# ON RESIDUAL COHOMOLOGY CLASSES ATTACHED TO RELATIVE RANK ONE EISENSTEIN SERIES FOR THE SYMPLECTIC GROUP

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Abstract. The cohomology of an arithmetically defined subgroup of the symplectic $\mathbb{Q}$–group $Sp_n$ is closely related to the theory of automorphic forms. This paper gives a structural account of that part of the cohomology which is generated by residues or derivatives of Eisenstein series of relative rank one. In particular we determine a set of conditions subject to which residues of Eisenstein series may give rise to non-vanishing cohomology classes. A non-vanishing condition on the value at $s = 1/2$ of certain automorphic $L$-functions which naturally appear in the constant terms of the Eisenstein series plays a major role.

## 1. Introduction

The cohomology of an arithmetically defined subgroup $\Gamma$ of the algebraic $\mathbb{Q}$–group $G = Sp_n$ of symplectic transformations on $\mathbb{Q}^{2n}$ with its standard alternating form can be interpreted in terms of the automorphic spectrum of $\Gamma$. With this framework in place, there is a sum decomposition of the cohomology into the cuspidal cohomology (i.e. classes represented by cuspidal automorphic forms) and the so–called Eisenstein cohomology constructed as the span of appropriate residues or derivatives of Eisenstein series. These are attached to cuspidal automorphic forms $\pi$ on the Levi components of proper parabolic $\mathbb{Q}$–subgroups of $G$. The main objective of this paper is to give a structural account of the building blocks of the Eisenstein cohomology which correspond to maximal parabolic $\mathbb{Q}$–subgroups.

Given a class $\{P\}$ of associate maximal parabolic $\mathbb{Q}$–subgroups of $G$ we describe in detail which types (in the sense of [30]) of Eisenstein cohomology classes occur and how their actual construction is related to the analytic properties of certain Euler products (or automorphic L–functions) attached to $\pi$. We exactly determine in which way residues of the Eisenstein series in question may give rise to non–trivial classes in the cohomology of $\Gamma$. The very existence of these residual Eisenstein cohomology classes is subject to a quite restrictive set of arithmetic conditions on the automorphic L–functions involved. In particular, a non–vanishing condition on the value of a certain Euler product $L(s, \pi)$ at $s = 1/2$ plays an important role in this discussion. These $L$–functions naturally appear in the constant terms of the Eisenstein series under consideration. Determining these conditions in an explicit form and viewing them in the cohomological context form the focal points of our investigation.

This work may be viewed as a contribution to the program, initiated by Harder in the case of $GL_2$ over a number field [11], [13], to understand that part of the cohomology of an arithmetic group which is related to the theory of Eisenstein series. The existence of these classes, in particular, their arithmetic nature and close relation to the theory of L–functions are the core issues of these investigations. There are some results for groups $G$ of small $\mathbb{Q}$-rank other than $GL_2$ or very specific types of Eisenstein series [12], [8], [24], [30], [32], [33]. In describing in the case of the symplectic group $Sp_n$ that part in the cohomology which is attached to relative rank one Eisenstein series this work concerns an algebraic group of an arbitrary $\mathbb{Q}$-rank, the case $n = 2$ being already treated in [31], [34].

Second author's work supported in part by FWF Austrian Science Fund, grant number P 21090 - N13.

We describe more precisely the results of this paper. As is the case in the theory of automorphic representations, the relation between the cohomology of arithmetically defined groups and the corresponding theory of automorphic forms for $G$ are best understood in terms of the cohomology $H^*(G,E)$ of $G$ defined as the inductive limit over congruence subgroups of $G(\mathbb{Q})$. The coefficient system $E$ is given by a finite–dimensional representation of $G(\mathbb{C})$ in a complex vector space. This cohomology group has an interpretation in relative Lie algebra cohomology. By [7, Theorem 18], this takes the form

$$H^*(G,E) = H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{A}_E \otimes_\mathbb{C} E),$$

where $\mathcal{A}_E$ denotes the space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ with respect to $(\nu, E)$ as defined in Section 2.

By a decisive result, first proved by Langlands [21], [3], and its refinement [8], the space $\mathcal{A}_E$ of automorphic forms permits a decomposition (as a direct sum of $(\mathfrak{sp}_n, K_\mathbb{R})$–modules) along their cuspidal support. More precisely, let $\mathcal{C}$ denote the set of classes of associate parabolic $\mathbb{Q}$–subgroups of $G$. Then we have

$$\mathcal{A}_E = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} \mathcal{A}_{E,\{P\},\phi},$$

where the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes $\phi = \{\phi_P\}_{P\in\{P\}}$ of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$. The space $\mathcal{A}_{E,\{P\},\phi}$ consists of all functions of uniform moderate growth whose constant term along each $P \in \{P\}$ belongs to the isotypic component attached to $\pi \in \phi_P$. This decomposition induces a direct sum decomposition

$$H^*(G,E) = \bigoplus_{\{P\}\in\mathcal{C}} \bigoplus_{\phi\in\Phi_{E,\{P\}}} H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes_\mathbb{C} E)$$

in cohomology. The summand in this decomposition indexed by $\{G\}$ is called the cuspidal cohomology of $G$ with coefficients in $E$, to be denoted $H^*_{cusp}(G,E)$. Due to the results in [7], the cohomology classes in the summands indexed by $\{P\} \in \mathcal{C}$, $P \neq G$, can be described by suitable derivatives of Eisenstein series or residues of these. These classes span the so called Eisenstein cohomology, to be denoted $H^*_{Eis}(G,E)$.

Our objects of concern are the families of summands $H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{A}_{E,\{P\},\phi} \otimes E)$ with $\{P\}$ an associate class of maximal parabolic $\mathbb{Q}$–subgroups of $G = Sp_n$. Up to conjugacy, a maximal parabolic $\mathbb{Q}$–subgroup has the form $P_r = L_r N_r$, $r = 1, \ldots, n$, with Levi component $L_r \cong GL_r \times Sp_{n-r}$ if $r < n$, and $L_n \cong GL_n$ if $r = n$, and $N_r$ the unipotent radical of $P_r$. Since such a maximal parabolic subgroup $P_r$ is conjugate to its opposite parabolic subgroup $P_r^{\mathrm{opp}}$, the conjugacy class $\mathcal{P}_r$ of $P_r$ is self–opposite, and the associate class $\{P_r\}$ coincides with $\mathcal{P}_r$.

Given an associate class $\{P_r\} \in \mathcal{C}$, with $P_r$ maximal parabolic, $r = 1, \ldots, n$, and $\phi \in \Phi_{E,\{P_r\}}$, the space $\mathcal{A}_{E,\{P_r\},\phi}$ has a two step filtration by the space $\mathcal{L}_{E,\{P_r\},\phi}$ consisting of square integrable automorphic forms in $\mathcal{A}_{E,\{P_r\},\phi}$. For a given automorphic realization $V_\pi$ of an irreducible cuspidal representation $\pi$ of the Levi factor of $P_r$, we introduce a subspace $\mathcal{L}_{E,\{P_r\},\phi,V_\pi}$ of $\mathcal{L}_{E,\{P_r\},\phi}$. For a precise definition of these subspaces see Section 5. Note that in the case $r = n$, due to multiplicity one for $GL_n(\mathbb{A})$, the spaces $\mathcal{L}_{E,\{P_n\},\phi}$ and $\mathcal{L}_{E,\{P_r\},\phi,V_\pi}$ coincide.

We are ready now to state the main results of the paper. They give a set of necessary conditions for non–vanishing of the Eisenstein cohomology classes in $H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{L}_{E,\{P_n\},\phi,V_\pi} \otimes E)$. These conditions are a subtle combination of arithmetic and geometric conditions. The former assure that the Eisenstein series in question has a pole, and the latter are the necessary conditions for the cohomology class so obtained to be non–vanishing. The first of the two theorems below refers to the case $P = P_n$ (cf. Theorem 8.2), and the second one to the case $P = P_r$ with $r < n$ (cf. Theorem 8.5).

**Theorem A.** *Let $E$ be the irreducible representation of $Sp_n(\mathbb{C})$ of highest weight*

$$\Lambda = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n,$$

*where all $\lambda_k$ are integers and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Let $\{P_n\}$ be the associate class of the standard maximal proper parabolic $\mathbb{Q}$–subgroup $P_n$ of $Sp_n$, with the Levi decomposition $P_n = L_n N_n$, where the Levi factor $L_n \cong GL_n$. Let $\phi$ be the associate class of a cuspidal automorphic representation $\tau$ of $L_n(\mathbb{A})$.*

*The cohomology space*

$$H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$$

*is trivial except possibly in the case where the following conditions are satisfied:*

1. *a cuspidal automorphic representation $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau) \neq 0$,*
2. *the $\mathbb{Q}$–rank $n$ of the algebraic group $Sp_n/\mathbb{Q}$ is even,*
3. *the highest weight $\Lambda$ of the irreducible representation $E$ satisfies $\lambda_{2l-1} = \lambda_{2l}$ for all $l = 1, 2, \ldots, n/2$,*
4. *the infinite component $\tau_\infty$ of $\tau$ has the infinitesimal character*

$$\chi_{\tau_\infty} = \sum_{l=1}^{n/2} \Big[ -\big(\mu_l + (n + 3/2 - 2l)\big)e_l + \big(\mu_l + (n + 3/2 - 2l)\big)e_{n+1-l} \Big],$$

*where $\mu_l = \lambda_{2l-1} = \lambda_{2l}$, i.e. $\tau_\infty$ is a tempered representation fully induced from $n/2$ unitary discrete series representations of $GL_2(\mathbb{R})$ having the lowest $O(2)$–types $2\mu_l + 2n - 4l + 4$ for $l = 1, \ldots, n/2$.*

In the case of an associate class $\{P_r\}$, $r \neq n$, the situation is even slightly more complicated. Thus we only consider the case of the trivial coefficient system $E = \mathbb{C}$, since this is the most interesting case in view of the results in [33].

**Theorem B.** *Let $E = \mathbb{C}$ be the trivial representation of $Sp_n(\mathbb{C})$. Let $r < n$, and let $\{P_r\}$ be the associate class of the standard maximal proper parabolic $\mathbb{Q}$–subgroup $P_r$ of $Sp_n$, with the Levi decomposition $P_r = L_r N_r$, where the Levi factor $L_r \cong GL_r \times Sp_{n-r}$. Let $\phi$ be the associate class of a cuspidal automorphic representation $\pi \cong \tau \otimes \sigma$ of $L_r(\mathbb{A})$ such that a fixed realization $V_\pi$ of $\pi$ in the space of cusp forms on $L_r(\mathbb{A})$ is globally $\psi$–generic (with respect to a fixed non–trivial additive character $\psi$ of $\mathbb{A}/\mathbb{Q}$).*

*Let*

$$\chi_{\pi_\infty} = \sum_{l=1}^{\lfloor r/2 \rfloor} (-x_l e_l + x_l e_{r+1-l}) - \sum_{l'=1}^{n-r} y_{l'} e_{r+l'}$$

*be the infinitesimal character of the Archimedean component $\pi_\infty$ of $\pi$, where $\lfloor x \rfloor$ denotes the greatest integer not greater than $x$. Then, the cohomology space*

$$H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{\mathbb{C},\{P_r\},\phi,V_\pi})$$

*is trivial except possibly in the case where one of the following two sets of conditions is satisfied:*

**(A)**
- (a1) *a cuspidal automorphic representation $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau \times \Pi_j) \neq 0$ for all $\Pi_j$ appearing in the global functorial lift of $\sigma$,*
- (a2) *$r$ is even,*
- (a3) *the coefficients $x_l$ of the infinitesimal character $\chi_{\pi_\infty}$ belong to the set*

$$x_l \in \{3/2, 5/2, \ldots, n - 1/2\},$$

*and $|x_{l_1} - x_{l_2}| \neq 0, 1$ for $l_1 \neq l_2$,*

(a4) *the coefficients $y_{l'}$ of the infinitesimal character $\chi_{\pi_\infty}$ are uniquely determined (up to sign) by the coefficients $x_l$ through the formula*

$$y_{l'} = n + 1 - k_{l'},$$

*for $l' = 1, \dots, n - r$, where*

$$k_{l'} \in S_n \setminus \{n - x_l + 1/2, n - x_l + 3/2 : l = 1, \dots, r/2\}.$$

**(B)** (b1) *a cuspidal automorphic representation $\tau$ is isomorphic to one of $\Pi_j$ appearing in the global functorial lift of $\sigma$ (this implies that $\tau$ is selfdual, and $r \leq \frac{2n+1}{3}$),*

(b2) *the coefficients $x_l$ of the infinitesimal character $\chi_{\pi_\infty}$ belong to the set*

$$x_l \in \begin{cases} \{2, 3, \dots, n\}, & \text{if } r \text{ is even,} \\ \{3, 4, \dots, n\}, & \text{if } r \text{ is odd,} \end{cases}$$

*and $|x_{l_1} - x_{l_2}| \neq 0, 2$ for $l_1 \neq l_2$,*

(b3) *the coefficients $y_{l'}$ of the infinitesimal character $\chi_{\pi_\infty}$ are uniquely determined (up to sign) by the coefficients $x_l$ through the formula*

$$y_{l'} = n + 1 - k_{l'}$$

*for $l' = 1, \dots, n - r$, where*

$$k_{l'} \in \begin{cases} S_n \setminus \{n - x_l, n - x_l + 2 : l = 1, \dots, r/2\}, & \text{if } r \text{ is even,} \\ S_n \setminus \{n, n - x_l, n - x_l + 2 : l = 1, \dots, \lfloor r/2 \rfloor\}, & \text{if } r \text{ is odd.} \end{cases}$$

Let us say a few words about the techniques applied in the proof of these theorems. The arithmetic conditions which provide a pole of the Eisenstein series are (1) in Theorem A, and (a1) and (b1) in Theorem B. Determination of the poles of the Eisenstein series relies on the Langlands spectral theory [20], [28]. The arithmetic conditions on the automorphic L–functions attached to $\pi$ are obtained by passing to the constant term of the Eisenstein series, and applying the Langlands–Shahidi method for the normalization of intertwining operators in terms of L–functions [37] (see also [5, Section 11]). This method assumes that $\pi$ is globally $\psi$–generic, and this is the only reason for such assumption in Theorem B.

However, the Langlands decomposition depends on the analytic properties of the L–functions involved. Those properties are not known for all the L–functions, but the geometric conditions reduce the consideration to the region where they are known. This seems to be a consequence of the subtle interplay between arithmetic and geometry.

The geometric conditions follow from the non–vanishing conditions for the Eisenstein cohomology space $H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{A}_{E,\{P_r\},\phi} \otimes_\mathbb{C} E)$. They give the remaining conditions in the theorems, and as already mentioned reduce the study of the poles of the Eisenstein series to the region where the poles of the automorphic L–functions in question are known. The geometric conditions are derived from the study of the way in which the Eisenstein series may give rise to a non–trivial cohomology class [24, Section 3], [33, Remark 4.12]. These are reduced to certain abstract equations involving the Weyl group and the root system. In unfolding the conditions the main tool is a variant of [40, Lemma 4.3] giving explicitly the action of the Weyl group.

In the case of $P = P_n$, we also obtain in Theorem 8.3 that the cohomology classes coming from $\mathcal{L}_{E,\{P_n\},\phi}$ are separated from the ones coming from $\mathcal{A}_{E,\{P_n\},\phi}/\mathcal{L}_{E,\{P_n\},\phi}$ by the degree in which they occur. More precisely, the residual cohomology classes (i.e. those coming from $\mathcal{L}_{E,\{P_n\},\phi}$) occur only in degrees strictly lower than half the dimension of the space $X_{Sp_n(\mathbb{R})} = Sp_n(\mathbb{R})/K_\mathbb{R}$, while the non-residual Eisenstein cohomology classes only occur in higher degrees.

Finally, let us describe the organization of the paper. In Section 2 the spaces of automorphic forms required in the sequel are introduced. In Section 3, we define the automorphic cohomology, and recall its decomposition along the cuspidal support. In Section 4 we turn our attention to the

symplectic group by reviewing its structure in order to fix the notation. Section 5 deals with the analytic properties of the Eisenstein series in question, and describes them in terms of the analytic properties of certain automorphic L–functions. Section 6 recalls the construction of Eisenstein cohomology classes and gives the non–vanishing conditions for those classes. Section 7 makes the non–vanishing conditions explicit for the case of maximal parabolic subgroups of the symplectic group. Finally, in Section 8 we combine the results of previous sections to obtain the main results of the paper, namely the theorems discussed above.

## Notation

(1) Let $\mathbb{Q}$ be the field of rational numbers. We denote by $V$ the set of places of $\mathbb{Q}$, and by $V_f$ the set of finite places. The Archimedean place is denoted by $v = \infty$. Let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ at $v$, and $\mathbb{Z}_v$ the ring of integers of $\mathbb{Q}_v$ for $v \in V_f$. Let $\mathbb{A}$ (resp. $\mathbb{I}$) be the ring of adeles (resp. the group of ideles) of $\mathbb{Q}$. We denote by $\mathbb{A}_f$ the finite adeles.

