vanishing of $H^3(SL_4(\mathbb{Z}))$ was also shown by completely different techniques in [Lee-Szc78]. In fact, their paper completely computes the cohomology of $SL_4(\mathbb{Z})$ in all degrees. See [Lee-Szc78], Thm. 2.

Remark 4.8. As $a(5) = \dim_{\mathbb{C}}\left((\mathrm{Eis}^6)^{SL_6(\hat{\mathbb{Z}})}\right) = 1$, combining Thm. 4.6 with Thm. 2.2 reestablishes the fact that the rank of the free part of the $\mathbb{Z}$-module $H^5(SL_6(\mathbb{Z}))$ is two, as already shown in [EVGS13], Thm. 7.3.

Remark 4.9. Moreover, Thm. 4.6 and Thm. 2.2 show that the rank of the free part of the $\mathbb{Z}$-module $H^9(SL_{10}(\mathbb{Z}))$ is either two or three.

4.4. **Non-trivial automorphic representatives.** In this section, we will explicitly exhibit, which automorphic forms represent the classes in the Eisenstein space $(\mathrm{Eis}^n)^{K_f}$. We recall from Thm. 4.6 that this space is non-trivial at least, if $n \in \{4, 6, 8, 10, 12, 16, 20, 24, 28, 32, 36, 40\}$. We use the notation regarding the Eisenstein series introduced in §2.2.2.

Theorem 4.10. *Let $n \geq 4$ be even. A non-trivial cohomology class in the Eisenstein space $(\mathrm{Eis}^n)^{K_f}$ for $H^{n-1}(SL_n(\mathbb{Z}), \mathbb{C})$, can be represented by an automorphic form obtained as the linear combination of the main values of the two degenerate Eisenstein series $E_{P_{\underline{n}}}(f^\circ, \lambda)$, constructed from the constant function f°, viewed as an element of the space $\mathcal{W}_{P_{\underline{n}}, \mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ associated to the trivial representation $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ of the Levi factor of $P_{\underline{n}}$, evaluated at the evaluation point $\lambda = \rho_{P_{\underline{n}}}$, where $\underline{n} \in \{(n-1, 1), (1, n-1)\}$.*

Proof. Let $n \geq 4$ be even. Prop. 3.7 shows that

$$(4.3) \qquad \bigoplus_{\underline{n}\in\{(n-1,1),(1,n-1)\}} \mathrm{Ind}_{P_{\underline{n}}(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)_f\rangle}\right) \cong H^{n-1}(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)})$$

where this isomorphism is constructed from the $(\mathfrak{g}, K, G(\mathbb{A}_f))$-morphisms (3.1) using the main values of the derivatives of degenerate Eisenstein series. Hence, the non-trivial cohomology classes in Eis^n can all be represented by linear combinations of the main values of the degenerate Eisenstein series $E_{P_{\underline{n}}}(f, \lambda)$, associated to the trivial representation $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ of the Levi factor $L_{\underline{n}}$ of $P_{\underline{n}}$, evaluated at $\lambda = \rho_{P_{\underline{n}}}$, where $\underline{n} \in \{(n-1, 1), (1, n-1)\}$. The function f ranges over the space $\mathcal{W}_{P_{\underline{n}}, \mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$. Observe that the main values are required, because there exist functions f, for which the Eisenstein series in question have a pole of order one at the point of evaluation. The residues actually span the trivial representation.

According to Thm. 2.2, in order to determine the representatives of non-trivial cohomology classes in $(\mathrm{Eis}^n)^{K_f}$, it remains to find the K_f-invariant representatives of non-trivial classes in Eis^n, i.e., from the representation theoretic point of view, the everywhere unramified vectors. The K_f-unramified component in each of the two summands in (4.3) is the unique quotient of the parabolically induced representation, which is of dimension one. The corresponding unramified Eisenstein series are obtained by choosing the constant functions $f = f^\circ$ from the space $\mathcal{W}_{P_{\underline{n}},L_{\underline{n}}(\mathbb{A})}$ in the Eisenstein series construction. Thus, the K_f-invariant representatives of cohomology classes in Eis^n are linear combinations of the main values of the two Eisenstein series as in the statement. □

5. Applications to degenerate Eisenstein classes below the tempered range

5.1. The non-trivial automorphic representatives of the class in $H^8(SL_6(\mathbb{Z}))$. In [EVGS13], Elbaz-Vincent, Gangl and Soulé have calculated the cohomology of $SL_n(\mathbb{Z})$ for $n = 5, 6, 7$. In particular, they found a non-trivial cohomology class of $SL_6(\mathbb{Z})$ in degree $q = 8$, cf. [EVGS13], Thm. 7.3.(ii), for whose existence, however, there seemed to be no proper conceptual explanation by the time of [EVGS13] and until very recent: We refer to Brown's recent preprint [Bro23], in particular to its Thm. 1.1 and Table 1, for a discussion of this phenomenon.

