数理解析研究所講究録
第1934巻 2015年 26-39

Analytic properties of automorphic L-functions and Arthur classification
保型L関数の解析的性質とArthur分類

Neven Grbac*
Department of Mathematics, University of Rijeka

Abstract

Langlands spectral theory describes the residual spectrum of a reductive group in terms of intertwining operators and analytic properties of automorphic L-functions. On the other hand, according to Arthur's endoscopic classification of automorphic representations of classical groups (due to Mok for quasi-split unitary groups), an Arthur parameter is attached to every residual representation. Comparing the two approaches yields information on the analytic properties of automorphic L-functions. In this expository paper we explain how we applied this general idea to prove the holomorphy in the critical strip of the complete symmetric and exterior square, and Asai L-functions.

Langlandsのスペクトル理論は、簡約群の留数スペクトルを絡作用素と保型L関数の解析性質により記述するものである。一方、Arthurによる古典群(およびMokによる準分裂ユニタリ群)の保型表現の内視論的分類によれば、全ての留数的表現に対してArthurパラメータが付随している。この2つのアプローチを比較することで保型L関数の解析的性質についての情報が得られる。本解説記事においては、この発想が、完備化された対称積L関数、2次外積L関数、浅井L関数の臨界帯領域における正則性の証明にとのように適用されるかを説明する。

1 Introduction

Besides the standard L-function, the three important automorphic L-functions attached to cuspidal automorphic representations of the general linear group are the symmetric square, exterior square and Asai L-function [24]. These L-functions have been extensively studied and many properties are known. In the paper [12] and the recent joint work with Shahidi [13], we proved the holomorphy of these L-functions in the critical strip, which was previously not known.

The purpose of this expository paper is to point out clearly the underlying general idea applied in [12, 13]. It does not contain any new results, we avoid many technical

*This work has been supported in part by Croatian Science Foundation under the project 9364 and by University of Rijeka research grant 13.14.1.2.02.

issues and give almost no proofs. We refer the interested reader to the original papers [12, 13] for more details.

The general idea in [12, 13] was to compare the endoscopic classification of Arthur [3], which is due to Mok [20] for quasi-split unitary groups, to the spectral theory of Langlands [18] (see also [19]). More precisely, according to endoscopic classification, every automorphic representation in the discrete spectrum of a classical group over a number field should have a square-integrable Arthur parameter. On the other hand, we can construct residual representations of a classical group using Langlands spectral theory, starting with a cuspidal automorphic representations of Levi factors. Since the construction in some cases depends on some unknown properties of these cuspidal automorphic representations, comparison with the possible Arthur parameters may lead to some new insight.

The paper is organized as follows. After this Introduction, we present in Sect. 2 the general idea of comparing the endoscopic classification and Langlands spectral decomposition. In Sect. 3 we introduce the automorphic L-functions considered in [12, 13], review their properties, and mention the main results of [12, 13]. Finally, in Sect. 4, we outline very roughly the proof for symmetric square L-functions by applying the general idea in the case of split odd special orthogonal group.

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This paper is an outgrowth of the talk under the same title given at the workshop *Automorphic Forms and Related Zeta Functions* held in January 2014 at the Research Institute for Mathematical Sciences (RIMS), Kyoto, Japan. We would like to thank the organizers Taku Ishii and Hiro-aki Narita for the invitation to give the talk. We are very grateful to Takayuki Oda for his invitation to visit Tokyo University, during which we also attended the workshop in Kyoto.

2 General idea

As already mentioned in the Introduction, we would like to use the work of Arthur [3, 2], that is, his endoscopic classification, to extract some extra information about automorphic representations. In particular, the holomorphy of certain automorphic L-functions attached to them. In this section we explain the general idea which is not new. It was already used in [12] and [13] to extract such information.

2.1 Automorphic forms

We begin with some notation. Let k be a number field with the ring of adèles $\mathbb{A}$. Let G be an algebraic group over k. Although at this point we could work with a reductive group, we assume for simplicity that G is *semi-simple*, thus avoiding the discussion of central characters. Later on there will be some furhter assumptions on G.