(2) Let $G$ be a connected reductive algebraic group defined over $\mathbb{Q}$. Suppose that a minimal parabolic $\mathbb{Q}$–subgroup $P_0$ of $G$ and a Levi decomposition $P_0 = L_0N_0$ of $P_0$ over $\mathbb{Q}$ have been fixed. By definition, a standard parabolic $\mathbb{Q}$–subgroup of $G$ is a parabolic $\mathbb{Q}$–subgroup $P$ of $G$ with $P_0 \subset P$. Then $P$ has a unique Levi decomposition $P = L_PN_P$ over $\mathbb{Q}$ such that $L_P \supset L_0$. When the dependency on the parabolic subgroup is clear from the context, we suppress the subscript $P$ from the notation.

Let $A_P$ be the maximal $\mathbb{Q}$–split torus in the center of $L_P$. In the case of the minimal parabolic $\mathbb{Q}$–subgroup $P_0$ we write $A_0 = A_{P_0}$. Then there is a unique Langlands decomposition $P = M_PA_PN_P$ with $M_P \supset M_0$ and $A_P \subset A_0$.

Two standard parabolic $\mathbb{Q}$-subgroups $P$ and $Q$ of $G$ are called associate if $A_P$ and $A_Q$ are conjugate in $G$ under an element in $G(\mathbb{Q})$.

Let $\mathfrak{g}, \mathfrak{p}, ...$ denote the Lie algebras of $G(\mathbb{R}), P(\mathbb{R}), \dots$. The Lie algebras of the factors in the Langlands decomposition of $P$ will be denoted by $\mathfrak{m}_P, \mathfrak{a}_P, \mathfrak{n}_P$, and $\mathfrak{l}_P = \mathfrak{m}_P + \mathfrak{a}_P$. We put $\check{\mathfrak{a}}_0 = X^*(P_0) \otimes \mathbb{R}$, where $X^*$ denotes the group of $\mathbb{Q}$–rational characters, and similarly for a given standard parabolic $\mathbb{Q}$–subgroup $P \supset P_0, \check{\mathfrak{a}}_P = X^*(P) \otimes \mathbb{R}$. Then $\mathfrak{a}_P = X_*(A_P) \otimes \mathbb{R}$, where $X_*$ denotes the group of $\mathbb{Q}$–rational cocharacters, and $\mathfrak{a}_0 = X_*(A_0) \otimes \mathbb{R}$ are in a natural way in duality with $\check{\mathfrak{a}}_P$ and $\check{\mathfrak{a}}_0$; the pairing is denoted by $\langle\ ,\ \rangle$. In particular, $\mathfrak{a}_P$ and $\mathfrak{a}_0$ are independent of the Langlands decomposition up to canonical isomorphism. The inclusion $A_P \subset A_0$ defines $\mathfrak{a}_P \to \mathfrak{a}_0$, and the restriction of characters of $P$ to $P_0$ defines $\check{\mathfrak{a}}_P \to \check{\mathfrak{a}}_0$ which is inverse to the dual of the previous map. Thus, one has direct sum decompositions $\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P$ and $\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P$ respectively. Let $\mathfrak{a}_P^Q$ be the intersection of $\mathfrak{a}_P$ and $\mathfrak{a}_0^Q$ in $\mathfrak{a}_0$. Similar notation is used for $\check{\mathfrak{a}}$. By $\mathfrak{m}_G$ we denote the intersection $\cap\ker(d\chi)$ of the kernels of the derivatives of all rational characters $\chi \in X^*(G)$. Then we put $\mathfrak{a}_P^G := \mathfrak{a}_P \cap \mathfrak{m}_G$; its dimension is called the rank of $P$. We denote by $\Phi \subset X^*(A_0) \subset \check{\mathfrak{a}}_0$ the set of roots of $A_0$ in $\mathfrak{g}$; it is a root system in the vector space $\check{\mathfrak{a}}_0$. The ordering on $\Phi$ is fixed so that $\Phi^+$ coincides with the set of roots of $A_0$ in $P_0$. Let $\Delta \subset \Phi$ be the set of simple positive roots. If $P$ is a standard parabolic $\mathbb{Q}$–subgroup of $G$ the Weyl group of $A_0$ in $L_P$ is denoted by $W_P$. In particular, we write $W = W_G$ for the Weyl group of the root system $\Phi$. Note that $W_P$ is a subgroup of $W$.

(3) Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Any element $D$ in $U(\mathfrak{g})$ defines a differential operator on the space $C^\infty(A_G(\mathbb{R})^0 \setminus G(\mathbb{A}))$ of smooth complex valued functions on $A_G(\mathbb{R})^0 \setminus G(\mathbb{A})$ by right differentiation with respect to the real component of $g \in G(\mathbb{A})$. This operator is denoted by $f \mapsto Df$. It commutes with the action of $G(\mathbb{R})$ given by left translation. If $D \in Z(\mathfrak{g})$ this operator also commutes with the action of $G(\mathbb{R})$ by right translation.

## 2. Spaces of Automorphic Forms

2.1. **Parabolic subgroups.** Let $G$ be a connected reductive algebraic group defined over $\mathbb{Q}$. Fix a minimal parabolic subgroup $P_0$ of $G$ defined over $\mathbb{Q}$ and a Levi subgroup $L_0$ of $P_0$ defined over $\mathbb{Q}$. One has the Levi decomposition $P_0 = L_0 N_0$ with unipotent radical $N_0$. By definition, a standard parabolic subgroup $P$ of $G$ is a parabolic subgroup $P$ of $G$ defined over $\mathbb{Q}$ that contains $P_0$. Analogously, a standard Levi subgroup $L$ of $G$ is a Levi subgroup of any standard parabolic subgroup $P$ of $G$ such that $L$ contains $L_0$. A given standard parabolic subgroup $P$ of $G$ has a unique standard Levi subgroup $L$. We denote by $P = LN$ the corresponding Levi decomposition of $P$ over $\mathbb{Q}$.

2.2. **Maximal compact subgroup.** By definition, the adele group $G(\mathbb{A})$ of the group $G$ is the restricted product $G(\mathbb{A}) = \prod_{v\in V}{}' G(\mathbb{Q}_v)$ with respect to the maximal compact subgroups $G(\mathbb{Z}_v) \subset G(\mathbb{Q}_v), v \in V_f$. The group $G(\mathbb{A})$ is the direct product of the group $G(\mathbb{R})$ of real points of $G$ and the restricted product $\prod'_{v\in V_f} G(\mathbb{Q}_v) =: G(\mathbb{A}_f)$. We fix a maximal compact subgroup $K$ of $G(\mathbb{A})$ subject to the following condition. Since it is of the form $K = \prod_{v\in V} K_v$ where $K_v$ is a maximal compact subgroup of $G(\mathbb{Q}_v), v \in V$, we suppose (as we may) that $K_v = G(\mathbb{Z}_v)$ for almost all finite places $v \in V_f$. If $v$ is archimedean we write $K_\mathbb{R}$ instead of $K_v$, and we write $K_f = \prod_{v\in V_f} K_v$.

We may assume that the group $K$ is in good position relative to $P_0$, that is, $K$ satisfies the following requirements:

- $G(\mathbb{A}) = P_0(\mathbb{A})K$,
- given a standard parabolic subgroup $P = LN$ of $G$ one has $P(\mathbb{A})\cap K = (L(\mathbb{A})\cap K)(N(\mathbb{A})\cap K)$ and $L(\mathbb{A}) \cap K$ is a maximal compact subgroup of $L(\mathbb{A})$.

For a given standard parabolic subgroup $P = LN$ of $G$ one has the Iwasawa decomposition $G(\mathbb{A}) = L(\mathbb{A})N(\mathbb{A})K$. Then we can define the standard height function $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ on $G(\mathbb{A})$ by $\prod_{v\in V} |\chi(l)|_v = e^{<\chi, H_P(lnk)>}$ for any character $\chi \in X^*(L) \subset \mathfrak{a}_P^*$.

2.3. **Lie algebras.** We denote by $M_G$ the connected component of the intersection of the kernels of all $\mathbb{Q}$ - rational characters of $G$, and by $\mathfrak{m}_G = \mathrm{Lie}(M_G(\mathbb{R}))$ the corresponding Lie algebra. Note that the maximal $\mathbb{Q}$ - split torus $A_G$ in the center of $G$ reduces to the identity if $G$ is a semi-simple group. In such a case, $\mathfrak{m}_G = \mathrm{Lie}(G(\mathbb{R}))$. In general, the Lie algebra $\mathfrak{g} = Lie(G(\mathbb{R}))$ decomposes as a direct sum of Lie algebras $\mathfrak{g} = \mathfrak{a}_G \oplus \mathfrak{m}_G$ where $\mathfrak{a}_G$ denotes the Lie algebra of $A_G(\mathbb{R})$. In particular, $\mathfrak{m}_G$ coincides with $\mathrm{Lie}(A_G(\mathbb{R})^0 \setminus G(\mathbb{R}))$. The maximal compact subgroup $K_\mathbb{R}$ of $G(\mathbb{R})$ may be viewed as a subgroup of $A_G(\mathbb{R})^0 \setminus G(\mathbb{R})$. A character $\chi \in X^*(G)$ defines a homomorphism $G(\mathbb{A}) \to \mathbb{I}$ of $G(\mathbb{A})$ into the group of ideles, also denoted by $\chi$. We denote by $G(\mathbb{A})^1$ the subgroup $\{g \in G(\mathbb{A}) || \chi(g)|_\mathbb{A} = 1, \chi \in X^*(G)\}$ of $G(\mathbb{A})$. One has a decomposition $G(\mathbb{A}) = A_G(\mathbb{R})^0 \times G(\mathbb{A})^1$ as a product, and the group $G(\mathbb{A})^1$ can be identified with $A_G(\mathbb{R})^0\backslash G(\mathbb{A})$. In an analogous way, $\mathfrak{m}_G$ can be identified with $\mathrm{Lie}(A_G(\mathbb{R})^0 \setminus (G(\mathbb{A}) \cap G(\mathbb{R})))$.

2.4. **Automorphic forms.** We fix a height $||\ ||$ on the adele group $G(\mathbb{A})$. By definition, a $C^\infty$–function $f : G(\mathbb{A}) \to \mathbb{C}$ is of uniform moderate growth on $G(\mathbb{Q})\backslash G(\mathbb{A})$ if

- $f$ is $K$–finite (i.e. the set $\{f_k, k \in K\}$, where $f_k(g) = f(gk)$, spans a finite–dimensional space)
- there exists a constant $c > 0$, $c \in \mathbb{R}$, such that for all elements $D \in U(\mathfrak{g})$ there is $r_D \in \mathbb{R}$ with $|Df(g)| \leq r_D||g||^c$ for all $g \in G(\mathbb{A})$.
- $f$ is invariant under left translation by elements of $G(\mathbb{Q})$.

We denote the space of such functions by $V_{umg}(G)$.

Let $A_G$ denote the maximal $\mathbb{Q}$ - split torus in the center $Z_G$ of $G$. We write

$$V_G = \ C^\infty_{umg}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))$$

for the space of smooth complex-valued functions of uniform moderate growth on $G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A})$. The space $V_G$ carries in a natural way the structure of a $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$–module.

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We call an element $f \in V_{umg}(G)$ an automorphic form on $G(\mathbb{A})$ if there exists an ideal $J \subset Z(\mathfrak{g})$ of finite codimension that annihilates $f$. We denote the space of automorphic forms on $G(\mathbb{A})$ by $\mathcal{A}(G)$.

2.5. **Constant term.** Let $P = LN$ be a standard parabolic $\mathbb{Q}$–subgroup of $G$. For a measurable locally integrable function $f$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$, the constant term of $f$ along $P$ is the function $f_P$ on $N(\mathbb{A})L(\mathbb{Q})\backslash G(\mathbb{A})$ defined by

$$f_P : g \longmapsto \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)dn \ , \ g \in G(\mathbb{A})$$

where the Haar measure $dn$ on $N(\mathbb{A})$ is normalized in such a way that one has $\mathrm{vol}_{dn}\ (N(\mathbb{Q})\backslash N(\mathbb{A})) = 1$. The assignment $f \longmapsto f_P$ is compatible with the actions of $\mathfrak{g}, K_\mathbb{R}$ and $G(\mathbb{A}_f)$ on these functions (if they are defined). If $f$ is smooth (or has moderate growth) then $f_P$ is smooth (or has moderate growth).

For an automorphic form $f \in \mathcal{A}(G)$ we say that $f$ is cuspidal if $f_P \equiv 0$ for all proper standard parabolic $\mathbb{Q}$ - subgroups of $G$.

2.6. **Decomposition over associate classes of parabolic subgroups.** Two parabolic subgroups $P$ and $P'$ of $G$ are said to be associate if their reductive components are conjugate by an element in $G(\mathbb{Q})$. This is equivalent to the condition that their split components are $G(\mathbb{Q})$-conjugate. This notion induces an equivalence relation on the set $\mathcal{P}(G)$ of parabolic $\mathbb{Q}$-subgroups of $G$. Given $P \in \mathcal{P}(G)$, we denote its equivalence class by $\{P\}$, to be called the associate class of $P$. Let $\mathcal{C}$ be the set of classes of associate parabolic $\mathbb{Q}$–subgroups of $G$. For $\{P\} \in \mathcal{C}$ denote by $V_G(\{P\})$ the space of elements in $V_G$ that are negligible along $Q$ for every parabolic $\mathbb{Q}$–subgroup $Q$ in $G$, $Q \notin \{P\}$, that is, given $Q = L_Q N_Q$, for all $g \in G(\mathbb{A})$ the function $l \mapsto f_Q(lg)$ is orthogonal to the space of cuspidal functions on $A_G(\mathbb{R})^0 L_Q(\mathbb{Q}) \setminus L_Q(\mathbb{A})$.

The space $V_G(\{P\})$, $\{P\} \in \mathcal{C}$, is a submodule in $V_G$ with respect to the $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$–module structure. It is known that the $\sum V_G(\{P\})$, $\{P\} \in \mathcal{C}$, forms a direct sum. Finally, one has a decomposition as a direct sum of $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$–modules

$$V_G = \bigoplus_{\{P\}\in\mathcal{C}} V_G(\{P\})).$$

This was first proved in [21], see [3, Theorem 2.4], for a variant of the original proof.

## 3. Automorphic cohomology

3.1. **Eisenstein cohomology.** We retain the notation of section 2. Let $(\nu, E)$ be a finite-dimensional algebraic representation of $G(\mathbb{C})$ in a complex vector space. We suppose that $A_G(\mathbb{R})^0$ acts by a character on $E$, to be denoted by $\chi^{-1}$. Let $J_E \subset Z(\mathfrak{g})$ be the annihilator of the dual representation of $E$ in $Z(\mathfrak{g})$. Let $\mathcal{A}_E \subset V_G = C^\infty_{umg}(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}))$ be the subspace of functions $f \in V_G$ which are annihilated by a power of $J_E$. Then the spaces $\mathcal{A}_E \otimes_\mathbb{C} E$ and $V_G \otimes_\mathbb{C} E$ both are naturally equipped with a $(\mathfrak{m}_G, K_\mathbb{R})$ - module structure. By [7, Theorem 18], the inclusion $\mathcal{A}_E \otimes_\mathbb{C} E \longrightarrow V_G \otimes_\mathbb{C} E$ of the space of automorphic forms on $G$ (with respect to $(\nu, E)$) in the

space of functions of uniform moderate growth induces an isomorphism on the level of $(\mathfrak{m}_G, K_\mathbb{R})$ - cohomology, that is,

$$H^*(\mathfrak{m}_G, K_\mathbb{R}, \mathcal{A}_E \otimes_\mathbb{C} E) \tilde{\longrightarrow} H^*(\mathfrak{m}_G, K_\mathbb{R}, V_G \otimes_\mathbb{C} E). \tag{3.1}$$

Both cohomology spaces carry a $G(\mathbb{A}_f)$-module structure induced by the one on $\mathcal{A}_E$ and $V_G$ respectively, and the isomorphism is compatible with this $G(\mathbb{A}_f)$-module structure. We define the automorphic cohomology of $G$ with coefficients in $E$ by

$$H^*(G, E) := H^*(\mathfrak{m}_G, K_\mathbb{R}, \mathcal{A}_E \otimes_\mathbb{C} E) \tag{3.2}$$

As explained in [8] we keep in mind that these cohomology groups have an interpretation as the inductive limit of the deRham cohomology groups $H^*(X_C, E)$ of the orbit space

$$X_C := G(\mathbb{Q}) A_G(\mathbb{R})^0 \setminus G(\mathbb{A}) / K_\mathbb{R} C$$

with coefficients in the local system given by the representation $(\nu, E)$ where $C$ ranges over the open compact subgroups of $G(\mathbb{A}_f)$.