We present here a structural reason, arising from the point of view of automorphic forms, for the existence of this non-trivial class, i.e., we will explain which automorphic forms represent the one-dimensional space $H^8(SL_6(\mathbb{Z}), \mathbb{C})$.

To this end, we first apply our Thm. 3.1 to the case $i = n - 2$, i.e., to the second last non-trivial step in Franke's filtration of $\mathcal{A}_{\{B\},\varphi(\chi)}$, $\chi = e^{\langle \rho_B, H_B(\cdot)\rangle}$. Its cohomology is then computed as

$$H^q(\mathfrak{g}, K, \mathcal{A}^{n-2}_{\{B\},\varphi(\chi)}/\mathcal{A}^{n-1}_{\{B\},\varphi(\chi)}) \cong \bigoplus_{\underline{n}=(n_1,n_2)} H^q\left(\mathfrak{g}, K, \mathrm{Ind}^{G(\mathbb{A})}_{P_{\underline{n}}(\mathbb{A})}\left(e^{\langle \rho_{P_{\underline{n}}}, H_{P_{\underline{n}}}(\cdot)\rangle}\right) \otimes S(\check{\mathfrak{a}}_{P_{\underline{n}},\mathbb{C}})\right), \tag{5.1}$$

which, invoking [Bor-Wal00], Thm. III.3.3 and [Fra98], p. 256, together with [Bor-Wal00], I.1.3.(2) and I.5.1.(4), is isomorphic as $G(\mathbb{A}_f)$-module to

$$\begin{aligned}\bigoplus_{\underline{n}=(n_1,n_2)} \bigoplus_{r+s=q-n_1n_2} &\Big(H^r(\mathfrak{sl}_{n_1}(\mathbb{R}), SO(n_1), \mathrm{sgn}^{n_2}) \\ &\otimes H^s(\mathfrak{sl}_{n_2}(\mathbb{R}), SO(n_2), \mathrm{sgn}^{n_1})\Big)^{S(O(n_1)\times O(n_2))/SO(n_1)\times SO(n_2)} \\ &\otimes \mathrm{Ind}^{G(\mathbb{A}_f)}_{P_{(n_1,n_2)}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(n_1,n_2)}}, H_{P_{(n_1,n_2)}}(\cdot)_f\rangle}\right).\end{aligned} \tag{5.2}$$

Put now $n = 6$ in (5.2). Then, $H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)})$ has five direct summands as due to (5.1), indexed by the partitions $(1,5)$, $(5,1)$, $(2,4)$, $(4,2)$, $(3,3)$. By equation (3.2), the partition $(3,3)$ only contributes to cohomology in degree $q \geq 3 \cdot 3 = 9$. Similarly, by (3.2) together with Lem. 3.2, the partitions $(1,5)$ and $(5,1)$ may only contribute to degrees $q = 5, 10, 14$. While for the same reason, the partitions $(2,4)$, $(4,2)$ may only contribute to degrees $q = 8, 10, 13, 15$. It therefore follows that

$$\begin{aligned}&H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \cong \\ &\quad\Big(H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C}) \otimes H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C})\Big)^{S(O(2)\times O(4))/SO(2)\times SO(4)} \\ &\qquad\otimes \mathrm{Ind}^{SL_6(\mathbb{A}_f)}_{P_{(2,4)}(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f\rangle}\right)\end{aligned}$$

$$\bigoplus \left(H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C})\right)^{S(O(4)\times O(2))/SO(4)\times SO(2)} \otimes \ \mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f\rangle}\right).$$