We consider the space of L^2 automorphic forms on $G(\mathbb{A})$ defined as

$$L^2(G) = L^2\left(G(k)\backslash G(\mathbb{A})\right) = \left\{ \begin{array}{l} \text{classes of measurable functions } f \text{ on } G(\mathbb{A}) \text{ such that} \\ f(\gamma g) = f(g) \text{ for } \gamma \in G(k) \text{ and } g \in G(\mathbb{A}), \text{ and} \\ \int_{G(k)\backslash G(A)} |f|^2 < \infty \end{array} \right\}.$$

This is a Hilbert space, and the action of the group $G(\mathbb{A})$ on $L^2(G)$ by right translation is a unitary representation. The goal of spectral decomposition for G is to decompose $L^2(G)$ with respect to that action.

As a first step in the $G(\mathbb{A})$-invariant decomposition, we have

$$L^2(G) = L^2_{\text{disc}}(G) \bigoplus L^2_{\text{cont}}(G),$$

where $L^2_{\text{disc}}(G)$ is the *discrete spectrum* of G spanned by all closed irreducible $G(\mathbb{A})$-invariant subspaces of $L^2(G)$, and $L^2_{\text{cont}}(G)$ is the *continuous spectrum* of G obtained as the orthogonal complement of the discrete spectrum. Classifying automorphic representations that appear in the discrete spectrum is the goal of Arthur's endoscopic classification. The continuous spectrum can be described in terms of direct integrals of Eisenstein series, which was already established by Langlands [18].

2.2 Arthur's endoscopic classification

Let now G be a *classical group* defined over a number field k. The endoscopic classification of automorphic representations should provide a parametrization of representations appearing as summands in $G(\mathbb{A})$-invariant decomposition of the discrete spectrum of G.

However, at the time of writing this paper, endoscopic classification still depends on the stabilization of the twisted trace formula for the general linear group. This problem is considered by Waldspurger [30, 31, 32], but not yet resolved.

We summarize the state-of-the-art of endoscopic classification, up to the stabilization mentioned above, as follows.

- For G a split symplectic or special orthogonal group, the classification is proved in Arthur's book [3].
- For G an inner form of split symplectic or special orthogonal group, the classification is stated without proof in the last section of Arthur's book [3].
- For G a quasi-split unitary group (given by a quadratic extension K/k), the classification is recently proved by Mok [20].

In what follows, we give a very rough description of the classification, sufficient only to express the general idea of our work. For a classical group G, a set $\Psi_2(G)$ of so-called square-integrable parameters is defined. A parameter $\psi \in \Psi_2(G)$ may be viewed as an isobaric sum of automorphic representations in $L^2_{\text{disc}}(GL_m)$ for some integers m. So in a sense, the endoscopic classification for a classical group G is given in terms of GL_m.

An Arthur packet Π_ψ is associated to every parameter $\psi \in \Psi_2(G)$. It is a set of nearly equivalent representations of $G(\mathbb{A})$ with certain properties. Packets associated to different parameters are disjoint. The endoscopic classification amounts to the decomposition

$$L^2_{\text{disc}}(G) \cong \bigoplus_{\psi\in\Psi_2(G)} \bigoplus_{\pi\in\Pi_\psi} m_{\text{disc}}(\pi)\, \pi,$$

where $m_{\text{disc}}(\pi) \geq 0$ is the multiplicity of π in $L^2_{\text{disc}}(G)$ given in terms of certain objects attached to ψ.

Observe that we use here the definition of Π_ψ without the condition that the multiplicity $m_{\text{disc}}(\pi)$ of $\pi \in \Pi_\psi$ is non-zero. In other words, our definition allows that Π_ψ contains representations π that do not appear in $L^2_{\text{disc}}(G)$. However, this is sufficient for our purposes.

2.3 Langlands spectral decomposition

We assume again that G is *semi-simple*, although the theory works for a reductive group as well. The Langlands spectral theory [18] is another way to construct representations in the non-cuspidal part of the discrete spectrum of G.