With this general framework in place, given a class $\{P\} \in \mathcal{C}$ of associate parabolic $\mathbb{Q}$-subgroups of $G$, one can set $\mathcal{A}_{E,\{P\}} = \mathcal{A}_E \cap V_G(\{P\})$. The spaces $\mathcal{A}_{E,\{P\}}$, $\{P\} \in \mathcal{C}$, form a direct sum, and one has a decomposition as a direct sum of $(\mathfrak{g}, K_\mathbb{R}, G(\mathbb{A}_f))$–modules

$$\mathcal{A}_E = \bigoplus_{\{P\} \in \mathcal{C}} \mathcal{A}_{E,\{P\}}. \tag{3.3}$$

This direct sum decomposition where the sum ranges over the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$ - subgroups of $G$ induces a direct sum decomposition

$$H^*(G, E) = \bigoplus_{\{P\} \in \mathcal{C}} H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\}} \otimes E) \tag{3.4}$$

in cohomology. The summand in the direct sum decomposition of the cohomology $H^*(G, E)$ that is indexed by the full group $\{G\}$ will be called the *cuspidal cohomology of $G$ with coefficients in $E$*, to be denoted $H^*_{cusp}(G, E)$.

The decompostion of $H^*(G, E)$ according to the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$ - subgroups of $G$ exhibits a natural complement to the cuspidal cohomology, namely the summands indexed by $\{P\} \in \mathcal{C}, \{P\} \neq \{G\}$. Due to the results in [7] that these cohomology classes can be described by suitable derivatives of Eisenstein series or residues of these, one calls this complement

$$H^*_{Eis}(G, E) := \bigoplus_{\{P\} \in \mathcal{C}, P \neq G} H^*(\mathfrak{m}_G, K_\mathbb{R}; \mathcal{A}_{E,\{P\}} \otimes E)$$

*the Eisenstein cohomology of $G$ with coefficients in $E$*. In addition, one can take into account the cuspidal support of each of these Eisenstein series. This results in an even finer decomposition of $H^*_{Eis}(G, E)$ to be discussed below.

3.2. **Decomposition along the cuspidal support.** Let $\{P\}$ be a class of associate parabolic $\mathbb{Q}$-subgroups of $G$, and let $\phi = \{\phi_P\}_{P \in \{P\}}$ be a class of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$ as defined in [8, Section 1.2.].

The set of all such collections $\phi = \{\phi_P\}_{P \in \{P\}}$ is denoted by $\Phi_{E,\{P\}}$. Given a class $\{P\}$ of associate parabolic $\mathbb{Q}$-subgroups of $G$, and any $\phi \in \Phi_{E,\{P\}}$, we let

$$\mathcal{A}_{E,\{P\},\phi} = \{f \in V_G(\{P\}) | f_P \in \bigoplus_{\pi \in \phi_P} L^2_{cusp,\pi}(L_P(\mathbb{Q}) \setminus L_P(\mathbb{A}))_{\chi_\pi} \otimes S(\check{\mathfrak{a}}^G_P)\}$$

be the space of functions of uniform moderate growth whose constant term along each $P \in \{P\}$ belongs to the isotypic components attached to the elements $\pi \in \phi_P$. Finally, we have the following

**Theorem 3.1.** *The automorphic cohomology $H^*(G,E)$ has a direct sum decomposition*

$$H^*(G,E)=\bigoplus_{\{P\}\in\mathcal{C}}\bigoplus_{\phi\in\Phi_{E,\{P\}}}H^*(\mathfrak{m}_G,K_{\mathbb{R}},\mathcal{A}_{E,\{P\},\phi}\otimes_{\mathbb{C}}E)$$

*where, given $\{P\}\in\mathcal{C}$, the second sum ranges over the set $\Phi_{E,\{P\}}$ of classes of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P\}$.*

For a proof of this result we refer to [8, Theorem 1.4 resp. 2.3], or [28, Theorem in III, 2.6], where a different approach to the decomposition of the space of automorphic forms along the cuspidal support is given.

## 4. The Symplectic Group

In the rest of the paper we consider the $\mathbb{Q}$–split simple simply connected symplectic group $Sp_n$ of $\mathbb{Q}$–rank $n$, where $n\geq 2$. Let $P_0$ be a minimal parabolic $\mathbb{Q}$–subgroup, and $P_0=L_0N_0$ its Levi decomposition, which are fixed throughout the paper. The maximal split torus $L_0$ is isomorphic to a product of $n$ copies of $\mathbb{G}_m/\mathbb{Q}$, and $N_0$ is the unipotent radical. Let $\Phi$, $\Phi^+$, $\Delta$ denote the corresponding sets of roots, positive roots, simple roots, respectively. If $e_i$ is the projection of $L_0$ to its $i^{\text{th}}$ component, then

$$\Delta=\{\alpha_1=e_1-e_2,\alpha_2=e_2-e_3,\ldots,\alpha_{n-1}=e_{n-1}-e_n,\alpha_n=2e_n\}.$$

The half–sum of the positive roots equals

$$\rho_{P_0}=\frac{1}{2}\sum_{\beta\in\Phi^+}\beta=ne_1+(n-1)e_2+\ldots+2e_{n-1}+e_n.$$

Let $W$ be the Weyl group of $G$ with respect to $L_0$.

The standard parabolic $\mathbb{Q}$–subgroups of $Sp_n$ are in bijection with the subsets of the set $\Delta$ of simple roots. For $\Theta\subset\Delta$, we denote by $P_\Theta$ the corresponding parabolic $\mathbb{Q}$–subgroup. Let $P_\Theta=L_\Theta N_\Theta$ be its Levi decomposition, where $L_\Theta$ is the Levi factor, i.e. the centralizer of $S_\Theta=(\cap_{\alpha\in\Theta}\mathrm{Ker}(\alpha))^\circ$, and $N_\Theta$ is the unipotent radical.

The bijection with the subsets of $\Delta$ gives rise to a bijection between standard parabolic $\mathbb{Q}$–subgroups of $Sp_n$ and the $l$–tuples of positive integers $(r_1,r_2,\ldots,r_l)$, where $r_i\geq 1$ for $i=1,\ldots,l$, and $r_1+\ldots+r_l\leq n$. Let $r_0=n-\sum_{i=1}^l r_i\geq 0$. The standard parabolic $\mathbb{Q}$–subgroup $P_{(r_1,r_2,\ldots,r_l)}$ corresponding to the $l$–tuple $(r_1,r_2,\ldots,r_l)$ is given by the subset $\Delta\setminus\{\alpha_{r_1+r_2+\ldots+r_i}:i=1,2,\ldots,l\}$ of $\Delta$. Its Levi factor is isomorphic to $GL_{r_1}\times GL_{r_2}\times\ldots\times GL_{r_l}\times Sp_{r_0}$.

In particular, for $r=1,2,\ldots,n$, the maximal proper standard parabolic $\mathbb{Q}$–subgroup $P_{\Delta\setminus\{\alpha_r\}}$ corresponding to the subset $\Delta\setminus\{\alpha_r\}$ of $\Delta$ is denoted shortly by $P_r$, and its Levi decomposition by $P_r=L_rN_r$, where $L_r$ is the Levi factor, and $N_r$ the unipotent radical. For $r<n$ we have $L_r\cong GL_r\times Sp_{n-r}$, and for $r=n$ we have $L_n\cong GL_n$. Observe that the parabolic subgroups $P_r$ are self–associate, i.e. $P_r$ itself is the only standard parabolic subgroup which is associate to $P_r$ (see Section 1.6). However, $P_r$ is conjugate to its opposite parabolic subgroup $P_r^{\text{opp}}$ by a representative of the unique non–trivial Weyl group element $w_0\in W$ with the property that $w_0(\Delta\setminus\{\alpha_r\})\subset\Delta$.

For $r=1,2,\ldots,n$, let $\rho_{P_r}$ be the half–sum of positive roots not being the positive roots of $L_r$. Then,

$$\rho_{P_r}=\frac{2n+1-r}{2}\sum_{i=1}^r e_i.$$

As a convenient basis for $\check{\mathfrak{a}}_{P_r,\mathbb{C}}\cong\mathbb{C}$ we choose

$$\widetilde{\rho}_{P_r}=\langle\rho_{P_r},\alpha_r^\vee\rangle^{-1}\rho_{P_r}=\sum_{i=1}^r e_i,$$

motivated by the work of Shahidi [37], where $\langle\cdot,\cdot\rangle$ is the natural pairing of $\check{\mathfrak{a}}_{P_0,\mathbb{C}}$ and $\mathfrak{a}_{P_0,\mathbb{C}}$, and we always identify accordingly $s\in\mathbb{C}$ with $\lambda_s=\widetilde{\rho}_{P_r}\otimes s\in\check{\mathfrak{a}}_{P_r,\mathbb{C}}$.

For later use we recall that the standard parabolic $\mathbb{Q}$–subgroups of a general linear group $GL_N/\mathbb{Q}$, split over $\mathbb{Q}$, are in bijection with the set of all partitions of $N$ into positive integers. The parabolic $\mathbb{Q}$–subgroup corresponding to partition $(d_1,\ldots,d_m)$, where $\sum_{l=1}^m d_l=N$, we denote by $P_{(d_1,\ldots,d_m)}$, and its Levi factor is isomorphic to $GL_{d_1}\times\ldots\times GL_{d_m}$.

## 5. Eisenstein Series of Relative Rank One

In this section, following the Langlands spectral theory [20] and [28], we study in some detail the main analytic properties of Eisenstein series attached to cuspidal automorphic representations on the Levi components of maximal proper parabolic $\mathbb{Q}$–subgroups of $Sp_n$ (for the approach to the residual spectrum via Arthur's conjectures see [26], [27]). Although the arguments and results apply over any number field, we work over $\mathbb{Q}$ having in mind the cohomological application. We retain the notation introduced in Section 4. In particular, $P_r$, for $r=1,\ldots,n$, denotes the standard maximal parabolic $\mathbb{Q}$–subgroup of $Sp_n$ which corresponds to $\Delta\setminus\{\alpha_r\}$. We write $P_r=L_rN_r$ for its Levi decomposition.

5.1. **Eisenstein series.** Let $\pi\cong\tau\otimes\sigma$ be a cuspidal automorphic representation of $L_r(\mathbb{A})$, where $\tau$ is a cuspidal automorphic representation of $GL_r(\mathbb{A})$ and $\sigma$ a cuspidal automorphic representation of $Sp_{n-r}(\mathbb{A})$. For $r=n$, we write $\pi\cong\tau$, where $\tau$ is a cuspidal automorphic representation of $GL_n(\mathbb{A})$. Throughout the paper by a cuspidal automorphic representation of $G(\mathbb{A})$, where $G$ is a $\mathbb{Q}$–split reductive group defined over $\mathbb{Q}$, we mean an irreducible $(\mathfrak{g},K_\mathbb{R};G(\mathbb{A}_f))$–module realized on a subspace of the space of cusp forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ (see [28, Section I.2.17]). We denote by $V_\pi$ the subspace of the space of cusp forms of $L_r(\mathbb{Q})\backslash L_r(\mathbb{A})$ on which $\pi$ acts.

When computing the Eisenstein cohomology, one considers only the real poles of the Eisenstein series. Hence, we make the following convention. We assume that $\pi$ is normalized in such a way that the differential of the restriction of the central character of $\pi$ to $A_{P_r}(\mathbb{R})^+$ is trivial. This assumption is just a convenient choice of coordinates, which makes the poles of the Eisenstein series attached to $\pi$ real. As explained in [8, Section 1.3], it can be achieved by replacing $\pi$ by an appropriate twist. The twist just moves the poles of the Eisenstein series along the imaginary axis.

As in [8, Section 1.3], consider the space $W_\pi$ of right K–finite smooth functions

$$f: N_r(\mathbb{A})L_r(\mathbb{Q})\backslash Sp_n(\mathbb{A})\to\mathbb{C}$$

such that for every $g\in Sp_n(\mathbb{A})$ the function $f_g(l)=f(lg)$ on $L_r(\mathbb{Q})\backslash L_r(\mathbb{A})$ belongs to the subspace $V_\pi$ of the space of cusp forms on $L_r(\mathbb{A})$. Then, for $f\in W_\pi$, and $\lambda_s\in\check{\mathfrak{a}}_{P_r,\mathbb{C}}$, and for each $g\in Sp_n(\mathbb{A})$, one defines (at least formally) the Eisenstein series as

$$E_{P_r}^{Sp_n}(f,\lambda_s)(g)=\sum_{\gamma\in P_r(k)\backslash Sp_n(k)}e^{\langle H_{P_r}(\gamma g),\lambda_s+\rho_{P_r}\rangle}f(\gamma g)=\sum_{\gamma\in P_r(k)\backslash Sp_n(k)}f_s(\gamma g),$$

where $f_s(g)=f(g)e^{\langle H_{P_r}(g),\lambda_s+\rho_{P_r}\rangle}$. This Eisenstein series converges absolutely and uniformly in $g$ if $Re(s)>\frac{2n+1-r}{2}$, and the assignment $s\longmapsto E_{P_r}^{Sp_n}(f,\lambda_s)(g)$ defines a map that is holomorphic in the region of absolute convergence of the defining series and has a meromorphic continuation to all of $\check{\mathfrak{a}}_{P_r,\mathbb{C}}$. It has a finite number of simple poles at in the real interval $0<\lambda_s\leq\rho_{P_r}$, i.e. $0<s\leq\frac{2n+1-r}{2}$ in the coordinate $\widetilde{\rho}_{P_r}$. All the remaining poles lie in the region $Re(s)<0$. The reference for these facts is [28, Section IV.1].

5.2. **Franke's filtration.** The space $\mathcal{A}_{E,\{P_r\}}$ introduced in Section 2 has a two–step filtration defined in [7, Section 6]. However, we use a slight modification as in [8, Section 5.2]. According to the decomposition of $\mathcal{A}_{E,\{P_r\}}$ along the cuspidal support as in Section 2, it suffices to give the filtration of the spaces $\mathcal{A}_{E,\{P_r\},\phi}$, where $\phi$ is the associate class of $\pi$. Then, the filtration is given by $\mathcal{L}_{E,\{P_r\},\phi} \subset \mathcal{A}_{E,\{P_r\},\phi}$, where $\mathcal{L}_{E,\{P_r\},\phi}$ is the subspace of $\mathcal{A}_{E,\{P_r\},\phi}$ consisting of square integrable automorphic forms. The space $\mathcal{L}_{E,\{P_r\},\phi}$ is spanned by the residues at $s > 0$ of the Eisenstein series attached to a function $f$ such that for every $g \in Sp_n(\mathbb{A})$ functions $f_g$ defined above belong to the $\pi$–isotypic subspace of the space of cusp forms on $L_r(\mathbb{A})$. Those residues are square–integrable automorphic forms by the Langlands criterion [28, Section I.4.11]. The quotient $\mathcal{A}_{E,\{P_r\},\phi}/\mathcal{L}_{E,\{P_r\},\phi}$ is spanned by the principal value of the derivatives of such Eisenstein series at $Re(s) \geq 0$.

We also consider a subspace of $\mathcal{L}_{E,\{P_r\},\phi}$ spanned by the residues at poles $s > 0$ of the Eisenstein series $E^{Sp_n}_{P_r}(f,\lambda_s)(g)$ attached as above to a fixed (irreducible) realization $V_\pi$ of a cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$. We denote that subspace by $\mathcal{L}_{E,\{P_r\},\phi,V_\pi}$. In the case $r = n$, i.e. $L_n \cong GL_n$, due to the multiplicity one theorem for cuspidal automorphic representations of $GL_n(\mathbb{A})$ (see [38], [29]), the $\pi$–isotypic subspace of the space of cusp forms on $L_n(\mathbb{A})$ is irreducible. Hence, if $r = n$, then $\mathcal{L}_{E,\{P_n\},\phi}$ and $\mathcal{L}_{E,\{P_n\},\phi,V_\pi}$ coincide. Otherwise, if $r < n$, it might not be the case.