The only non-trivial element of $S(O(2)\times O(4))/SO(2)\times SO(4)$ (resp. $S(O(4)\times O(2))/SO(4)\times SO(2)$) operates trivially on the one-dimensional spaces $H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C})\otimes H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C})$ (resp. $H^0(\mathfrak{sl}_4(\mathbb{R}), SO(4), \mathbb{C}) \otimes H^0(\mathfrak{sl}_2(\mathbb{R}), SO(2), \mathbb{C})$), hence

$$H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \cong \mathrm{Ind}_{P_{(2,4)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f\rangle}\right) \oplus \mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f\rangle}\right).$$

If we plug this (and the knowledge on $H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) = H^q(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathbf{1}_{SL_6(\mathbb{A})})$, which is given by Lem. 3.2) into the long exact sequence in cohomology, which comes from the short exact sequence of $(\mathfrak{sl}_6(\mathbb{R}), SO(6), SL_6(\mathbb{A}_f))$-modules

$$\{0\} \to \mathcal{A}^5_{\{B\},\varphi(\chi)} \to \mathcal{A}^4_{\{B\},\varphi(\chi)} \to \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)} \to \{0\}.$$

i.e., into the exact sequence of $SL_6(\mathbb{A}_f)$-modules

$$\cdots \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) \to$$
$$\to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}/\mathcal{A}^5_{\{B\},\varphi(\chi)}) \to H^9(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^5_{\{B\},\varphi(\chi)}) \to \dots,$$

we obtain an exact sequence of $SL_6(\mathbb{A}_f)$-modules

$$\{0\} \to H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) \to$$
$$\to \mathrm{Ind}_{P_{(2,4)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f\rangle}\right) \oplus \mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f\rangle}\right) \to \mathbf{1}_{SL_6(\mathbb{A}_f)} \to \cdots$$

Recalling that both $\mathrm{Ind}_{P_{(2,4)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(2,4)}}, H_{P_{(2,4)}}(\cdot)_f\rangle}\right)$ and $\mathrm{Ind}_{P_{(4,2)}(\mathbb{A}_f)}^{SL_6(\mathbb{A}_f)}\left(e^{\langle \rho_{P_{(4,2)}}, H_{P_{(4,2)}}(\cdot)_f\rangle}\right)$ contain $\mathbf{1}_{SL_6(\mathbb{A}_f)}$ with multiplicity one as a quotient, it follows that $H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)})$ contains (at least) one copy of $\mathbf{1}_{SL_6(\mathbb{A}_f)}$.

In order to determine automorphic forms that represent a non-trivial class in $H^8(SL_6(\mathbb{Z}), \mathbb{C})$, it hence suffices to show by Thm. 2.2 that

$$H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)}) = H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}_{\{B\},\varphi(\chi)}).$$

But this is clear, once we realize that all the other quotients $\mathcal{A}^{6-k}_{\{B\},\varphi(\chi)}/\mathcal{A}^{6-k+1}_{\{B\},\varphi(\chi)}$ $k \geq 3$, will only have non-trivial $(\mathfrak{sl}_6(\mathbb{R}), SO(6))$-cohomology in degrees $q \geq 9$ by inserting into (3.2). Therefore, in summary, as Hecke-modules

$$H^8(SL_6(\mathbb{Z}), \mathbb{C}) \cong H^8(\mathfrak{sl}_6(\mathbb{R}), SO(6), \mathcal{A}^4_{\{B\},\varphi(\chi)})^{K_f}.$$

To conclude, the same argument as in the proof of Thm. 4.10 now shows that a non-trivial class in $H^8(SL_6(\mathbb{Z}), \mathbb{C})$ is necessarily represented by a linear combination of main values of degenerate Eisenstein series $E_{P_{\underline{n}}}(f^\circ, \lambda)$, constructed from the constant function f°, viewed as an element of the space $\mathcal{W}_{P_{\underline{n}}, \mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ associated to the trivial representation $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ of the Levi factor of $P_{\underline{n}}$, evaluated at $\lambda = \rho_{P_{\underline{n}}}$, where $\underline{n} \in \{(4, 2), (2, 4)\}$.

Remark 5.1. Shortly after our "automorphic explanation" of the existence of the non-trivial class in $H^8(SL_6(\mathbb{Z}),\mathbb{C})$ was communicated to the public, Ash-Miller-Patzt could also describe it, following a completely different approach, as a certain product of classes coming from $GL_2(\mathbb{Z})$ and $GL_4(\mathbb{Z})$, by putting a Hopf algebra structure on the cohomology of $GL_{2n}(\mathbb{Z})$ with coefficients in the Steinberg module. We refer to [AMP24] and also to Brown-Hu-Panzer, [BHP24], table 1, where the non-trivial class in $H^8(SL_6(\mathbb{Z}),\mathbb{C})$ is written explicitly as Pfaffians.