More precisely, let

$$L^2_{\text{cusp}}(G) = \left\{ f \in L^2(G) \; : \; \int_{N(k)\backslash N(\mathbb{A})} f(ng)dn = 0, \; \begin{array}{l} \text{for almost all } g \in G(\mathbb{A}), \text{ and} \\ \text{for unipotent radicals } N \text{ of all} \\ \text{proper parabolic subgroups of } G \end{array} \right\}$$

be the *cuspidal spectrum* of G. It is a closed $G(\mathbb{A})$-invariant subspace of $L^2(G)$. By a fundamental result of Gelfand, Graev, Piatetski-Shapiro [10] and Langlands [18] (see also [11]), it decomposes into a Hilbert space direct sum of irreducible invariant subspaces with finite multiplicities. In particular, $L^2_{\text{cusp}}(G)$ is a subspace of the discrete spectrum.

The *residual spectrum* $L^2_{\text{res}}(G)$ is the orthogonal complement of the cuspidal spectrum $L^2_{\text{cusp}}(G)$ inside the discrete spectrum $L^2_{\text{disc}}(G)$. Thus, we have a decomposition

$$L^2_{\text{disc}}(G) = L^2_{\text{cusp}}(G) \bigoplus L^2_{\text{res}}(G).$$

The Langlands spectral decomposition is a decomposition of the residual spectrum along the cuspidal support. Besides the original book of Langlands [18], main references are [19] and [9].

Let P be a standard proper parabolic subgroup of G, and $\{P\}$ its associate class.[1] Let σ be a cuspidal automorphic representation of the Levi factor $M_P(\mathbb{A})$, normalized in such a way that the poles of Eisenstein series and L-functions attached to σ are real.[2] We denote by $\{\sigma\}$ the associate class of σ.[3] Given the associate classes $\{P\}$ and $\{\sigma\}$, let $L^2_{\text{res},\{P\},\{\sigma\}}(G)$ be the (possibly trivial) space spanned by square-integrable iterated residues of Eisenstein series on $G(\mathbb{A})$ constructed from σ. The residual spectrum decomposes into

$$L^2_{\text{res}}(G) = \bigoplus_{\{P\}} \bigoplus_{\{\sigma\}} L^2_{\text{res},\{P\},\{\sigma\}}(G).$$

Hence, a way to construct automorphic representations in the residual spectrum, is to study the poles of Eisenstein series.

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[1]Here we implicitly assume that a minimal parabolic subgroup of G defined over k has been fixed. By definition, a parabolic subgroup is *standard*, if it contains a fixed minimal parabolic subgroup. Parabolic subgroup Q is *associate* to P if their Levi factors are $G(k)$-conjugate.

[2]More precisely, we always assume that σ is trivial on the connected component of the archimedean part of the maximal split torus in the center of M_P. This is not restrictive, because it can be obtained by twisting σ with a unitary character of $M_P(\mathbb{A})$. Hence, it is just a convenient normalization, which makes the poles of Eisenstein series and L-functions attached to σ real.

[3]Associate class of σ is a family of (non-empty) finite sets φ_Q, indexed by $Q \in \{P\}$, consisting of cuspidal automorphic representations of the Levi factor $M_Q(\mathbb{A})$ obtained as conjugates of σ by elements of $G(k)$.

For simplicity of exposition, we do not define the Eisenstein series here. We also assume that P is self-associate.[4] We simply let $E(s,g)$ be an Eisenstein series on $G(\mathbb{A})$ constructed using a section of representations parabolically induced from σ tensored by a character of $M_P(\mathbb{A})$ which depends on the complex parameter $s \in \mathbb{C}^r$.[5] The general theory of Eisenstein series implies that the poles of the Eisenstein series $E(s,g)$ coincide with the poles of its constant term along P, which is defined as

$$E(s,g)_P = \int_{N_P(k)\backslash N_P(\mathbb{A})} E(s,ng)dn,$$

where N_P is the unipotent radical of P. On the other hand, the constant term can be written as

$$E(s,g)_P = \sum_{\text{finite}} \text{standard intertwining operators},$$

where standard intertwining operators are certain integral operators acting on the section of induced representations from which $E(s,g)$ is constructed. They are defined as the analytic continuation from the domain of convergence of certain integrals (see [25, 19] for more details). The study of poles of the Eisenstein series reduces to that of finite sums of standard intertwining operators.