5.3. **Normalization of intertwining operators.** Since $P_r$ is self–associate, the poles of the Eisenstein series coincide with the poles of its constant term $E^{Sp_n}_{P_r}(f,\lambda_s)_{P_r}$ along $P_r$ (see [28, Section II.1.7]). The constant term along $P_r$ is given by

$$E^{Sp_n}_{P_r}(f,\lambda_s)_{P_r}(g) = f_s(g) + M(\lambda_s,\pi,w_0)f_s(g), \tag{5.1}$$

where $w_0 \in W$ is the unique non–trivial Weyl group element such that $w_0(\Delta \setminus \{\alpha_r\}) \subset \Delta$, and $M(\lambda_s,\pi,w_0)$ is the standard intertwining operator defined as the analytic continuation from the domain of convergence of the integral

$$M(\lambda_s,\pi,w_0)f_s(g) = \int_{N_r(\mathbb{A})} f_s(\tilde{w}_0^{-1}ug)du, \tag{5.2}$$

where $\tilde{w}_0$ is the representative for $w_0$ in $Sp_n(\mathbb{Q})$ chosen as in [35]. Away from the poles it intertwines the induced representation

$$\begin{aligned} I(\lambda_s,\pi) &= \mathrm{Ind}_{P_r(\mathbb{A})}^{Sp_n(\mathbb{A})}\left(\tau e^{\langle H_{P_r}(\cdot),\lambda_s\rangle}\otimes\sigma\right) \\ &\cong \left\{f_s = f\cdot e^{\langle H_{P_r}(\cdot),\lambda_s+\rho_{P_r}\rangle} : f\in W_\pi\right\} \end{aligned}$$

and $I(\lambda_{-s},w_0(\pi))$, where the action of $w_0$ on $\pi$ is given by $w_0(\pi)(l) = \pi(\tilde{w}_0^{-1}l\tilde{w}_0)$ for $l \in L_r(\mathbb{A})$. Let $\widetilde{\tau}$ denote the contragredient of $\tau$. If $r = n$, then $\sigma$ does not appear in the above equation, and $w_0(\pi) = w_0(\tau) \cong \widetilde{\tau}$. If $r < n$, then $w_0(\pi) \cong w_0(\tau\otimes\sigma) \cong \widetilde{\tau}\otimes\sigma$. Observe that in our notation $\mathrm{Ind}_{P_r(\mathbb{A})}^{Sp_n(\mathbb{A})}$ includes the normalization by $\rho_{P_r}$, and thus $\rho_{P_r}$ does not appear in the first line but appears in the second line of the above equation.

The poles of the constant term $E^{Sp_n}_{P_r}(f,\lambda_s)_{P_r}(g)$ of the Eisenstein series coincide with the poles of $M(\lambda_s,\pi,w_0)f_s(g)$. As explained below, for a globally $\psi$–generic cuspidal automorphic representation $\pi$, the poles of the standard intertwining operator $M(\lambda_s,\pi,w_0)$ for $s \geq 0$ coincide with the poles of certain automorphic L–functions. Therefore, in what follows we assume that $\pi$, as a cuspidal automorphic representation on a subspace $V_\pi$ of the space of cusp forms on $L_r(\mathbb{Q})\backslash L_r(\mathbb{A})$, is globally generic with respect to a fixed non–trivial continuous additive character $\psi$ of $\mathbb{Q}\backslash\mathbb{A}$. In other words, there exists a cusp form in $V_\pi$ such that its $\psi$–Fourier coefficient along the minimal

parabolic $\mathbb{Q}$–subgroup is non–trivial. Since all cuspidal automorphic representations of $GL_r(\mathbb{A})$ are globally generic, this assumption applies only for $r < n$, and is in fact a condition on $\sigma$.

Let $\pi \cong \otimes_v \pi_v \cong \otimes_v (\tau_v \otimes \sigma_v)$ be the decomposition into a restricted tensor product as in [6], where $\pi_v \cong \tau_v \otimes \sigma_v$ is a unitary irreducible representation of $L_r(\mathbb{Q}_v)$, and $\tau_v$ and $\sigma_v$ are unitary irreducible representations of $GL_r(\mathbb{Q}_v)$ and $Sp_{n-r}(\mathbb{Q}_v)$, respectively. At almost all non–Archimedean places $v \in V_f$, $\pi_v$ is unramified, and we denote by $f^\circ_{s,v}$ the unique $K_v$–invariant vector in $I(\lambda_s, \pi_v)$ normalized by the condition $f^\circ_{s,v}(e) = 1$, where $e$ is the identity in $Sp_n(\mathbb{Q}_v)$. By [19, Section 5], the standard local intertwining operator $A(\lambda_s, \pi_v, w_0)$, defined as the analytic continuation of the local analogue of the integral (5.2), acts at an unramified place $v \in V_f$ on $f^\circ_{s,v}$ as

$$A(\lambda_s, \pi_v, w_0) f^\circ_{s,v} = r(\lambda_s, \pi_v, w_0) \widetilde{f}^\circ_{-s,v},$$

where $r(\lambda_s, \pi_v, w_0)$ is the local normalizing factor given as a certain ratio of the local L–functions, and $\widetilde{f}^\circ_{-s,v}$ is the normalized $K_v$–invariant vector in $I(\lambda_{-s}, w_0(\pi_v))$. If $f_s = \otimes_v f_{s,v}$ is decomposable, let $S$ be the finite set of places which contains all Archimedean places $V_\infty$ and such that $f_{s,v} = f^\circ_{s,v}$ for all $v \in V_f \setminus S$. Then the global standard intertwining operator acts on $f_s$ as

$$M(\lambda_s, \pi, w_0) f_s = [\otimes_{v \in S} A(\lambda_s, \pi_v, w_0) f_{s,v}] \otimes r^S(\lambda_s, \pi, w_0) \left[ \otimes_{v \notin S} \widetilde{f}^\circ_{-s,v} \right],$$

where

$$r^S(\lambda_s, \pi, w_0) = \prod_{v \notin S} r(\lambda_s, \pi_v, w_0)$$

is a certain ratio of partial L–functions attached to $\pi$.

In [37], the local normalizing factors $r(\lambda_s, \pi_v, w_0)$ are defined at all places for a $\psi_v$–generic representation $\pi_v$. Let $N(\lambda_s, \pi_v, w_0)$ be the local normalized intertwining operator defined by

$$A(\lambda_s, \pi_v, w_0) = r(\lambda_s, \pi_v, w_0) N(\lambda_s, \pi_v, w_0).$$

It intertwines the induced representations $I(\lambda_s, \pi_v)$ and $I(\lambda_{-s}, w_0(\pi_v))$. Note that at a place $v \in V_f$ where $\pi_v$ is unramified $N(\lambda_s, \pi_v, w_0)$ maps $f^\circ_{s,v}$ to $\widetilde{f}^\circ_{-s,v}$. Hence,

$$M(\lambda_s, \pi, w_0) f_s = r(\lambda_s, \pi, w_0) \left[ \otimes_{v \in S} N(\lambda_s, \pi_v, w_0) f_{s,v} \right] \otimes \left[ \otimes_{v \notin S} \widetilde{f}^\circ_{-s,v} \right], \tag{5.3}$$

where

$$r(\lambda_s, \pi, w_0) = \prod_v r(\lambda_s, \pi_v, w_0)$$

is the global normalizing factor given as a certain ratio of automorphic L–functions attached to $\pi$. This ratio is made precise in Theorem 5.1 below.

**Theorem 5.1.** *Let $r \leq n$ be a positive integer, and $P_r = L_r N_r$ the maximal proper standard parabolic $\mathbb{Q}$–subgroup of $Sp_n$. Let $\pi \cong \tau \otimes \sigma$ be a cuspidal automorphic representation of the Levi factor $L_r(\mathbb{A})$, where $\sigma$ does not appear if $r = n$. If $r < n$, assume that $\sigma$ is globally generic with respect to $\psi$. Then:*

1. *There is a global functorial lift $\Pi$ of $\sigma$ to $GL_{2(n-r)+1}(\mathbb{A})$, where $\Pi$ is an automorphic representation of $GL_{2(n-r)+1}(\mathbb{A})$ such that there exists a standard parabolic subgroup $P_d = P_{(d_1,d_2,\ldots,d_m)}$ of $GL_{2(n-r)+1}$, where $d_1 + d_2 + \ldots + d_m = 2(n-r)+1$, and cuspidal automorphic representations $\Pi_j$ of $GL_{d_j}(\mathbb{A})$ satisfying the following:*
   - $\Pi \cong \mathrm{Ind}_{P_d(\mathbb{A})}^{GL_{2(n-r)+1}(\mathbb{A})} (\Pi_1 \otimes \Pi_2 \otimes \ldots \otimes \Pi_m)$,
   - *each $\Pi_j$ is selfdual, i.e. $\widetilde{\Pi}_j \cong \Pi_j$, where $\widetilde{\Pi}_j$ denotes the contragredient representation of $\Pi_j$,*
   - *$\Pi_j \ncong \Pi_{j'}$ for $j \neq j'$, i.e. $\Pi_j$ are pairwise non–isomorphic,*

- *the symmetric square L–function* $L(s,\Pi_j,Sym^2)$ *has a simple pole at* $s=1$ *for all* $j=1,\dots,m$,
- *the central character of* $\Pi$ *is trivial.*

(2) *The Rankin–Selberg L–function for* $\tau$ *and* $\sigma$ *equals*

$$L(s,\tau\times\sigma)=\prod_{j=1}^{m}L(s,\tau\times\Pi_j),$$

*where the L–functions on the right hand side are the Rankin–Selberg ones for* $\tau$ *and* $\Pi_j$.

(3) *The local normalized intertwining operator* $N(\lambda_s,\pi_v,w_0)$ *is holomorphic and non–vanishing for* $s\geq 0$, *and thus the possible poles of the standard intertwining operator* $M(\lambda_s,\pi,w_0)$ *for* $s\geq 0$ *coincide with the poles of the global normalizing factor which is given by*

$$r(\lambda_s,\pi,w_0)=\begin{cases}\frac{L(s,\tau\times\sigma)}{L(1+s,\tau\times\sigma)\varepsilon(s,\tau\times\sigma)}\cdot\frac{L(2s,\tau,\wedge^2)}{L(1+2s,\tau,\wedge^2)\varepsilon(2s,\tau,\wedge^2)}, & \text{for } r<n,\\ \frac{L(s,\tau)}{L(1+s,\tau)\varepsilon(s,\tau)}\cdot\frac{L(2s,\tau,\wedge^2)}{L(1+2s,\tau,\wedge^2)\varepsilon(2s,\tau,\wedge^2)}, & \text{for } r=n,\end{cases}$$

*where* $L(s,\tau,\wedge^2)$ *is the exterior square L–function, and for* $r=1$ *we make a convention that* $L(s,\tau,\wedge^2)\equiv 1$ *for any Hecke character* $\tau$ *of* $\mathbb{Q}^\times\backslash\mathbb{I}$.

*All these L–functions are defined as the product over all places of the local ones, which are defined in* [37] *for generic representations.*

*Proof.* All the assertions are given in [5]. The first one is Theorem 7.2 of [5] describing the image of the global functorial lift of the globally generic representation of a split symplectic group. In fact, that image was already described in [9] and [39] before its construction in [5]. The second assertion follows from the proof of Lemma 7.1 of [5]. The claim on the local normalized intertwining operator in the third assertion is Theorem 11.1 of [5], and the rest follows from (5.3). The formula for the global normalizing factor $r(\lambda_s,\pi,w_0)$ is given in [36]. □

*Remark* 5.2. The non–vanishing of $N(\lambda_s,\pi_v,w_0)$ at $s=s_0$ means that its image is non–trivial, i.e. there is a section $f_{s,v}\in I(\lambda_s,\pi_v)$ such that $N(\lambda_{s_0},\pi_v,w_0)f_{s_0,v}\not\equiv 0$. This assures that the residues of the Eisenstein series in Theorems 5.6 and 5.7 below are non–trivial.

5.4. **Analytic properties of automorphic L–functions.** Next we recall the analytic properties of the L–functions appearing in the normalizing factors.

**Theorem 5.3.** *Let* $r$ *and* $r'$ *be positive integers. Let* $\tau$ *and* $\tau'$ *be cuspidal automorphic representations of* $GL_r(\mathbb{A})$ *and* $GL_{r'}(\mathbb{A})$, *respectively. We assume, as explained above, that* $\tau$ *and* $\tau'$ *are normalized to be trivial on* $A_{GL_r}(\mathbb{R})^+$ *and* $A_{GL_{r'}}(\mathbb{R})^+$. *Then:*

(1) *If* $r=1$, *i.e.* $\tau$ *is a Hecke character of* $\mathbb{Q}^\times\backslash\mathbb{I}$, *then the Hecke L–function* $L(s,\tau)$ *is entire if* $\tau$ *is non–trivial, while it has simple poles at* $s=0$ *and* $s=1$ *and is holomorphic elsewhere if* $\tau$ *is trivial.*

(2) *If* $r>1$, *then the principal L–function* $L(s,\tau)$ *is entire.*

(3) *If either* $r\neq r'$, *or* $r=r'$ *and* $\tau\not\cong\widetilde{\tau}'$, *then the Rankin–Selberg L–function* $L(s,\tau\times\tau')$ *is entire, while if* $r=r'$ *and* $\tau\cong\widetilde{\tau}'$, *then it has simple poles at* $s=0$ *and* $s=1$ *and is holomorphic elsewhere.*

(4) *If* $r>1$ *and* $\tau$ *is not selfdual, then the exterior square L–function* $L(s,\tau,\wedge^2)$ *is entire, while if* $r>1$ *and* $\tau$ *is selfdual, then it is holomorphic for* $s>1$ *and* $s<0$, *and has simple poles at* $s=0$ *and* $s=1$ *if and only if the symmetric square L–function* $L(s,\tau,Sym^2)$ *is holomorphic at* $s=0$ *and* $s=1$.

(5) *All the L–functions involved are non–zero for* $Re(s)\geq 1$ *and* $Re(s)\leq 0$.

*Proof.* Most of the analytic properties of the theorem are well–known. Property (1) is obtained in [41], and (2) and (3) follow from the integral representations for the L–functions developed in [14], [15], [16]. The property (5), i.e. non–vanishing for $Re(s) \geq 1$ and $Re(s) \leq 0$ of all the L–functions involved, follows from [36]. The analytic properties in that region for the exterior and symmetric square L–functions follow from the relation $L(s, \tau \times \tau) = L(s, \tau, \wedge^2)L(s, \tau, Sym^2)$ and properties (3) and (5). The holomorphy of $L(s, \tau, \wedge^2)$ inside the strip $0 < s < 1$ for non–selfdual $\tau$ follows from section IV.3.12 of [28], thus giving property (4). □

5.5. **Residues of Eisenstein series.** Before discussing the poles of the Eisenstein series we introduce some more notation. We consider the local normalized intertwining operator $N(\lambda_{s_0}, \pi_v, w_0)$ acting on the induced representation $I(\lambda_{s_0}, \pi_v)$, where either $s_0 = 1/2$, or $r < n$ and $s_0 = 1$. Let $W(\lambda_{s_0}, \pi_v) \subset I(\lambda_{s_0}, \pi_v)$ denote its kernel, and $J(\lambda_{s_0}, \pi_v)$ its image.

**Proposition 5.4.** *Let either $s_0 = 1/2$, or $r < n$ and $s_0 = 1$. Let $\pi_\infty \cong \tau_\infty \otimes \sigma_\infty$ be the Archimedean local component of a globally generic (with respect to $\psi$) cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$. Assume that $\tau_\infty$ is tempered. Then, the image $J(\lambda_{s_0}, \pi_\infty)$ of the local normalized intertwining operator $N(\lambda_{s_0}, \pi_\infty, w_0)$ is irreducible.*

*Proof.* Since $\tau_\infty$ is tempered, there are a standard parabolic $\mathbb{Q}$–subgroup $P_{(r_1,r_2,\ldots,r_k)}$ of $GL_r$, and irreducible unitary square–integrable representations $\delta_i$ of $GL_{r_i}(\mathbb{R})$ (thus $r_i = 1$ or $r_i = 2$) such that $\tau_\infty$ is the fully induced representation

$$\tau_\infty \cong \mathrm{Ind}_{P_{(r_1,r_2,\ldots,r_k)}(\mathbb{R})}^{GL_r(\mathbb{R})} \left(\delta_1 \otimes \delta_2 \otimes \ldots \otimes \delta_k\right).$$

On the other hand, with regard to the representation $\sigma_\infty$ of $Sp_{n-r}(\mathbb{R})$, in Section 10 of [5] a bound on the exponents of the local components of a globally generic cuspidal automorphic representation is given. More precisely, following the description of the generic unitary dual given in [23], there are

- a standard parabolic $\mathbb{Q}$–subgroup $P_{(r'_1,r'_2,\ldots,r'_l)}$ of $Sp_{n-r}$, where $r'_1 + r'_2 + \ldots + r'_l \leq n - r$,
- irreducible unitary square integrable representations $\delta'_j$ of $GL_{r'_j}(\mathbb{R})$ (thus $r'_j = 1$ or $r'_j = 2$),
- an irreducible tempered representation $\sigma_0$ of $Sp_{r_0}(\mathbb{R})$, where $r_0 = n - r - \sum_{j=1}^{l} r'_j$,
- and real exponents $t_j$, where $1/2 > t_1 \geq t_2 \geq \ldots \geq t_l > 0$,

such that $\sigma_\infty$ is the fully induced representation

$$\sigma_\infty \cong \mathrm{Ind}_{P_{(r'_1,r'_2,\ldots,r'_l)}(\mathbb{R})}^{Sp_{n-r}(\mathbb{R})} \left(\delta'_1 |\det|^{t_1} \otimes \delta'_2 |\det|^{t_2} \otimes \ldots \otimes \delta'_l |\det|^{t_l} \otimes \sigma_0\right).$$

By induction in stages, the induced representation $I(\lambda_{s_0}, \pi_\infty)$ on which $N(\lambda_{s_0}, \pi_\infty, w_0)$ acts is the fully induced representation

$$I(\lambda_{s_0}, \pi_\infty) \cong \mathrm{Ind}_{Q(\mathbb{R})}^{Sp_n(\mathbb{R})} \left(\delta_1 |\det|^{s_0} \otimes \ldots \otimes \delta_k |\det|^{s_0} \otimes \delta'_1 |\det|^{t_1} \otimes \ldots \otimes \delta'_l |\det|^{t_l} \otimes \sigma_0\right),$$

where $Q = P_{(r_1,\ldots,r_k,r'_1,\ldots,r'_l)}$ is a standard parabolic $\mathbb{Q}$–subgroup of $Sp_n$. Denote by $\delta$ the representation of the Levi factor $L_Q(\mathbb{R})$ of $Q(\mathbb{R})$ appearing on the right hand side.