5.2. **Two non-trivial classes in $H^{15}(SL_8(\mathbb{Z}))$ and a question of A. Ash.** As communicated to the second named author by Brown, A. Ash has asked for a description of the cohomology of $SL_8(\mathbb{Z})$. Among others, degree $q=15$ was of particular interest. Here we show by an automorphic argument that $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$ is two-dimensional, and we describe, which automorphic forms of $SL_8(\mathbb{A})$ represent the non-trivial classes in $H^{15}(SL_8(\mathbb{Z}),\mathbb{C})$.

We put $n=8$ in (5.2). By the analogous arguments as presented in §5.1 above, i.e., by recalling Lem. 3.2 and using the long exact sequence in cohomology, that stems from Franke's filtration, we obtain an isomorphism of $SL_8(\mathbb{A}_f)$-modules

$$H^{15}(\mathfrak{sl}_8(\mathbb{R}),SO(8),\mathcal{A}^6_{\{B\},\varphi(\chi)})\cong \mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(3,5)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(3,5)}},H_{P_{(3,5)}}(\cdot)_f\rangle}\right)\oplus\mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(5,3)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(5,3)}},H_{P_{(5,3)}}(\cdot)_f\rangle}\right).$$

Once more we use Lem. 3.2 and (3.2) and deduce that

$$H^{15}(\mathfrak{sl}_8(\mathbb{R}),SO(8),\mathcal{A}^6_{\{B\},\varphi(\chi)})\cong H^{15}(\mathfrak{sl}_8(\mathbb{R}),SO(8),\mathcal{A}_{\{B\},\varphi(\chi)}).$$

Hence, invoking Thm. 2.2 and the fact that the induced representation $\mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(3,5)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(3,5)}},H_{P_{(3,5)}}(\cdot)_f\rangle}\right)$ as well as $\mathrm{Ind}^{SL_8(\mathbb{A}_f)}_{P_{(5,3)}(\mathbb{A}_f)}\left(e^{\langle\rho_{P_{(5,3)}},H_{P_{(5,3)}}(\cdot)_f\rangle}\right)$ contain $\mathbf{1}_{SL_8(\mathbb{A}_f)}$ with multiplicity one as a quotient, it follows that

$$H^{15}(SL_8(\mathbb{Z}),\mathbb{C})\cong H^{15}(\mathfrak{sl}_8(\mathbb{R}),SO(8),\mathcal{A}^6_{\{B\},\varphi(\chi)})^{K_f}\cong\mathbb{C}^2$$

as modules under the Hecke algebra attached to $K_f=SL_8(\hat{\mathbb{Z}})$. Following the argument of the proof of Thm. 4.10, the cohomology classes in this case are represented by the main values of degenerate Eisenstein series $E_{P_{\underline{n}}}(f^\circ,\lambda)$, constructed from the constant function f°, viewed as an element of the space $\mathcal{W}_{P_{\underline{n}},\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}}$ associated to the trivial representation $\mathbf{1}_{L_{\underline{n}}(\mathbb{A})}$ of the Levi factor of $P_{\underline{n}}$, evaluated at $\lambda=\rho_{P_{\underline{n}}}$, where $\underline{n}\in\{(5,3),(3,5)\}$.

5.3. **A final remark on $H^{m^2-1}(SL_{2m}(\mathbb{Z}),\mathbb{C})$.** Let now $n=2m$ be an arbitrary positive even number. It is well-known (cf. [Spe83b]) that every cohomological irreducible cuspidal automorphic representation of $SL_{2m}(\mathbb{A})$ is tempered at infinity (as it is obtained by restriction from a cuspidal automorphic – and hence globally generic, cf. [Sha74] – representation of $GL_{2m}(\mathbb{A})$), and therefore, by [Bor-Wal00], Prop. I.5.3 the lowest degree in which it may have non-zero cohomology is given by $q=m^2$.