At this point we assume that G is *quasi-split* and σ *globally generic* (with respect to some fixed non-trivial additive character of $k\backslash\mathbb{A}$), so that we may apply the Langlands-Shahidi method [25, 24]. In that case, we still have some serious difficulties to overcome.

1. The poles of standard intertwining operators can be determined, in principle, from poles (and zeros) of certain automorphic L-functions. But these could be unknown.

2. Even when the poles of individual standard intertwining operators are known, taking the residue of a finite sum could lead to cancelations. Understanding the cancelations involves demanding combinatorial considerations.

3. Finally, the residues are isomorphic to images of some intertwining operators. In particular, they are isomorphic to a constituent of some induced representation.

2.4 The strategy

The very rough and incomplete description of the endoscopic classification and the Langlands spectral theory, given in previous sections, is sufficient to explain our strategy. Now G is a *quasi-split classical* group so that we may apply both approaches to the discrete spectrum.

By the Langlands spectral theory, we can use Eisenstein series to construct an automorphic representation π in the discrete spectrum $L^2_{\text{disc}}(G)$. The construction will depend

[4]A standard parabolic subgroup is *self-associate* if it is the only standard parabolic in its associate class.

[5]Let f_s be a "good" section of the induced representation $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes \chi_s)$, where χ_s is a character of $M_P(\mathbb{A})$ that depends on the complex parameter $s \in \mathbb{C}^r$. The Eisenstein series $E(f_s,g)$ is defined as the analytic continuation from the domain of convergence of the series $E(f_s,g) = \sum_{\gamma\in P(k)\backslash G(k)} f_s(\gamma g)$, where $g \in G(\mathbb{A})$. For simplicity, we write $E(s,g)$ for such Eisenstein series constructed from some section f_s. See [19, 9] for more details.

on some properties of the cuspidal automorphic representation σ of a Levi factor in G from which the Eisenstein series is constructed. For example, certain automorphic L-function attached to σ should have a pole.

Having constructed π in $L^2_{\text{disc}}(G)$, the endoscopic classification implies that there should be a square-integrable parameter $\psi \in \Psi_2(G)$, such that π belongs to the packet Π_ψ. If we can prove that such ψ does not exist, then π does not appear in $L^2_{\text{disc}}(G)$. Hence, the properties of σ on which the construction of π depends are not satisfied. In the example mentioned above, this means that the L-function in question does not have a pole.

We applied this simple strategy in [12] and [13] to prove holomorphy of certain automorphic L-functions attached to σ for some values of the complex parameter.

3 Automorphic L-functions

In this section we introduce the automorphic L-function considered in this paper, and review their analytic properties. We state the main results of [12] and [13]. It is the holomorphy of these L-functions in the critical strip, which was not known before using other methods.

3.1 Definition

Let k and $\mathbb{A}$ be a number field and its ring of adèles, as before. For a place v of k, we denote by k_v the completion of k at v. Let σ be a cuspidal automorphic representation of $GL_n(\mathbb{A})$. Let v be a non-archimedean place of k such that the local component σ_v of σ is an unramified representation of $GL_n(k_v)$. We denote by $c(\sigma_v)$ its Satake parameter [22, 8] in $GL_n(\mathbb{C})$. Let r be a finite-dimensional algebraic representation of $GL_n(\mathbb{C})$, which is the dual group of GL_n. To these data, Langlands attached in [17] the local unramified L-function

$$L_v(s,\sigma,r) = L(s,\sigma_v,r) = \det\left(I - r(c(\sigma_v))q_v^{-s}\right)^{-1},$$

where I is the identity matrix of appropriate size, and q_v the cardinality of the residue field of k_v.

In this paper we are interested in three automorphic L-functions. Two of them are the symmetric and exterior square L-functions of σ, attached to the symmetric square $r = Sym^2$ and exterior square $r = \wedge^2$ of the standard representation of $GL_n(\mathbb{C})$, respectively.