Let $\underline{s} = (s_0, \ldots, s_0, t_1, t_2, \ldots, t_l) \in \check{\mathfrak{a}}_{Q,\mathbb{C}}$. Observe that $s_0 > t_1 \geq t_2 \geq \ldots \geq t_l > 0$, i.e. $\underline{s}$ satisfies the condition of the Langlands classification for $Sp_n(\mathbb{R})$ (see [22]). Let $w_l$ be the longest element in the Weyl group $W_{L_r}$ of the Levi factor $L_r$ modulo the Weyl group $W_{L_Q}$ of the Levi $L_Q$ of $Q$. Then the local normalized intertwining operator $N(\underline{s}, \delta, w_l)$ is an isomorphism because it is in fact an intertwining operator for the group $L_r(\mathbb{R})$ acting on the irreducible induced representation. Hence, by the decomposition of local intertwining operators according to a decomposition of the Weyl group element, the image $J(\lambda_{s_0}, \pi_v)$ is isomorphic to the image of the composition

$$N(\lambda_s, \pi_v, w_0) N(\underline{s}, \delta, w_l) = N(\underline{s}, \delta, w_0 w_l).$$

However, $w_0 w_l$ is the longest element of the Weyl group $W$ of $Sp_n$ modulo the Weyl group $W_{L_Q}$, and $\underline{s}$ satisfies the inequality of the Langlands classification. Therefore, the image of $N(\underline{s}, \delta, w_0 w_l)$ is irreducible by the Langlands classification. Thus $J(\lambda_{s_0}, \pi_v)$ is indeed irreducible. □

*Remark* 5.5. For any place $v$, if we assume that $\tau_v$ is tempered, the proof of the Lemma applies showing $J(\lambda_{s_0}, \pi_v)$ is irreducible. However, since the Ramanujan conjecture for cuspidal automorphic representations of $GL_r(\mathbb{A})$ is not proved, the assumption of temperedness might not be satisfied. It is not difficult to see, using the theory of $R$ groups of [17], [10] (see also [23]), that there are non–tempered unitary generic representations of $GL_r(\mathbb{Q}_v)$ such that $J(\lambda_{s_0}, \pi_v)$ is not irreducible.

Nevertheless, the cuspidal automorphic representations of $GL_r(\mathbb{A})$ having a non–tempered local component at an Archimedean place are of no interest in the application to automorphic cohomology. The reason is that only cuspidal representations having tempered $\tau_v$ may give a non–trivial cohomology class ([32, Section §3]).

**Theorem 5.6** (Case $r = n$)**.** *Let $r = n$, $\pi \cong \tau$ as above, and $s_0 \geq 1/2$.*

(1) *The Eisenstein series $E_{P_n}^{Sp_n}(f, \lambda_s)$ is holomorphic at $s = s_0$ unless $s_0 = 1/2$, $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau) \neq 0$. In this case, the map*

$$f \cdot e^{\langle H_{P_n}(\cdot), \lambda_{s_0} + \rho_{P_n} \rangle} \mapsto E_{P_n}^{Sp_n}(f, \lambda_{s_0})$$

*is an embedding of the induced representation $I(\lambda_{s_0}, \tau)$ into the space of automorphic forms on $Sp_n(\mathbb{Q}) \backslash Sp_n(\mathbb{A})$.*

(2) *Moreover, if $s_0 = 1/2$, $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau) \neq 0$, but $f = \otimes_v f_v$ has at least one local component $f_v$ in the kernel $W(\lambda_{1/2}, \tau_v)$ of the local normalized intertwining operator $N(\lambda_{1/2}, \tau_v, w_0)$, then the Eisenstein series $E_{P_n}^{Sp_n}(f, \lambda_s)$ is holomorphic at $s = s_0 = 1/2$ as well.*

(3) *Finally, if $s_0 = 1/2$, $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau) \neq 0$, and $f = \otimes_v f_v$ is such that for all places $v$ its local component $f_v$ is not in the kernel $W(\lambda_{1/2}, \tau_v)$ of the local normalized intertwining operator $N(\lambda_{1/2}, \tau_v, w_0)$, then the Eisenstein series $E_{P_n}^{Sp_n}(f, \lambda_s)$ has a simple pole at $s = s_0 = 1/2$. The map*

$$f \cdot e^{\langle H_{P_n}(\cdot), \lambda_{1/2} + \rho_{P_n} \rangle} \mapsto (s - 1/2) E_{P_n}^{Sp_n}(f, \lambda_s)\Big|_{s=1/2}$$

*is an intertwining of the induced representation $I(\lambda_{1/2}, \tau)$ and the space of automorphic forms on $Sp_n(\mathbb{Q}) \backslash Sp_n(\mathbb{A})$. Its image is non–trivial, isomorphic to $J(\lambda_{1/2}, \tau) \cong \otimes_v J(\lambda_{1/2}, \tau_v)$, and consists of square integrable automorphic forms.*

**Theorem 5.7** (Case $r < n$)**.** *Let $r < n$, $\pi \cong \tau \otimes \sigma$ as above, and $s_0 \geq 1/2$.*

(1) *The Eisenstein series $E_{P_r}^{Sp_n}(f, \lambda_s)$ is holomorphic at $s = s_0$ unless*
- *either $s_0 = 1/2$, $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau \times \Pi_j) \neq 0$ for all $\Pi_j$ appearing in the global functorial lift of $\sigma$,*
- *or $s_0 = 1$, and $\tau \cong \Pi_j$ for some $\Pi_j$ appearing in the global functorial lift of $\sigma$.*

*In this case, the map*

$$f \cdot e^{\langle H_{P_r}(\cdot), \lambda_{s_0} + \rho_{P_r} \rangle} \mapsto E_{P_r}^{Sp_n}(f, \lambda_{s_0})$$

*is an embedding of the induced representation $I(\lambda_{s_0}, \pi)$ into the space of automorphic forms on $Sp_n(\mathbb{Q}) \backslash Sp_n(\mathbb{A})$.*

(2) *Moreover, if*

- *either* $s_0 = 1/2$, $\tau$ *is selfdual,* $L(s, \tau, \wedge^2)$ *has a pole at* $s = 1$, *and* $L(1/2, \tau \times \Pi_j) \neq 0$ *for all* $\Pi_j$, *but* $f = \otimes_v f_v$ *has at least one local component* $f_v$ *in the kernel* $W(\lambda_{1/2}, \pi_v)$ *of the local normalized intertwining operator* $N(\lambda_{1/2}, \pi_v, w_0)$,
- *or* $s_0 = 1$ *and* $\tau \cong \Pi_j$ *for some* $\Pi_j$, *but* $f = \otimes_v f_v$ *has at least one local component* $f_v$ *in the kernel* $W(\lambda_1, \pi_v)$ *of the local normalized intertwining operator* $N(\lambda_1, \pi_v, w_0)$,

*then the Eisenstein series* $E_{P_r}^{Sp_n}(f, \lambda_s)$ *is holomorphic at* $s = s_0$ *as well.*

(3) *Finally, if*

- *either* $s_0 = 1/2$, $\tau$ *is selfdual,* $L(s, \tau, \wedge^2)$ *has a pole at* $s = 1$, *and* $L(1/2, \tau \times \Pi_j) \neq 0$ *for all* $\Pi_j$, *and* $f = \otimes_v f_v$ *is such that for all places* $v$ *its local component* $f_v$ *is not in the kernel* $W(\lambda_{1/2}, \pi_v)$ *of the local normalized intertwining operator* $N(\lambda_{1/2}, \pi_v, w_0)$,
- *or* $s_0 = 1$, $\tau \cong \Pi_j$ *for some* $\Pi_j$, *and* $f = \otimes_v f_v$ *is such that for all places* $v$ *its local component* $f_v$ *is not in the kernel* $W(\lambda_1, \pi_v)$ *of the local normalized intertwining operator* $N(\lambda_1, \pi_v, w_0)$,

*then the Eisenstein series* $E_{P_r}^{Sp_n}(f, \lambda_s)$ *has a simple pole at* $s = s_0$. *The map*

$$f \cdot e^{\langle H_{P_r}(\cdot), \lambda_{s_0} + \rho_{P_r} \rangle} \mapsto (s - s_0) E_{P_r}^{Sp_n}(f, \lambda_s)\Big|_{s=s_0}$$

*is an intertwining of the induced representation* $I(\lambda_{s_0}, \pi)$ *and the space of automorphic forms on* $Sp_n(\mathbb{Q})\backslash Sp_n(\mathbb{A})$. *Its image is non–trivial, isomorphic to* $J(\lambda_{s_0}, \pi) \cong \otimes_v J(\lambda_{s_0}, \pi_v)$, *and consists of square–integrable automorphic forms.*

*Proof.* We prove Theorems 5.6 and 5.7. By claim (3) of Theorem 5.1 and the expression (5.1) for the constant term of the Eisenstein series which relates the poles of the Eisenstein series to the poles of the standard intertwining operator, the poles of the Eisenstein series $E_{P_r}^{Sp_n}(f, \lambda_s)$ at $s = s_0 \geq 0$ coincide with those of the normalizing factor $r(\lambda_s, \pi, w_0)$, unless there is a place $v$ where $f_v$ is in the kernel of the local normalized intertwining operator. Then, claim (2) of Theorem 5.1, and the analytic properties of L–functions of Theorem 5.3, imply the conditions for the pole given in the theorems. The description of the spaces of automorphic forms so obtained follows by looking at the expression (5.1) for the constant term. The space $J(\lambda_{s_0}, \pi)$ is non–trivial because $J(\lambda_{s_0}, \pi_v)$ is non–trivial for all places $v$ due to the non–vanishing of the normalized intertwining operator $N(\lambda_{s_0}, \pi_v, w_0)$ in claim (3) of Theorem 5.1 (see also Remark 5.2). □

*Remark* 5.8. Observe that if $r < n$, the two poles $s_0 = 1/2$ and $s_0 = 1$ cannot both occur for a fixed selfdual representation $\tau$. Indeed, for the pole at $s_0 = 1/2$ it is necessary that $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, while for the pole at $s_0 = 1$ it is necessary that $\tau \cong \Pi_j$ for some $j$, which implies $L(s, \tau, Sym^2) = L(s, \Pi_j, Sym^2)$ has a pole at $s = 1$ by claim (1) of Theorem 5.1. However, by (4) of Theorem 5.3, the exterior and symmetric square L–functions $L(s, \tau, \wedge^2)$ and $L(s, \tau, Sym^2)$ cannot both have a pole at $s = 1$.

*Remark* 5.9. In both Theorems we consider only $s_0 \geq 1/2$ because the condition for the non–vanishing of the cohomology studied in Section 6 excludes the strip $0 \leq s_0 < 1/2$ as possible evaluation points. However, the proof of the Theorems applies for $s_0 > 0$ up to the analytic properties of the exterior square L–function $L(s, \tau, \wedge^2)$ inside the strip $0 < s < 1$. The holomorphy of that L–function inside $0 < s < 1$ would follow from Arthur's conjectural description, given in section 30 of [1], of the discrete spectrum for $\mathbb{Q}$–split connected classical groups.

**Corollary 5.10.** *Let* $\mathcal{L}_{E,\{P_r\},\phi,V_\pi}^{s_0 \geq 1/2}$ *be the subspace of* $\mathcal{L}_{E,\{P_r\},\phi,V_\pi}$ *spanned by the square integrable automorphic forms which are obtained as the residues at* $s_0 \geq 1/2$ *of the Eisenstein series* $E_{P_r}^{Sp_n}(f, \lambda_{s_0})$ *attached to* $V_\pi$. *In the case* $r = n$, *we use the notation* $\mathcal{L}_{E,\{P_n\},\phi}^{s_0 \geq 1/2}$.

(1) *In the case $r = n$, the space $\mathcal{L}^{s_0\geq 1/2}_{E,\{P_n\},\phi}$ is non–trivial if and only if $\tau$ is selfdual, $L(s,\tau,\wedge^2)$ has a pole at $s = 1$, and $L(1/2,\tau) \neq 0$. If non–trivial, $\mathcal{L}^{s_0\geq 1/2}_{E,\{P_n\},\phi}$ is spanned by the residues of the Eisenstein series attached to $\tau$ at the pole $s_0 = 1/2$, and it is isomorphic to the image $J(\lambda_{1/2},\tau)$ of the normalized intertwining operator $N(\lambda_{1/2},\tau,w_0)$.*

(2) *In the case $r < n$, the space $\mathcal{L}^{s_0\geq 1/2}_{E,\{P_r\},\phi,V_\pi}$ is non–trivial if and only if*
   (a) *either $\tau$ is selfdual, $L(s,\tau,\wedge^2)$ has a pole at $s = 1$, and $L(1/2,\tau\times\Pi_j) \neq 0$ for all $\Pi_j$ appearing in the lift of $\sigma$ (see Theorem 5.1); in this case $\mathcal{L}^{s_0\geq 1/2}_{E,\{P_r\},\phi,V_\pi}$ is spanned by the residues of the Eisenstein series attached to $V_\pi$ at the pole $s_0 = 1/2$, and it is isomorphic to the image $J(\lambda_{1/2},\pi)$ of the normalized intertwining operator $N(\lambda_{1/2},\tau,w_0)$,*
   (b) *or $\tau$ is selfdual, and $\tau \cong \Pi_j$ for some $\Pi_j$ appearing in the lift of $\sigma$; in this case $\mathcal{L}^{s_0\geq 1/2}_{E,\{P_r\},\phi,V_\pi}$ is spanned by the residues of the Eisenstein series attached to $V_\pi$ at the pole $s_0 = 1$, and it is isomorphic to the image $J(\lambda_1,\pi)$ of the normalized intertwining operator $N(\lambda_1,\pi,w_0)$.*

## 6. Evaluation Points and Non–vanishing Conditions

Given an associate class $\{P_r\} \in \mathcal{C}$ of maximal parabolic $\mathbb{Q}$–subgroups in $Sp_n$, we now analyze the actual construction of cohomology classes in the corresponding summand $H^*(\mathfrak{sp}_n, K_\mathbb{R}, \mathcal{A}_{E,\{P_r\}}\otimes_\mathbb{C} E)$ of the Eisenstein cohomology $H^*_{Eis}(Sp_n, E)$. By Theorem 3.1, the latter space decomposes as

$$H^*(\mathfrak{sp}_n, K_\mathbb{R}, \mathcal{A}_{E,\{P_r\}} \otimes_\mathbb{C} E) = \bigoplus_{\phi\in\Phi_{E,\{P_r\}}} H^*(\mathfrak{sp}_n, K_\mathbb{R}, \mathcal{A}_{E,\{P_r\},\phi} \otimes_\mathbb{C} E),$$

where the sum ranges over the set $\Phi_{E,\{P_r\}}$ of classes $\phi = \{\phi_Q\}_{Q\in\{P_r\}}$ of associate irreducible cuspidal automorphic representations of the Levi components of elements of $\{P_r\}$.

Suppose $\pi \in \phi_{P_r}$ is an irreducible cuspidal automorphic representation of the Levi component $L_r(\mathbb{A})$ on the subspace $V_\pi$ of the space of cusp forms on $L_r(\mathbb{A})$. By carrying through the construction of residues or derivatives of Eisenstein series attached to $(\pi, V_\pi)$ (as in [24], Section 3), the corresponding contribution to $H^*(\mathfrak{sp}_n, K_\mathbb{R}, \mathcal{A}_{E,\{P_r\},\phi} \otimes_\mathbb{C} E)$ is embodied in the cohomology

$$H^*\left(\mathfrak{sp}_n, K_\mathbb{R}; \mathrm{Ind}_{P_r(\mathbb{A}_f)}^{Sp_n(\mathbb{A}_f)} \mathrm{Ind}_{(\mathfrak{sp}_n\cap\mathfrak{p}_r, K_\mathbb{R}\cap L_r(\mathbb{R}))}^{(\mathfrak{sp}_n, K_\mathbb{R})} \left(V_\pi \otimes E \otimes S(\check{\mathfrak{a}}_{P_r}^{Sp_n})\right)\right),$$

where $S(\check{\mathfrak{a}}_{P_r}^{Sp_n})$ is the symmetric algebra of $\check{\mathfrak{a}}_{P_r}^{Sp_n}$ with the $(\mathfrak{sp}_n, K_\mathbb{R})$–module structure as defined on page 218 of [7] (see also Section 3.1 of [24]).