Our Thm. 4.10 (for $n=4$) together with our considerations of §5.1 and §5.2 above, may therefore be viewed as a first description of the non-constant automorphic functions, which represent a non-trivial cohomology class of $SL_{2m}(\mathbb{Z})$ "right below" the cuspidal range for $m=2,3,4$.

It is a very recent result of Ash-Miller-Patzt (see Thm. B in [AMP24] and apply Borel-Serre duality) and also of Brown-Chan-Galatius-Payne's (see Cor. 1.10 in [BCGP24]) that the phenomenon of non-vanishing of $H^{m^2-1}(SL_{2m}(\mathbb{Z}),\mathbb{C})$ persists for all $m \geq 5$. It would hence be worthwhile to give a structural description of the automorphic representatives for these classes as well.

However, in higher rank, the problem gets more and more complicated. The possible contributions to cohomology in degree m^2-1 of the quotients of Franke's filtration associated to parabolic subgroups of lower rank cannot be excluded by a simple argument based on the length of the Kostant representative. In the cases of $m = 2, 3$ there were no such contributions, and in the case of $m = 4$, the only possible contributions arise from the associate class of the parabolic subgroup $P_{(1,1,6)}$, but it cannot contribute to degree $q = m^2-1 = 15$ by the Poincaré polynomial, cf. Lem. 3.2. As m grows, the rank of parabolic subgroups associated to the quotients of Franke's filtration that may contribute to the cohomology in the considered degree can be bounded, but the bound is slightly larger than $m/2$, which gives quite a lot of possibilities, and Lem. 3.2 cannot exclude all of them. Therefore, although the problem is a natural generalization of our results, it seems that it is still out of reach.

References

Ash24. A. Ash, *On the cohomology of $SL_n(\mathbb{Z})$*, preprint (2024) arXiv:2402.08840

AMP24. A. Ash, J. Miller, P. Patzt, *Hopf algebras, Steinberg modules and the unstable cohomology of $SL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z})$*, preprint (2024) arXiv:2404.13776

Ber-Lap23. J. Bernstein, E. Lapid, On the meromorphic continuation of Eisenstein series, to appear in *J. Amer. Math. Soc.* (2023) DOI: https://doi.org/10.1090/J. Amer. Math. Soc./1020

Bor74. A. Borel, Stable real cohomology of arithmetic groups, *Ann. Sci. de l'ENS* **7** (1974), 235–272

Bor-Jac79. A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979) 189-202

BLS96. A. Borel, J.-P. Labesse, J. Schwermer, On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields, *Comp. Math.*, **102** (1996), pp. 1-40

Bor-Wal00. A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies **94**, (Princeton Univ. Press, New Jersey, 2000)

Bro23. F. Brown, *Bordifications of the moduli spaces of tropical curves and abelian varieties, and unstable cohomology of $GL_g(\mathbb{Z})$ and $SL_g(\mathbb{Z})$*, preprint (2023) arXiv:2309.12753

BCGP24. F. Brown, M. Chan, S. Galatius, S. Payne, *Hopf algebras in the cohomology of $\mathcal{A}_g$, $GL_n(\mathbb{Z})$, and $SL_n(\mathbb{Z})$*, preprint (2024) arXiv:2405.11528

BHP24. F. Brown, S. Hu, E. Panzer, *Unstable cohomology of $GL_{2n}(\mathbb{Z})$ and the odd commutative graph complex*, preprint (2024) arXiv:2406.12734

Che-Lan19. G. Chenevier, J. Lannes, *Automorphic Forms and Even Unimodular Lattices – Kneser Neighbors of Niemeier Lattices*, Ergebnisse der Mathematik und ihrer Grenzgebiete, (Springer, 2019)

Che-Tai20. G. Chenevier, O. Taibï, Discrete series multiplicities for classical groups over $\mathbb{Z}$ and level 1 algebraic cusp forms, *Publ. Math. IHES* **131** (2020) 261-323

CFP14. T. Church, B. Farb, A. Putman, A stability conjecture for the unstable cohomology of $SL_n(\mathbb{Z})$, mapping class groups, and $\mathrm{Aut}(\mathrm{F}_n)$, in: "Algebraic Topology: Applications and New Directions", *Contemp. Math.* **620** (2014) 55–70

Chu-Put17. T. Church, A. Putman, The codimension-one cohomology of $SL_n(\mathbb{Z})$, *Geom. Topol.* **21** (2017) 999–1032