The third L-function that we consider in this paper is the so-called Asai L-function, as it generalizes the case considered by Asai in [4]. It is attached to a cuspidal automorphic representation σ of $GL_n(\mathbb{A}_K)$, where K/k is a quadratic extension of number fields, and $\mathbb{A}_K$ is the ring of adèles of K. We are skipping the details here,[6] but for a place v of k

[6]To precisely define the Asai L-function in the setting of Langlands [17], one should view σ as a representation of the group $H(\mathbb{A}_k)$, where $H = Res_{K/k}GL_n$ is the algebraic group over k obtained from GL_n/K by restriction of scalars. For an unramified place v of k, the local L-function is attached to a finite-dimensional representation r_v of the L-group of H viewed as a k_v-group. This is the reason for a distinction between split and inert places in the definition. If v splits in K, and w_1 and w_2 are the two places of K lying over v, the local L-function at v is in fact the local unramified Rankin-Selberg L-function attached to the pair σ_{w_1} and σ_{w_2}. If v is inert in K, and w is the unique place of K lying over v, then the local unramified L-function at v is the Asai L-function.

Table 1: Automorphic L-functions in constant terms of Eisenstein series

r	G
Sym^2	split odd special orthogonal group SO_{2n+1}
$\wedge^2$	split even special orthogonal group SO_{2n} split symplectic group Sp_{2n}
$Asai$	quasi-split unitary group U_n

which is unramified in K and such that the local components of σ at places of K lying over v are unramified, one can define the local Asai L-function at v. The local Asai L-function at v attached to σ is denoted by

$$L_v(s, \sigma, Asai)$$

in this paper.

Let $r = Sym^2, \wedge^2$ or $Asai$. Let S be a finite set of places of k, containing all archimedean places, and such that the local L-function $L_v(s, \sigma, r)$ attached to σ is defined for all $v \notin S$. The *partial L-function* attached to σ and r is defined as

$$L^S(s, \sigma, r) = \prod_{v \notin S} L_v(s, \sigma, r).$$

The product on the right-hand side converges absolutely in some right half-plane.

These partial L-functions appear in the constant term of the Eisenstein series on the appropriate (quasi-)split classical group G. The groups G in question for $r = Sym^2, \wedge^2, Asai$ are listed in Table 1. As G is (quasi-)split and σ is globally generic,[7] Shahidi defined in [24] the local L-functions $L_v(s, \sigma, r)$ at ramified non-archimedean places. At archimedean places $L_v(s, \sigma, r)$ is just the Artin L-function attached to the Langlands parameter of σ_v as in [23]. Hence, we can define the *complete automorphic L-functions* attached to σ and r as

$$L(s, \sigma, r) = \prod_{v} L_v(s, \sigma, v),$$

where the product over all places on the right-hand side converges in some right half-plane.

3.2 Review of analytic properties

The three automorphic L-functions $L(s, \sigma, r)$, defined above, have been extensively studied and many analytic properties are known. We summarize these properties below. In what follows it is convenient to use the notation

$$\sigma^* = \begin{cases} \widetilde{\sigma}, & \text{if } r = Sym^2, \wedge^2, \\ \widetilde{\sigma}^\theta, & \text{if } r = Asai, \end{cases}$$

where $\widetilde{\sigma}$ is the representation contragredient to σ, and θ is the unique non-trivial element in the Galois group of the quadratic extension K/k. We denote by σ^θ the representation conjugate to σ by the Galois automorphism θ.

[7]Every cuspidal automorphic representation of $GL_n(\mathbb{A})$ is globally generic. This follows from the Fourier expansion of cuspidal automorphic forms on $GL_n(\mathbb{A})$, see [21, 26].

1. The defining product of $L^S(s,\sigma,r)$, and thus also $L(s,\sigma,r)$, has the analytic continuation to a meromorphic function in the whole complex plane[8] (cf. [17, 25]).

2. The complete L-functions $L(s,\sigma,r)$ satisfy a functional equation[9] relating the values at s and $1-s$ (cf. [24, 25]).

3. Holomorphy of the complete L-function $L(s,\sigma,r)$ always implies holomorphy for the partial L-function $L^S(s,\sigma,r)$, since local L-functions have no zeros. But the converse is not true, due to ramification and problems at archimedean places.