Using Frobenius reciprocity, the study of this space is reduced to an analysis of the $Sp_n(\mathbb{A}_f)$–module

$$\mathrm{Ind}_{P_r(\mathbb{A}_f)}^{Sp_n(\mathbb{A}_f)} H^*\left(\mathfrak{l}_r, K_\mathbb{R}\cap L_r(\mathbb{R}); V_\pi \otimes H^*(\mathfrak{n}_r, E) \otimes S(\check{\mathfrak{a}}_{P_r}^{Sp_n})\right). \tag{6.1}$$

Following Kostant ([18], Thm. 5.13), the Lie algebra cohomology $H^*(\mathfrak{n}_r, E)$ of $\mathfrak{n}_r$ with coefficients in the irreducible representation $(\nu, E)$ of $Sp_n(\mathbb{C})$ is given as a $(\mathfrak{l}_r, K_\mathbb{R}\cap L_r(\mathbb{R}))$–module as the sum

$$H^*(\mathfrak{n}_r, E) = \bigoplus_{w\in W^{P_r}} F_{\mu_w}$$

where the sum ranges over $w$ in the set $W^{P_r}$ of the minimal coset representatives for the left cosets of $W$ modulo the Weyl group $W_{P_r}$ of the Levi factor $L_r$ of $P_r$, and $F_{\mu_w}$ denotes the irreducible finite–dimensional $(\mathfrak{l}_r, K_\mathbb{R}\cap L_r(\mathbb{R}))$–module of highest weight

$$\mu_w = w(\Lambda + \rho_{P_0}) - \rho_{P_0}, \tag{6.2}$$

where $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$ is the highest weight of $(\nu, E)$. The weights $\mu_w$ are all dominant and distinct and, given a fixed degree $q$, only the weights $\mu_w$ with length $\ell(w) = q$ occur in the decomposition of $H^q(\mathfrak{n}_r, E)$ into irreducibles. As in [30, Section 3.2], we call a cohomology class in (6.1) which gives rise to a non–trivial class in

$$H^*\left(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w}\right)$$

a class of type $(\pi, w)$, $w \in W^{P_r}$. If the infinitesimal character $\chi_{\pi_\infty}$ of the Archimedean component $\pi_\infty$ of $\pi$ does not coincide with the infinitesimal character of the representation contragredient to $F_{\mu_w}$, the cohomology space $H^*\left(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w}\right)$ vanishes, that is, there are no classes of type $(\pi, w)$.

Moreover, if $F_{\mu_w}$ is not isomorphic to its complex conjugate contragredient $\overline{F}^*_{\mu_w}$, then the cohomology space $H^*\left(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w}\right) = (0)$, since this condition implies that the complex contragredient of $F_{\mu_w}$ and $V_\pi$ have distinct infinitesimal character. Following [2, Section §1], $F_{\mu_w} \ncong \overline{F}^*_{\mu_w}$ is equivalent to the condition that $-w_{l,L_r}(\mu_w\big|_{\check{\mathfrak{a}}^{P_r}_{P_0}})$ is distinct from $\mu_w\big|_{\check{\mathfrak{a}}^{P_r}_{P_0}}$, where $w_{l,L_r}$ is the longest element in the Weyl group $W_{P_r}$ of the Levi component $L_r$. We recall that the transformation $-w_{l,L_r}$ maps the highest weight of an irreducible $\mathfrak{l}_{r,\mathbb{C}}$–module into that of the contragredient one.

Suppose there is a non–trivial cohomology class of type $(\pi, w)$, $w \in W^{P_r}$. In order to understand the cohomological contribution of the corresponding Eisenstein series $E^{Sp_n}_{P_r}(f, \lambda_s)$ or a residue of such in $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}, \mathcal{A}_{E,\{P_r\},\phi} \otimes_{\mathbb{C}} E)$, following [30, Corollary 3.5], we have to analyze the analytic behaviour of $E^{Sp_n}_{P_r}(f, \lambda_s)$ at the point

$$\lambda_w = -w(\Lambda + \rho_{P_0})\Big|_{\check{\mathfrak{a}}_{P_r}}. \tag{6.3}$$

This evaluation point is real and uniquely determined by the datum $(\pi, w)$. It only depends on $w$ and the highest weight $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$. As a consequence of the description of the space $\mathcal{A}_{E,\{P_r\},\phi}$ of automorphic forms in Section 1.3 of [8], only the points $\lambda_w$ with $\langle \lambda_w, \alpha_r^\vee \rangle \geq 0$ matter in our analysis. In other words, it suffices to consider only the evalution points $\lambda_w$ such that in the basis $\widetilde{\rho}_{P_r}$ of $\check{\mathfrak{a}}_{P_r}$ we have $\lambda_w = \lambda_{s_w} = \widetilde{\rho}_{P_r} \otimes s_w$ with $s_w \geq 0$.

In the following, under the assumption that $H^*\left(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w}\right)$ is non–trivial for a given $\{P_r\} \in \mathcal{C}$, and a pair $(\pi, w)$, we make explicit the two necessary conditions this assumption implies by the discussion above, namely:

$$-w_{l,L_r}\left(\mu_w\Big|_{\check{\mathfrak{a}}^{P_r}_{P_0}}\right) = \mu_w\Big|_{\check{\mathfrak{a}}^{P_r}_{P_0}}, \tag{6.4}$$

and

$$\chi_{\pi_\infty} = -w\left(\Lambda + \rho_{P_0}\right)\Big|_{\check{\mathfrak{a}}^{P_r}_{P_0}}. \tag{6.5}$$

In a next step, given a non–trivial cohomology class of type $(\pi, w)$, $w \in W^{P_r}$, we determine the corresponding evaluation point $\lambda_w = \widetilde{\rho}_{P_r} \otimes s_w$. Finally, this allows us to decide for which minimal coset representatives $w \in W^{P_r}$, the corresponding point $\lambda_w$ takes the value $s_w = 1/2$ or $s_w = 1$. In Section 7 it will turn out that the condition (6.4) is never satisfied for $w \in W^{P_r}$ such that $0 \leq s_w < 1/2$. Therefore, in view of the results in Section 5 concerning the analytic behaviour of the Eisenstein series in question, the evaluation points $s_w = 1/2$ and $s_w = 1$ (the latter in the case of $P_r$ with $r < n$) are decisive for the eventual construction of residues of Eisenstein series and related cohomology classes.

For later use we introduce the following notation. As before, let $(\nu, E)$ be an irreducible representation of $Sp_n(\mathbb{C})$ with highest weight $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$. If we write $\Lambda = \sum_{i=1}^n \lambda_i e_i$, with $e_i$ the projection

of $L_0$ to its $i^{th}$ component as in Section 4 then all $\lambda_i$ are integers and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. On the other hand, if we write $\Lambda = \sum_{i=1}^{n} c_i\omega_i$, where $\omega_i = \sum_{j=1}^{i} e_j$ is the $i^{\text{th}}$ fundamental weight, then all $c_i$ are non–negative integers. The relationship between the two expressions for $\Lambda$ is given by

$$\lambda_i = \sum_{j=i}^{n} c_j \quad \text{and} \quad c_i = \begin{cases} \lambda_i - \lambda_{i+1}, & \text{for } i = 1, 2, \ldots, n-1, \\ \lambda_n, & \text{for } i = n. \end{cases}$$

## 7. Non–vanishing Cohomology for Maximal Parabolic Subgroups

Let $(\nu, E)$ be a finite–dimensional irreducible representation of $Sp_n(\mathbb{C})$ of highest weight $\Lambda \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$. For $r = 1, \ldots, n$, let $P_r = L_r N_r$ be the Levi decomposition of the standard parabolic $\mathbb{Q}$–subgroup of $Sp_n$ corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of the set of simple roots. Recall that we assume $n \geq 2$.

Let $\pi \cong \tau \otimes \sigma$ be a cuspidal automorphic representation of $L_r(\mathbb{A}) \cong GL_r(\mathbb{A}) \times Sp_{n-r}(\mathbb{A})$. In this section we explicitly determine, for $w \in W^{P_r}$ such that the corresponding evaluation point is $s_w = 1/2$ or $s_w = 1$, the two necessary conditions (6.4) and (6.5) implied by the assumption that the space $H^*(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w})$ is non–trivial. We also show that for $w \in W^{P_r}$ such that $0 \leq s_w < 1/2$ that space is trivial since the condition (6.4) is never satisfied.

7.1. **Action of elements of $W^{P_r}$.** The calculations in this section are based on Lemma 7.1 which gives the explicit formula for the action of $w \in W^{P_r}$ on $\check{\mathfrak{a}}_{P_0}$. This Lemma is a variant of [40, Lemma 4.3].

**Lemma 7.1.** *In the case of the maximal proper parabolic subgroup $P_r$ with $r = 1, \ldots, n$, there is a bijection*

$$(I, J) \mapsto w_{I,J}$$

*between the family $\mathcal{S}$ of all ordered pairs $(I, J)$ of disjoint subsets $I$ and $J$ of the set $S_n = \{1, 2, \ldots, n\}$ such that their union $I \cup J$ contains exactly $r$ elements, and the set $W^{P_r}$ of minimal coset representatives for $W_{P_r} \setminus W$. The bijection is defined as follows: let*

$$\begin{aligned} I &= \{i_1, i_2, \ldots, i_{|I|}\}, \text{ where } i_1 < i_2 < \ldots < i_{|I|}, \\ J &= \{j_1, j_2, \ldots, j_{|J|}\}, \text{ where } j_1 < j_2 < \ldots < j_{|J|}, \\ S_n \setminus (I \cup J) &= \{k_1, k_2, \ldots, k_{n-r}\}, \text{ where } k_1 < k_2 < \ldots < k_{n-r}, \end{aligned}$$

*where $|I|$ and $|J|$ denote the cardinality of $I$ and $J$, respectively. Then, $w_{I,J}$ is defined by its action on $e_1, e_2, \ldots, e_n \in X^*(A_0)$ as*

$$\begin{aligned} w_{I,J}(e_{i_{l_1}}) &= -e_{r+1-l_1}, \text{ for } l_1 = 1, 2, \ldots, |I|, \\ w_{I,J}(e_{j_{l_2}}) &= e_{l_2}, \text{ for } l_2 = 1, 2, \ldots, |J|, \\ w_{I,J}(e_{k_{l_3}}) &= e_{r+l_3}, \text{ for } l_3 = 1, 2, \ldots, n-r. \end{aligned}$$

*In other words,*

$$w_{I,J}\left(\sum_{l=1}^{n} s_l e_l\right) = \sum_{l_2=1}^{|J|} s_{j_{l_2}} e_{l_2} - \sum_{l_1=1}^{|I|} s_{i_{|I|+1-l_1}} e_{|J|+l_1} + \sum_{l_3=1}^{n-r} s_{k_{l_3}} e_{r+l_3},$$

*where $s_l \in \mathbb{C}$. In particular, in the case $r = n$, we have $J = S_n \setminus I$ for any pair $(I, J) \in \mathcal{S}$.*

*Proof.* The assignment $(I, J) \mapsto w_{I,J}$ obviously defines an injective map $\mathcal{S} \to W$. However, one needs to check that $w_{I,J} \in W^{P_r}$. By [18, Theorem 5.13], the set $W^{P_r}$ consists of all $w \in W$ such

that $w^{-1}(\Delta \setminus \{\alpha_r\}) \subset \Phi^+$, i.e. $w^{-1}(\alpha)$ is a positive root for all simple roots $\alpha$ in the subset $\Delta \setminus \{\alpha_r\}$ of $\Delta$ corresponding to the parabolic subgroup $P_r$. The action of $w_{I,J}^{-1}$ is given by

$$w_{I,J}^{-1}(e_l) = \begin{cases} e_{j_l}, & \text{for } l = 1, \dots, |J|, \\ -e_{i_{r+1-l}}, & \text{for } l = |J|+1, \dots, r \\ e_{k_{l-r}}, & \text{for } l = r+1, \dots, n. \end{cases}$$

Hence,

$$w_{I,J}^{-1}(\alpha_m) = \begin{cases} e_{j_m} - e_{j_{m+1}}, & \text{for } m = 1, \dots, |J|-1, \\ e_{j_{|J|}} + e_{i_{|I|}}, & \text{for } m = |J|, \\ e_{i_{r-m}} - e_{i_{r-m+1}}, & \text{for } m = |J|+1, \dots, r-1, \\ e_{k_{m-r}} - e_{k_{m-r+1}}, & \text{for } m = r+1, \dots, n-1, \\ 2e_{k_{n-r}}, & \text{for } m = n, \end{cases}$$

and all the roots on the right hand side are positive. Note that for $r = n$ the last two cases do not exist.

For surjectivity we prove that $\mathcal{S}$ and $W^{P_r}$ have the same cardinality. The number of ordered pairs $(I, J)$ of disjoint subsets of $S_n$ such that $I \cup J$ has exactly $r$ elements is counted as follows. In the first step we choose a subset of $r$ elements inside $S_n$ to be $I \cup J$. This step can be done in $\binom{n}{r}$ ways. Next, in the second step, we choose $I$ to be any subset of already chosen $I \cup J$. This step can be done in $2^r$ ways, since that is the number of subsets of a set of $r$ elements. Hence, there are $\binom{n}{r} \cdot 2^r$ ordered pairs $(I, J)$. On the other hand, the number of representatives in $W^{P_r}$ is obtained as the quotient of $|W| = n! \cdot 2^n$ and $|W_{P_r}| = r! \cdot (n-r)! \cdot 2^{n-r}$, which is the same. □

**Lemma 7.2.** *Let the notation be as in the previous Lemma. For $a \in S_n$, let*

$$S_{I,J}(a) = \{x \in S_n \setminus (I \cup J) \quad : \quad x < a\},$$

*and $m_{I,J}(a)$ the cardinality of $S_{I,J}(a)$. Then, the length of $w_{I,J}$ is given by*

$$\ell(w_{I,J}) = \sum_{i \in I} (n+1-i) + \sum_{j \in J} m_{I,J}(j) + |I|(n-r).$$

*In particular, for $r = n$, we have $\ell(w_{I,S_n \setminus I}) = \sum_{i \in I} (n+1-i)$.*

*Proof.* We compute the length of $w_{I,J}$ by writing its reduced decomposition in simple reflections. This is achieved in three steps.

In the first step we move the $e_i$ with $i \in I$ to the end and change the sign starting with the largest index in $I$. This step is in fact the only step in the $r = n$ case. For every $i \in I$ we apply $n - i$ simple transpositions and one sign change which gives the total of $\sum_{i \in I} (n+1-i)$ simple reflections.

In the second step we move the $e_j$ with $j \in J$ to the beginning, starting with the smallest index in $J$, but without changing their order. Since the elements of $I$ are already at the end, for every $j \in J$ one has to apply as many simple transpositions as there are indices smaller than $j$ which are neither in $I$ nor in $J$. That number is denoted by $m_{I,J}(j)$, and hence in this step we use $\sum_{j \in J} m_{I,J}(j)$ simple reflections.

Finally, in the third step, we move the $e_i$ with $i \in I$, without changing their order, from their position at the end to the places just after the elements $e_j$ of $J$ which occupy the first $|J|$ places. During this step we apply for every $i \in I$ as many simple transpositions as there are indices outside both $I$ and $J$ which is $n - r$. Hence, in this step we use $|I| \cdot (n-r)$ simple reflections.