EVGS13. P. Elbaz-Vincent, H. Gangl, C. Soulé, Perfect forms, K-theory and the cohomology of modular groups, *Adv. Math.* **245** (2013) 587?624

Fra98. J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. de l'ENS* **31** (1998) 181–279

Fra08. J. Franke, *A topological model for some summand of the Eisenstein cohomology of congruence subgroups*, in: *Eisenstein Series and Applications*, W.T. Gan, S.S. Kudla, Y. Tschinkel eds., *Progr. Math.* **258**, Birkhäuser Boston, Boston, 2008, pp. 27–85

Fra-Schw98. J. Franke, J. Schwermer, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.* **311** (1998), pp. 765-790

Grb12. N. Grbac, The Franke filtration of the spaces of automorphic forms supported in a maximal proper parabolic subgroup, *Glas. Mat. Ser. III* **47(67)** (2012), no. 2, 351-372

Grb23. N. Grbac, The Franke filtration of the spaces of automorphic forms on the symplectic group of rank two, *Mem. Amer. Math. Soc.*, to appear

Grb-Gro13. N. Grbac, H. Grobner, The residual Eisenstein cohomology of Sp_4 over a totally real number field, *Trans. Amer. Math. Soc.* **365** (2013), no. 10, 5199-5235

Grb-Gro24. N. Grbac, H. Grobner, Some unexpected phenomena in Franke's filtration of the space of automorphic forms of the general linear group, *Israel J. Math.* **263** (2024) 301–347

GHV76. W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature, and Cohomology. Volume III: Cohomology of principal bundles and homogeneous spaces*, Pure and Applied Math., Vol. **47**, Academic Press (1976)

Gro13. H. Grobner, Residues of Eisenstein series and the automorphic cohomology of reductive groups, *Compos. Math.* **149** (2013) 1061–1090

Gro23. H. Grobner, *Smooth-automorphic forms and smooth-automorphic representations*, *Series on Number Theory and Its Applications* **17** (WorldScientific, 2023), 264 pages, https://doi.org/10.1142/12523

Gro-Lin21. H. Grobner, J. Lin, Special values of L-functions and the refined Gan-Gross-Prasad conjecture, *Amer. J. Math.* **143** (2021) 859–937

Har87. G. Harder, Eisenstein cohomology of arithmetic groups. The case GL_2, *Invent. Math.* **89** (1987) 37–118

Har90. G. Harder, *Some results on the Eisenstein cohomology of arithmetic subgroups of GL_n*, in: *Cohomology of Arithmetic Groups and Automorphic Forms*, eds. J.-P. Labesse, J. Schwermer, Lecture Notes in Math. **1447** (Springer, 1990) 85–153

KMP21. A. Kupers, J. Miller, P. Patzt, Improved homological stability for certain general linear groups, *Proc. London Math. Soc.* **125** (2021) 511–542

Lan76. R. P. Langlands, *On the Functional Equations Satisified by Eisenstein Series*, Lect. Notes Math. **544**, Springer (1976)

Lee-Szc78. R. Lee, R. H. Szczarba, On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$, *Duke Math. J.* **45** (1978) 101–129

Mœ-Wal95. C. Mœglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Univ. Press (1995)

Pla-Rap94. V. P. Platonov, A. Rapinchuck, *Algebraic groups and number theory*, Pure Appl. Math. **139**, Academic Press (1994)

PSS20. A. Putman, S. Sam, A. Snowden, Stability in the homology of unipotent groups, *Alg. Number Th.* **14** (2020) 119–154.

Ser79. J.-P. Serre, *Arithmetic Groups*, in: *Homological Group Theory*, *London Math. Soc. Lect. Notes Series* **36**, ed. C.T.C. Wall (1979), pp. 77–159

Sha74. J. A. Shalika, The multiplicity one theorem for GL_n, *Ann. Math.* **100** (1974) 171–193

Spe83a. B. Speh, *A Note on Invariant Forms on Locally Symmetric Spaces*, in: Representation Theory of Reductive Groups - Proceedings of the University of Utah Conference 1982, (1983) 197 – 206

Spe83b. B. Speh, Unitary Representations of $Gl(n,\mathbb{R})$ with non-trivial $(\mathfrak{g},K)$-cohomology, *Invent. Math.* **71** (1983) 443 – 465

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