4. If σ is *not isomorphic* to its (Galois conjugate) contragredient σ^*, then the L-function $L(s,\sigma,r)$, and thus also the partial L-function $L^S(s,\sigma,r)$, is entire.[10]

5. For σ *isomorphic* to σ^*, i.e., σ (Galois conjugate) self-dual, the situation for complete L-functions is presented in Fig. 1.

   (a) For $Re(s) \geq 1$ and $Re(s) \leq 0$ the complete L-function $L(s,\sigma,r)$ is holomorphic and non-zero, except for possible poles at $s=0$ and $s=1$ of order at most one.

   (b) In the critical strip, i.e., for $0 < Re(s) < 1$, the holomorphy of the *partial* L-function $L^S(s,\sigma,r)$ is established using the integral representation approach in [27, 28, 29, 7] for $r = Sym^2$, in [6, 16, 5] for $r = \wedge^2$, and in [1] for $r = Asai$.

   (c) In the critical strip, i.e., for $0 < Re(s) < 1$, the holomorphy of the *complete* L-function $L(s,\sigma,r)$ was *not known*, and the goal of this paper is to explain how the general idea of Sect. 2 is used in [12] and [13] to prove this fact.

3.3 Holomorphy in the critical strip

The holomorphy of the *complete* L-functions $L(s,\sigma,r)$ in the critical strip was the main new result, for which a crucial insight from the endoscopic classification was required, in [12] for $r = Sym^2, \wedge^2$, and in [13] for $r = Asai$. For convenience, we state these results in separate theorems below.

Theorem A (G. (2011), [12]). *Let σ be a cuspidal automorphic representation of $GL_n(\mathbb{A})$. Then the complete symmetric square L-function $L(s,\sigma,Sym^2)$ and the complete exterior square L-function $L(s,\sigma,\wedge^2)$ are holomorphic in the critical strip, i.e., for $0 < Re(s) < 1$.*

Theorem B (G. and Shahidi (preprint, 2014), [13]). *Let σ be a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$, where K/k is a quadratic extension of number fields. Then the complete Asai L-function $L(s,\sigma,Asai)$ is holomorphic in the critical strip, i.e., for $0 < Re(s) < 1$.*

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[8]The proof is essentially the same as for the Riemann ζ-function, once a uniform bound for Satake parameters over unramified places is known. See [25, Appendix 2.5] for details.

[9]The functional equation is $L(s,\sigma,r) = \varepsilon(s,\sigma,r)L(1-s,\sigma^*,r)$, where $\varepsilon(s,\sigma,r)$ is the ε-factor.

[10]This is a consequence of the theory of Eisenstein series. See Remark at the end of [19, Section IV.3.12].

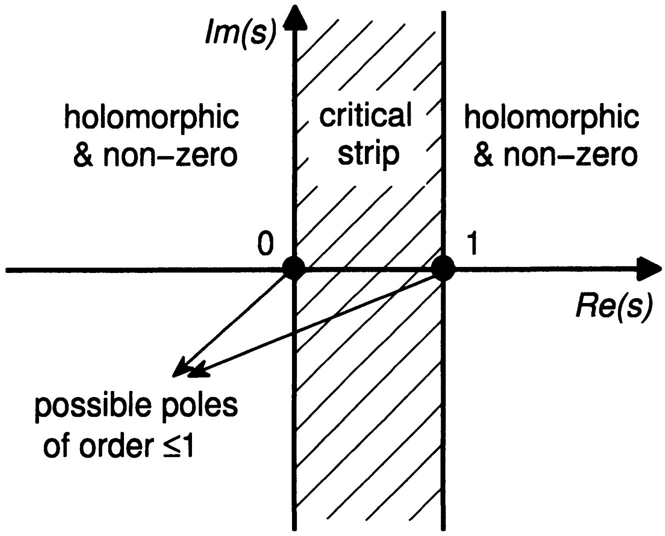

Figure 1: Analytic properties of symmetric square, exterior square and Asai complete automorphic L-functions

4 About the proof

We finish this paper with a few words about the proof of Theorem A and B. The idea is to apply the strategy outlined in Sect. 2.4, that is, compare the Langlands spectral decomposition and Arthur classification for an appropriate quasi-split classical group. For the complete proof, we refer the interested reader to the original papers [12] for Theorem A and [13] for Theorem B, in which these two theorems are proved.