Summing up the total number of simple reflections used in each step gives precisely the formula for the length given in the Lemma. □

7.2. **Evaluation point.**

**Lemma 7.3.** *Let $E$ be an irreducible representation of $Sp_n(\mathbb{C})$ with highest weight $\Lambda = \sum_{k=1}^{n} \lambda_k e_k \in \check{\mathfrak{a}}_{P_0,\mathbb{C}}$, where $\lambda_k$ are integers, and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Then*

$$\begin{aligned} -w_{I,J}\left(\Lambda + \rho_{P_0}\right) = & - \sum_{l_2=1}^{|J|} \left[\lambda_{j_{l_2}} + (n+1-j_{l_2})\right] e_{l_2} \\ & + \sum_{l_1=1}^{|I|} \left[\lambda_{i_{|I|+1-l_1}} + (n+1-i_{|I|+1-l_1})\right] e_{|J|+l_1} \\ & - \sum_{l_3=1}^{n-r} \left[\lambda_{k_{l_3}} + (n+1-k_{l_3})\right] e_{r+l_3}, \end{aligned}$$

*where the notation is as in Lemma 7.1.*

*Proof.* This is a straightforward computation using the formula for the action of $w_{I,J}$ given in Lemma 7.1. □

**Corollary 7.4.** *In the notation of Lemma 7.1, the evaluation point $\lambda_{w_{I,J}} = \lambda_{s_{w_{I,J}}}$ for $w_{I,J} \in W^{P_r}$ corresponds to the real number*

$$s_{w_{I,J}} = \frac{1}{r}\left[\sum_{i\in I}\lambda_i - \sum_{j\in J}\lambda_j - \sum_{i\in I} i + \sum_{j\in J} j + (|I|-|J|)(n+1)\right].$$

*Moreover, $r s_{w_{I,J}}$ is always an integer. In particular, if $s_{w_{I,J}} = 1/2$, then $r$ is necessarily even.*

*Proof.* The restriction to $\check{\mathfrak{a}}_{P_r}$ in the formula (6.3) for $\lambda_{w_{I,J}}$ is obtained by replacing every $e_k$ for $k = 1, 2, \ldots, r$ by $\frac{1}{r}\sum_{l=1}^{r} e_l = \frac{1}{r}\widetilde{\rho}_{P_r} \in \check{\mathfrak{a}}_{P_r}$, and every $e_k$ for $k = r+1, \ldots, n$ by 0. In other words, $s_{w_{I,J}}$ is the arithmetic mean of the first $r$ coefficients in the expression for $-w_{I,J}(\Lambda + \rho_{P_0})$ obtained in Lemma 7.3. These are in fact the coefficients appearing in the first two sums. Thus, a straightforward computation gives the formula. Finally, $r s_{w_{I,J}}$ is an integer because all the terms in the square bracket are integers. □

7.3. **Non–vanishing condition.**

**Lemma 7.5.** *Let $E$ be as in Lemma 7.3. Then, in the notation of Lemma 7.1,*

$$\mu_{w_{I,J}} = w_{I,J}\left(\Lambda + \rho_{P_0}\right) - \rho_{P_0} \in \check{\mathfrak{a}}_{P_0}$$

*is given by the formula*

$$\begin{aligned} \mu_{w_{I,J}} = & \sum_{l_2=1}^{|J|} \left[\lambda_{j_{l_2}} - j_{l_2} + l_2\right] e_{l_2} \\ & - \sum_{l_1=1}^{|I|} \left[\lambda_{i_{|I|+1-l_1}} + \left(n+1-i_{|I|+1-l_1}\right) + (n+1-|J|-l_1)\right] e_{|J|+l_1} \\ & + \sum_{l_3=1}^{n-r} \left[\lambda_{k_{l_3}} - k_{l_3} + r + l_3\right] e_{r+l_3}. \end{aligned}$$

*Proof.* This is a direct computation using Lemma 7.1. □

**Proposition 7.6.** *Let $E$ be of highest weight $\Lambda = \sum_{l=1}^{n} \lambda_l e_l$ as above. Let $w_{I,J} \in W^{P_r}$ be such that $s_{w_{I,J}} = 1/2$. Thus, by Corollary 7.4, $r$ is necessarily even. Let $\pi \cong \tau \otimes \sigma$ be a cuspidal automorphic representation of $L_r(\mathbb{A}) \cong GL_r(\mathbb{A}) \times Sp_{n-r}(\mathbb{A})$, where $\tau$ and $\sigma$ are cuspidal automorphic representations of $GL_r(\mathbb{A})$ and $Sp_{n-r}(\mathbb{A})$, respectively. Then, if a non–trivial cohomology class in $H^*(\mathfrak{l}_r, K_{\mathbb{R}} \cap L_r(\mathbb{R}); V_\pi \otimes H^*(\mathfrak{n}_r, E))$ of type $(\pi, w_{I,J})$ exists, the following holds:*

- *$w_{I,J}$ corresponds to a pair of disjoint subsets $(I, J)$ of the form*

$$\begin{aligned} I &= \{i_1, i_2, \dots, i_{r/2}\}, \\ J &= \{i_1 + 1, i_2 + 1, \dots, i_{r/2} + 1\}, \end{aligned}$$

  *where the form of $I$ and $J$ implies that neither of them contains a pair of consecutive integers,*
- *the coefficients of $\Lambda$ satisfy $\lambda_i = \lambda_{i+1}$ for all $i \in I$,*
- *the infinitesimal character $\chi_{\pi_\infty}$ of the infinite component $\pi_\infty$ of $\pi$ equals*

$$\begin{aligned} \chi_{\pi_\infty} &= \sum_{l=1}^{r/2} \Big[ -\left(\mu_l + (n + 1/2 - i_l)\right) e_l + \left(\mu_l + (n + 1/2 - i_l)\right) e_{r+1-l} \Big] \\ &- \sum_{l'=1}^{n-r} \left(\lambda_{k_{l'}} + (n + 1 - k_{l'})\right) e_{r+l'}, \end{aligned}$$

  *where $\mu_l = \lambda_{i_l} = \lambda_{i_l+1}$ for $l = 1, 2, \dots, r/2$.*

*In particular, for $r = n$, there is a unique $w_{I_0, S_n \setminus I_0} \in W^{P_n}$ satisfying those conditions. It corresponds to $I_0 = \{1, 3, \dots, n-1\}$. Here $n$ is necessarily even.*

*Proof.* We first make the non–vanishing condition (6.4) more explicit. Let $\sum_{l=1}^{n} s_l e_l \in \check{\mathfrak{a}}_{P_0}$. As in the proof of Corollary 7.4, its restriction to $\check{\mathfrak{a}}_{P_r}$ is just $\sum_{l=1}^{r} \overline{s} e_l = \overline{s} \widetilde{\rho}_{P_r} \in \check{\mathfrak{a}}_{P_r}$, where $\overline{s} = \frac{1}{r} \sum_{l=1}^{r} s_l$ is the arithmetic mean of the coefficients $s_1, s_2, \dots, s_r$. Note that here $\overline{s}$ is the arithmetic mean of the first $r$ coefficients. Hence, the restriction to $\check{\mathfrak{a}}_{P_0}^{P_r}$ equals

$$\sum_{l=1}^{r} (s_l - \overline{s})\, e_l + \sum_{k=r+1}^{n} s_k e_k.$$

Since $L_r \cong GL_r \times Sp_{n-r}$, the longest Weyl group element $w_{l,L_r} \in W_{P_r}$ acts as

$$w_{l,L_r} \left( \sum_{l=1}^{r} (s_l - \overline{s})\, e_l + \sum_{k=r+1}^{n} s_k e_k \right) = \sum_{l=1}^{r} (s_{r+1-l} - \overline{s})\, e_l - \sum_{k=r+1}^{n} s_k e_k.$$

Therefore, condition (6.4) is in fact

$$s_l + s_{r+1-l} = 2\overline{s} \tag{7.1}$$

for $l = 1, 2, \dots, r$. Observe that the condition is only on the first $r$ coefficients of $\mu_{w_{I,J}}$.

From this point on, let $s_l$ denote the coefficient of $e_l$ in the expression for $\mu_{w_{I,J}}$ in Lemma 7.5, and $\overline{s}$ the arithmetic mean of the first $r$ coefficients. Direct computation gives that

$$\overline{s} = \frac{1}{r} \left[ \sum_{j \in J} \lambda_j - \sum_{i \in I} \lambda_i + \sum_{i \in I} i - \sum_{j \in J} j + \frac{r(r+1)}{2} - 2|I|(n+1) \right],$$

which can be written in terms of the evaluation point $s_{w_{I,J}}$ using the formula of Corollary 7.4 as

$$\overline{s} = -\left( s_{w_{I,J}} + n - \frac{r-1}{2} \right).$$

In this Proposition we study the case $s_{w_{I,J}} = 1/2$. Hence,

$$2\overline{s} = -2n + r - 2.$$

We consider separately the three cases depending on the size of $I$. Recall that $r$ is even.

*Case 1:* $|I| < r/2$. In this case $|J| > r/2$, and hence there is an index $m \in S_n$ such that $m \leq |J|$ and $r + 1 - m \leq |J|$. Thus, the coefficients $s_m$ and $s_{r+1-m}$ are both in the first sum in the formula for $\mu_{w_{I,J}}$ of Lemma 7.5. Their sum equals

$$s_m + s_{r+1-m} = \lambda_{j_m} + \lambda_{j_{r+1-m}} - (j_m + j_{r+1-m}) + r + 1.$$

By (7.1) this sum should be equal to $2\overline{s} = -2n + r - 2$ which gives the condition

$$\lambda_{j_m} + \lambda_{j_{r+1-m}} + 2n + 3 = j_m + j_{r+1-m},$$

which is never satisfied since the left hand side is at least $2n + 3$, while the right hand side is not greater than $2n$.

*Case 2:* $|I| > r/2$. In this case $|J| < r/2$. Hence, for $l = 1, 2, \ldots, |J|$ the coefficient $s_l$ is in the first, and the coefficient $s_{r+1-l}$ in the second sum of the formula for $\mu_{w_{I,J}}$ of Lemma 7.5. Their sum equals

$$\begin{aligned} s_l + s_{r+1-l} &= (\lambda_{j_l} - j_l + l) - (\lambda_{i_l} + (n + 1 - i_l) + (n - r + l)) \\ &= -\left[(\lambda_{i_l} - \lambda_{j_l}) + (j_l - i_l)\right] - 2n + r - 1. \end{aligned}$$

By (7.1), this sum should be equal to $2\overline{s} = -2n + r - 2$, which gives

$$(\lambda_{i_l} - \lambda_{j_l}) + (j_l - i_l) = 1.$$

Recall that $i_l \neq j_l$, and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Hence, if $i_l > j_l$, then $\lambda_{i_l} - \lambda_{j_l} \leq 0$, and the condition is not satisfied since the left hand side is negative. If $i_l < j_l$, then $\lambda_{i_l} - \lambda_{j_l} \geq 0$, and the left hand side is strictly positive. It equals 1 if and only if $j_l = i_l + 1$ and $\lambda_{i_l} = \lambda_{j_l}$ for all $l = 1, 2, \ldots, n - |I|$. The subsets $I$ and $J$ are of the form

$$\begin{aligned} I &= \left\{i_1, i_2, \ldots, i_{|J|}, i_{|J|+1}, \ldots, i_{|I|}\right\}, \\ J &= \left\{i_1 + 1, i_2 + 1, \ldots, i_{|J|} + 1\right\}. \end{aligned}$$

We denote by

$$I_{\text{end}} = \left\{i_{|J|+1}, \ldots, i_{|I|}\right\}$$

the set containing the last $|I| - |J|$ elements of $I$. In this case $I_{\text{end}}$ is not empty. Finally, consider the condition $s_{w_{I,J}} = 1/2$. By Corollary 7.4, it can be written as

$$\sum_{i \in I} \lambda_i - \sum_{j \in J} \lambda_j = r/2 + \sum_{i \in I} i - \sum_{j \in J} j - (|I| - |J|)(n + 1).$$

Since $\lambda_{i_l} = \lambda_{j_l}$ for $l = 1, \ldots, |J|$, the left hand side of this condition equals

$$\sum_{i \in I_{\text{end}}} \lambda_i \geq 0.$$

Using $j_l = i_l + 1$ for $l = 1, \ldots, |J|$, and $|I| + |J| = r$, the right hand side becomes

$$-(|I| - |J|)(n + 1/2) + \sum_{i \in I_{\text{end}}} i.$$

Since $|I_{\text{end}}| = |I| - |J| > 0$, the right hand side is not greater than

$$-(|I| - |J|)(n + 1/2) + (|I| - |J|)\, n = -(|I| - |J|)/2 < 0.$$

This shows that the condition $s_{w_{I,J}} = 1/2$ and (6.4) are never simultaneously satisfied in this case.

*Case 3:* $|I| = r/2$. The first part of the argument in Case 2 applies to obtain $j_l = i_l + 1$ and $\lambda_{i_l} = \lambda_{j_l}$ for $l = 1, 2, \ldots, r/2$, and the disjoint subsets $I$ and $J$ are of the form

$$\begin{aligned} I &= \left\{ i_1, i_2, \ldots, i_{r/2} \right\}, \\ J &= \left\{ i_1 + 1, i_2 + 1, \ldots, i_{r/2} + 1 \right\}. \end{aligned}$$

Their disjointness implies that neither of them contains a pair of consecutive integers. All the $w_{I,J}$ corresponding to such a pair of subsets $(I, J)$ satisfies the condition (6.4) and $s_{w_{I,J}} = 1/2$.

It remains to compute $\chi_{\pi_\infty}$ from the condition (6.5) for $w_{I,J}$ as above. The right hand side of (6.5) equals

$$-w_{I,J} \left( \Lambda + \rho_{P_0} \right) - \left( -w_{I,J} \left( \Lambda + \rho_{P_0} \right) \Big|_{\mathfrak{a}_{P_0}} \right),$$

where the first term is given in Lemma 7.3, while the second one is just $s_{w_{I,J}} \widetilde{\rho}_{P_0} = 1/2 \sum_{l=1}^{r} e_l$. Hence, using the form of $I$ and $J$, $\lambda_i = \lambda_{i+1}$, and $|I| = |J| = r/2$ gives the expression for $\chi_{\pi_\infty}$. □

**Corollary 7.7.** *In the notation as in Proposition 7.6, assume that, for $w_{I,J} \in W^{P_r}$ such that $s_{w_{I,J}} = 1/2$, there exists a non–trivial cohomology class of type $(\pi, w_{I,J})$. Then, the length of $w_{I,J}$ equals*

$$\ell(w_{I,J}) = \frac{r(4n - 3r + 2)}{4}.$$

*Recall that $r$ is necessarily even. In particular, for $r = n$, we have $\ell(w_{I_0, S_n \setminus I_0}) = \frac{n(n+2)}{4}$.*

*Proof.* By Proposition 7.6, the existence of a non–trivial cohomology class of type $(\pi, w_{I,J})$ implies the form of the subsets $I$ and $J$. Thus, using the formula for the length of $w_{I,J}$ obtained in Lemma 7.2, gives the expression for $\ell(w_{I,J})$. Note that for $j_l = i_l + 1$, where $l = 1, 2, \ldots, r/2$, the term $m_{I,J}(j)$ in the formula equals $m_{I,J}(j_l) = i_l - 2l + 1$, because there are $l$ elements of $I$ and $l - 1$ elements of $J$ which are smaller than $j_l$. □

*Remark* 7.8. Observe that the condition of Corollary 7.4 on $w_{I,J} \in W^{P_r}$ and the highest weight $\Lambda$, which gives $s_{w_{I,J}} = 1/2$ could be in general satisfied for a large number of $w_{I,J}$. For example, in the case $P = P_n$, and the trivial coefficient system, the condition $s_{w_{I,S_n \setminus I}} = 1/2$ is equivalent to $\sum_{i \in I} (n + 1 - i) = \frac{n(n+2)}{4}$. Since assignment $i \mapsto n + 1 - i$ defines a permutation of $S_n$, this condition in fact shows that the number of $w_{I,S_n \setminus I} \in W^{P_n}$ with $s_{w_{I,S_n \setminus I}} = 1/2$ for the trivial coefficient system is the same as the number of ways to write $\frac{n(n+2)}{4}$ as the sum of different positive integers not greater than $n$.

However, as proved in Proposition 7.6, the necessary condition for the existence of a non–trivial cohomology class of type $(\tau, w_{I,S_n \setminus I})$ singles out at most one among $w_{I,S_n \setminus I}$ such that $s_{w_{I,S_n \setminus I}} = 1/2$. Even more, it implies a condition on the highest weight $\Lambda$ and the infinitesimal character of the infinite component $\tau_\infty$ of $\tau$.

**Proposition 7.9.** *Let $E$ be of highest weight $\Lambda = \sum_{l=1}^{n} \lambda_l e_l$ as above. Let $w_{I,J} \in W^{P_r}$ be such that $0 \leq s_{w_{I,J}} < 1/2$. Then, for any cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$, a non–trivial cohomology class in $H^*(\mathfrak{l}_r, K_\mathbb{R} \cap L_r(\mathbb{R}); V_\pi \otimes H^*(\mathfrak{n}_r, E))$ of type $(\pi, w_{I,J})$ does not exist.*

*Proof.* Consider first the case $r = 1$. Then, either $I = \{i\}$ is a singleton and $J = \emptyset$, or $I = \emptyset$ and $J = \{j\}$ is a singleton. For the two possibilities, using the formula of Corollary 7.4, we compute

$$s_{w_{I,J}} = \begin{cases} \lambda_i + (n + 1 - i) \geq 1, & \text{for } I = \{i\} \text{ and } J = \emptyset, \\ -(\lambda_j + (n + 1 - j)) \leq -1, & \text{for } I = \emptyset \text{ and } J = \{j\}. \end{cases}$$

This shows that for $r = 1$ the evaluation point $s_{w_{I,J}}$ is never inside the interval $0 \leq s_{w_{I,J}} < 1/2$, and the claim trivially holds.