The simplest case for the presentation is the case of symmetric square L-function $L(s, \sigma, Sym^2)$, where σ is a cuspidal automorphic representation of $GL_n(\mathbb{A})$. Other cases are proved in a similar way.

4.1 Step 1

From Table 1, we know that the symmetric square L-function $L(s, \sigma, Sym^2)$ appears in the constant term of the Eisenstein series for $G = SO_{2n+1}$. Hence, let SO_{2n+1} be the split odd special orthogonal group defined over a number field k. Let P be the Siegel maximal proper standard parabolic subgroup[11] of SO_{2n+1}. This is the standard parabolic subgroup with the Levi factor M_P isomorphic to GL_n. Consider σ as a cuspidal automorphic representation of $M_P(\mathbb{A}) \cong GL_n(\mathbb{A})$.

For $s \in \mathbb{C}$, let

$$I(s, \sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{SO_{2n+1}(\mathbb{A})} (\sigma |\det|^s)$$

be the parabolically induced representation, where the induction is normalized, det is the

[11]Here we implicitly assume that a Borel subgroup B of SO_{2n+1} has been fixed, and by definition, P is standard, if it contains B.

determinant on $M_P(\mathbb{A})$ and $|\cdot|$ is the adèlic absolute value. Taking a good[12] section f_s of $I(s,\sigma)$, the Eisenstein series constructed from f_s is defined as the analytic continuation from the domain of convergence of the series

$$E(f_s, g) = \sum_{\gamma \in P(k)\backslash SO_{2n+1}(k)} f_s(\gamma g)$$

for $g \in SO_{2n+1}(\mathbb{A})$.

As explained in Sect. 2.3, the poles of $E(f_s, g)$ are determined by those of the constant term $E(f_s, g)_P$ along P. Note that P is self-associate. The constant term in this case is the sum of two terms

$$E(f_s, g)_P = f_s(g) + \big(\text{std intertw op}\big) f_s(g).$$

But the first term is just the identity operator. Hence, the poles are the same as the poles of a single standard intertwining operator. In particular, there is no problem with cancelations in the sum mentioned in Sect. 2.3.

It turns out that the standard intertwining operator, appearing in the constant term, has the same poles for $Re(s) > 0$ as the L-function $L(2s, \sigma, Sym^2)$. Observe the factor two appearing in the argument. Hence, for $Re(s) > 0$ and the appropriate choice of f_s, the poles of the Eisenstein series $E(f_s, g)$ are the same as the poles of the L-function $L(2s, \sigma, Sym^2)$.

Conclusion. Assume that the L-function $L(z, \sigma, Sym^2)$ has a pole at $z = 2s_0$ such that $0 < 2s_0 < 1$. Then $E(f_s, g)$ would have a pole at $s = s_0$ for appropriate choices of f_s. By the square-integrability criterion of Langlands (see [18, p. 104] or [14, p. 187]), the residues of $E(f_s, g)$ at the pole $s = s_0$ would span an automorphic representation π in the residual spectrum $L^2_{\text{res}}(SO_{2n+1})$. This representation π would be isomorphic to a constituent of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{SO_{2n+1}(\mathbb{A})} \left(\sigma |\det|^{s_0}\right),$$

since it is isomorphic to the image of certain normalized intertwining operator, as mentioned in Sect. 2.3.

4.2 Step 2

In the previous step we constructed the representation π in $L^2_{\text{res}}(SO_{2n+1})$, under the assumption that $L(s, \sigma, Sym^2)$ has a pole in the critical strip. Hence, according to Sect. 2.2, it should belong to the Arthur packet Π_ψ for some square-integrable parameter $\psi \in \Psi_2(SO_{2n+1})$. To see what are the possibilities, we now give an incomplete definition of square-integrable parameters for SO_{2n+1}, which will be sufficient for our purposes.

A square-integrable Arthur parameter $\psi \in \Psi_2(SO_{2n+1})$ is an unordered formal sum of formal tensor products

$$\psi = \big(\mu_1 \boxtimes \nu(n_1)\big) \boxplus \cdots \boxplus \big(\mu_\ell \boxtimes \nu(n_\ell)\big),$$

where

[12]This has a precise meaning, which we do not recall here. See, for example, [19] or [9].

(i) $\mu_i \cong \widetilde{\mu}_i$ is a self-dual cuspidal automorphic representation of $GL_{m_i}(\mathbb{A})$ for some positive integer m_i,

(ii) n_i are positive integers such that $\sum_{i=1}^{\ell} m_i n_i = 2n$,

(iii) $\nu(n_i)$ is the unique irreducible n_i-dimensional algebraic representation of $SL_2(\mathbb{C})$,

(iv) the formal sum is multiplicity free,

(v) certain condition on central characters of μ_i,

(vi) a technical condition given in terms of the twisted endoscopic datum associated to μ_i.

The conditions (iv)-(vi) in this definition are not given precisely, as we will not need them in this paper.

As explained in Sect. 2.2, Arthur packet Π_ψ, associated with $\psi \in \Psi_2(SO_{2n+1})$, is a set of nearly equivalent representations of $SO_{2n+1}(\mathbb{A})$. If $\pi \cong \otimes'_v \pi_v \in \Pi_\psi$, then π_v is unramified at almost all places, and its Satake parameter $c(\pi_v) \in Sp_{2n}(\mathbb{C})$, viewed via inclusion as an element in $GL_{2n}(\mathbb{C})$, is at almost all places determined by the Satake parameter of ψ, which is given as

$$c_v(\psi) = \bigoplus_{i=1}^{\ell} \left[c(\mu_{i,v}) \otimes \operatorname{diag}\left(q_v^{\frac{n_i-1}{2}}, q_v^{\frac{n_i-3}{2}}, \dots, q_v^{-\frac{n_i-1}{2}} \right) \right] \in GL_{2n}(\mathbb{C}),$$

where $c(\mu_{i,v})$ is the Satake parameter of $\mu_{i,v}$, and q_v the cardinality of the residue field of k_v.

Conclusion. Given an automorphic representation $\pi \cong \otimes'_v \pi_v$ of $SO_{2n+1}(\mathbb{A})$ appearing in $L^2_{\text{disc}}(SO_{2n+1})$, the Satake parameters $c_v(\psi)$ of some square-integrable parameter $\psi \in \Psi_2(SO_{2n+1})$ should determine the Satake parameters $c(\pi_v)$ of π_v at almost all places.

4.3 End of proof

Finally, we apply the strategy proposed in Sect. 2.4. In step 1, we constructed an automorphic representation π in the residual spectrum $L^2_{\text{res}}(SO_{2n+1})$, which is a constituent of the induced representation

$$\operatorname{Ind}_{P(\mathbb{A})}^{SO_{2n+1}(\mathbb{A})} (\sigma |\det|^{s_0}),$$

with $0 < 2s_0 < 1$, under the assumption that $L(z, \sigma, Sym^2)$ has a pole in the critical strip at $z = 2s_0$. Hence, we can determine the Satake parameters $c(\pi_v)$ at almost all places.

Since π is in the discrete spectrum, it should have a square-integrable parameter $\psi \in \Psi_2(SO_{2n+1})$, such that the Satake parameters of π_v and ψ match at almost all places. Looking at the possible Satake parameters $c_v(\psi)$ of a square-integrable parameter ψ, and viewing parameters as induced representations of $GL_{2n}(\mathbb{A})$, the strong multiplicity one for general linear groups [15] implies that necessarily $\ell = 1$, $\mu_1 = \sigma$ with $m_1 = n$, and $n_1 = 2$, that is,

$$\psi = \sigma \boxtimes \nu(2),$$

and thus $s_0 = 1/2$. In other words, there is no Arthur parameter for π, because the only possible candidate is such that $2s_0 = 1$, and we assumed $0 < 2s_0 < 1$. Therefore, π can not exist, and our assumption that $L(z, \sigma, Sym^2)$ has a pole in the critical strip was wrong. This proves the holomorphy of $L(s, \sigma, Sym^2)$ in the critical strip.

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Department of Mathematics, University of Rijeka
Radmile Matejčić 2
HR-51000 Rijeka
CROATIA
E-mail address: neven.grbac@math.uniri.hr