A similar argument shows that for $r>1$, if either $|I|=r$ and $J=\emptyset$, or $I=\emptyset$ and $|J|=r$, then the formula for $s_{w_{I,J}}$ of Corollary 7.4 gives

$$s_{w_{I,J}}=\begin{cases} \frac{1}{r}\left[\sum_{i\in I}\lambda_i+\sum_{i\in I}(n+1-i)\right]\geq 1, & \text{for } |I|=r \text{ and } J=\emptyset,\\ -\frac{1}{r}\left[\sum_{j\in J}\lambda_j+\sum_{j\in J}(n+1-j)\right]\leq -1, & \text{for } I=\emptyset \text{ and } |J|=r.\end{cases}$$

Thus, the claim again trivially holds.

It remains to consider for $r>1$ the case when both $I$ and $J$ are not empty. Then, in the expression for $\mu_{w_{I,J}}$ in Lemma 7.5 the coefficient $s_1$ of $e_1$ is in the first, while $s_r$ of $e_r$ is in the second sum. In the notation of the proof of Proposition 7.9, the existence of the cohomology class of type $(\pi,w_{I,J})$ would imply that

$$s_1+s_r=2\overline{s}=-2s_{w_{I,J}}-2n+r-1.$$

As in Case 2 of the proof of Proposition 7.6 this condition is equivalent to

$$(\lambda_{i_1}-\lambda_{j_1})+(j_1-i_1)=2s_{w_{I,J}},$$

which is impossible for $0\leq s_{w_{I,J}}<1/2$ since the left hand side is a non–zero integer. □

*Remark* 7.10. The condition of Corollary 7.4 allows the evaluation point $s_{w_{I,J}}$ to be less than $1/2$, and moreover, there are several $w_{I,J}\in W^{P_r}$ giving every such point. However, as in Remark 7.8, the necessary condition for the existence of a non–trivial cohomology class of type $(\tau,w_{I,J})$ rules out all those possibilities in the cohomological context. See also Remark 5.9.

**Proposition 7.11.** *Let $E$ be of highest weight $\Lambda=\sum_{l=1}^{n}\lambda_l e_l$, as above. Consider the case of the parabolic subgroup $P_r$ with $r<n$. Let $w_{I,J}\in W^{P_r}$ be such that $s_{w_{I,J}}=1$. Let $\pi\cong\tau\otimes\sigma$ be a cuspidal automorphic representation of $L_r(\mathbb{A})\cong GL_r(\mathbb{A})\times Sp_{n-r}(\mathbb{A})$, where $\tau$ and $\sigma$ are cuspidal automorphic representations of $GL_r(\mathbb{A})$ and $Sp_{n-r}(\mathbb{A})$, respectively. Then, if a non–trivial cohomology class in $H^*(\mathfrak{l}_r,K_{\mathbb{R}}\cap L_r(\mathbb{R});V_\pi\otimes H^*(\mathfrak{n}_r,E))$ of type $(\pi,w_{I,J})$ exists the following holds:*

- *$w_{I,J}$ corresponds to a pair of disjoint subsets $(I,J)$ of the form*

$$\begin{aligned} I &= \begin{cases}\{i_1,i_2,\dots,i_{r/2}\}, & \text{if } r \text{ is even},\\ \{i_1,i_2,\dots,i_{\lfloor r/2\rfloor},n\}, & \text{if } r \text{ is odd},\end{cases}\\ J &= \{i_1+\epsilon_1,i_2+\epsilon_2,\dots,i_{\lfloor r/2\rfloor}+\epsilon_{\lfloor r/2\rfloor}\},\end{aligned}$$

  *where $\epsilon_l\in\{1,2\}$, and $\lfloor x\rfloor$ is "the floor" of $x$, i.e. the greatest integer not strictly greater than $x$,*
- *the coefficients of $\Lambda$ satisfy $\lambda_{i_l}=\lambda_{i_l+\epsilon_l}+2-\epsilon_l$, for $l=1,\dots,\lfloor r/2\rfloor$, and in the case of $r$ odd $\lambda_n=0$,*
- *the infinitesimal character $\chi_{\pi_\infty}$ of the infinite component $\pi_\infty$ of $\pi$ equals*

$$\begin{aligned}\chi_{\pi_\infty} &= \sum_{l=1}^{\lfloor r/2\rfloor}\Big[-(\lambda_{i_l}+n-i_l)\,e_l+(\lambda_{i_l}+n-i_l)\,e_{r+1-l}\Big]\\ &- \sum_{l'=1}^{n-r}\left(\lambda_{k_{l'}}+n+1-k_{l'}\right)e_{r+l'},\end{aligned}$$

*Proof.* The proof goes along the same lines as the proof of Proposition 7.6. Since $s_{w_{I,J}}=1$, we have

$$2\overline{s}=-2n+r-3,$$

and the condition (6.4) is equivalent to

$$s_l+s_{r+1-l}=-2n+r-3, \tag{7.2}$$

for $l = 1, \dots, r$.

The same argument as in Case 1 in Proposition 7.6 shows that if $|I| < r/2$, the condition (6.4) is never satisfied. Hence, let $|I| \geq r/2$. As in Case 2 in Proposition 7.6, we obtain

$$(\lambda_{i_l} - \lambda_{j_l}) + (j_l - i_l) = 2$$

for $l = 1, \dots, |J|$. If $j_l < i_l$, then $\lambda_{i_l} \leq \lambda_{j_l}$, and the left hand side is negative. If $j_l > i_l$, then $\lambda_{i_l} \geq \lambda_{j_l}$, and both brackets in the above equation are non–negative integers. Hence, the solutions satisfy $j_l - i_l = \epsilon_l \in \{1, 2\}$, and $\lambda_{i_l} - \lambda_{j_l} = 2 - \epsilon_l$.

Write

$$\begin{aligned} I &= \{i_1, \dots, i_{|J|}, i_{|J|+1}, \dots, i_{|I|}\}, \\ J &= \{i_1 + \epsilon_1, \dots, i_{|J|} + \epsilon_{|J|}\}, \end{aligned}$$

and let $I_{\text{end}} = \{i_{|J|+1}, \dots, i_{|I|}\}$. Inserting $s_{w_{I,J}} = 1$, $|I_{\text{end}}| = |I| - |J|$, $|I| + |J| = r$, and taking into account the relationship between $i_l$ and $j_l$, and $\lambda_{i_l}$ and $\lambda_{j_l}$ for $l = 1, \dots, |J|$, the formula for $s_{w_{I,J}}$ of Corollary 7.4 gives

$$\sum_{i \in I_{\text{end}}} i = |I_{\text{end}}| \cdot n + \sum_{i \in I_{\text{end}}} \lambda_i.$$

Since $\lambda_i \geq 0$, and $i \in I_{\text{end}}$ are distinct and not greater than $n$, this equation has a solution only if either $I_{\text{end}}$ is empty (thus, $r$ is even), or $I_{\text{end}} = \{n\}$ and $\lambda_n = 0$ (thus, $r$ is odd). This gives the first two conditions of the proposition.

The remaining condition on the infinitesimal character $\chi_{\pi_\infty}$ of $\pi_\infty$ follows from (6.5) using the description of $w_{I,J}$ and $\Lambda$. □

## 8. Residual Eisenstein Cohomology

Let $\phi$ be the associate class of a cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$. There is a two step filtration of the space $\mathcal{A}_{E,\{P_r\},\phi}$ by the subspace $\mathcal{L}_{E,\{P_r\},\phi}$ spanned by the square integrable automorphic forms. Let $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi}$ be the subspace of $\mathcal{L}_{E,\{P_r\},\phi}$ spanned by the residues at $s_0 \geq 1/2$ of the Eisenstein series attached to the $\pi$–isotypic subspace of the space of cusp forms on $L_r(\mathbb{A})$. The following lemma shows that the two spaces give the same contribution to cohomology.

**Lemma 8.1.** *Let $E$ be an irreducible representation of $Sp_n(\mathbb{C})$. Let $\{P_r\}$ be the associate class of the standard parabolic $\mathbb{Q}$–subgroup, corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of the set of simple roots, with the Levi factor $L_r$. Let $\phi$ be the associate class of a cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$. Then, the map*

$$H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi} \otimes_\mathbb{C} E) \to H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{L}_{E,\{P_r\},\phi} \otimes_\mathbb{C} E)$$

*induced on the cohomology by the inclusion $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi} \subset \mathcal{L}_{E,\{P_r\},\phi}$ is an isomorphism.*

*Proof.* The injectivity of the map in the cohomology is a consequence of the fact that $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi}$ is a direct summand in the space of square integrable automorphic forms $\mathcal{L}_{E,\{P_r\},\phi}$. The surjectivity follows from Proposition 7.9. Those propositions show that even if the Eisenstein series, attached to a cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$, had a pole at $0 \leq s_0 < 1/2$, its contribution to the cohomology is trivial because, for $w \in W^{P_r}$ such that $s_w = s_0$, a non–trivial cohomology class in $H^*(\mathfrak{l}_r, K_\mathbb{R} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes H^*(\mathfrak{n}_r, E))$ of type $(\pi, w)$ does not exist. □

8.1. **The case $P = P_n$.**

**Theorem 8.2.** *Let $E$ be the irreducible representation of $Sp_n(\mathbb{C})$ of highest weight $\Lambda = \sum_{k=1}^{n} \lambda_k e_k$, where all $\lambda_k$ are integers and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Let $\{P_n\}$ be the associate class of the standard maximal proper parabolic $\mathbb{Q}$–subgroup $P_n$ of $Sp_n$, corresponding to the subset $\Delta \setminus \{\alpha_n\}$ of the set of simple roots, and with the Levi decomposition $P_n = L_n N_n$, where the Levi factor $L_n \cong GL_n$. Let $\phi$ be the associate class of a cuspidal automorphic representation $\tau$ of $L_n(\mathbb{A})$.*

*The cohomology space*

$$H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$$

*is trivial is trivial except possibly in the case where the following conditions are satisfied:*

1. *a cuspidal automorphic representation $\tau$ is selfdual, $L(s, \tau, \wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau) \neq 0$,*
2. *the $\mathbb{Q}$–rank $n$ of the algebraic group $Sp_n/\mathbb{Q}$ is even,*
3. *the highest weight $\Lambda$ of the irreducible representation $E$ satisfies $\lambda_{2l-1} = \lambda_{2l}$ for all $l = 1, 2, \ldots, n/2$,*
4. *the infinite component $\tau_\infty$ of $\tau$ has the infinitesimal character*

$$\chi_{\tau_\infty} = \sum_{l=1}^{n/2} \Big[ -\big(\mu_l + (n + 3/2 - 2l)\big)e_l + \big(\mu_l + (n + 3/2 - 2l)\big)e_{n+1-l} \Big],$$

*where $\mu_l = \lambda_{2l-1} = \lambda_{2l}$, i.e. $\tau_\infty$ is a tempered representation fully induced from $n/2$ unitary discrete series representations of $GL_2(\mathbb{R})$ having the lowest $O(2)$–types $2\mu_l + 2n - 4l + 4$ for $l = 1, \ldots, n/2$.*

*Proof.* By Lemma 8.1, if the cohomology space $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}^{s_0 \geq 1/2}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$ is trivial, then the space $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$ is trivial as well. By Corollary 5.10, condition (1) assures that the space $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_n\},\phi}$ is non–trivial. Moreover, it is spanned by the residues of the Eisenstein series $E^{Sp_n}_{P_n}(f, \lambda_s)$ at $s_0 = 1/2$. Hence, in order to have a non–trivial cohomology class in $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}^{s_0 \geq 1/2}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$, it is necessary to have a non–trivial cohomology class in $H^*(\mathfrak{l}_n, K_{\mathbb{R}} \cap L_n(\mathbb{Q}); V_{\tau_\infty} \otimes F_{\mu_{w_{I,S_n \setminus I}}})$ of type $(\tau, w_{I,S_n \setminus I})$, such that the corresponding evaluation point for the Eisenstein series is $s_{w_{I,S_n \setminus I}} = 1/2$. By Proposition 7.6, this gives the remaining three conditions, and only the classes of type $(\tau, w_{I_0,S_n \setminus I_0})$ with $I_0 = \{1, 3, \ldots, n-1\}$ may exist. □

**Theorem 8.3.** *Suppose that the necessary conditions for a non–trivial cohomology class in*

$$H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{E,\{P_n\},\phi} \otimes_{\mathbb{C}} E)$$

*given in Theorem 8.2 are satisfied with $E = \mathbb{C}$ the trivial representation. In particular, $n$ is even. Then the map*

$$H^q(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{\mathbb{C},\{P_n\},\phi}) \to H^q(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{A}_{\mathbb{C},\{P_n\},\phi})$$

*is the trivial map for $q \geq \frac{n(n+1)}{2}$, and an epimorphism for $q < \frac{n(n+1)}{2}$. In other words, the cohomology classes coming from $\mathcal{L}_{\mathbb{C},\{P_n\},\phi}$ are separated from the ones coming from $\mathcal{A}_{\mathbb{C},\{P_n\},\phi}/\mathcal{L}_{\mathbb{C},\{P_n\},\phi}$ by the degree. Note that $\frac{n(n+1)}{2} = \frac{1}{2}\dim X_{Sp_n(\mathbb{R})}$ is half of the dimension of the space $X_{Sp_n(\mathbb{R})} = Sp_n(\mathbb{R})/K_{\mathbb{R}}$.*

*Proof.* We have to analyze in which cohomological degree the spaces $\mathcal{L}_{\mathbb{C},\{P_n\},\phi}$ and $\mathcal{A}_{\mathbb{C},\{P_n\},\phi}/\mathcal{L}_{\mathbb{C},\{P_n\},\phi}$ have a non–vanishing relative Lie algebra cohomology. Let $\phi$ be the associate class of a cuspidal automorphic representation $\tau$ of $L_n(\mathbb{A})$ subject to the necessary conditions of Theorem 8.2. In view of the actual construction of elements in $\mathcal{A}_{\mathbb{C},\{P_n\},\phi}$ and $\mathcal{L}_{\mathbb{C},\{P_n\},\phi}$ as described in Section 5,

we have to determine the range in which the following relative Lie algebra cohomology spaces are non–trivial:

$$H^*\left(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathrm{Ind}_{P_n(\mathbb{R})}^{Sp_n(\mathbb{R})}\left(\tau_\infty e^{\langle H_{P_n}(\cdot),\lambda_{s_w}\rangle}\right)\right), \tag{8.1}$$

for $w \in W^{P_n}$ such that there is a non–trivial cohomology class of type $(\tau, w)$, and respectively

$$H^*\left(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathrm{Ind}_{P_n^{\mathrm{opp}}(\mathbb{R})}^{Sp_n(\mathbb{R})}\left(\tau_\infty e^{\langle H_{P_n}(\cdot),\lambda_{-1/2}\rangle}\right)\right). \tag{8.2}$$

In the latter case we have a class of type $(\tau, w)$, with $w \in W^{P_n}$, and $\lambda_w = \lambda_{1/2}$, i.e. $s_w = 1/2$, and thus $\ell(w) = \frac{n(n+2)}{4}$ by Corollary 7.4. Following Proposition 7.6, this element $w \in W^{P_n}$ is uniquely determined, and it is the element $w_{I_0, S_n\setminus I_0}$ corresponding to the subset $I_0 = \{1, 3, \ldots, n-1\}$ in the notation of Lemma 7.1.

In our computation we use the following notation: suppose $L$ is a Lie group with finitely many connected components, and let $K_L$ be a maximal compact subgroup of $L$. Suppose the Lie algebra $\mathfrak{l}$ of $L$ is reductive. Write

$$2q(L) := \dim L - \dim K_L$$

for the dimension of the corresponding space $X_L = L/K_L$ of maximal compact subgroups of $L$. Set $\ell_0(L) := \mathrm{rk}(L) - \mathrm{rk}(K_L)$, and write

$$q_0(L) = \frac{1}{2}\left(2q(L) - \ell_0(L)\right) = \frac{1}{2}\left(\dim X_L - \ell_0(L)\right).$$

The rank and the dimension of a reductive Lie algebra are congruent modulo 2. Thus, $q_0(L)$ is an integer. The following result [24, Proposition 4.4] is decisive: let $(\delta, H_\delta)$ be an irreducible unitary tempered representation of $L$, and let $(\mu, F)$ be a finite–dimensional representation of $L$. Then, $H^j(\mathfrak{l}, K_L; H_\delta \otimes F) = (0)$ if $j \notin [q_0(L), q_0(L) + \ell_0(L)]$.

In our case at hand, by [4, Section III, 3.3], the cohomology (8.1) is the tensor product of

$$H^*\left(\mathfrak{l}_n, K_{\mathbb{R}} \cap L_n(\mathbb{R}); V_{\tau_\infty} \otimes F_{\mu_w}\right)$$

by $\Lambda^*\check{\mathfrak{a}}_{P_r,\mathbb{C}}$, up to a shift in degrees by $\ell(w)$. Thus, since $\tau_\infty$ is a unitary tempered representation of $L_n(\mathbb{R}) \cong GL_n(\mathbb{R})$, the cohomology space (8.1) vanishes in degrees outside of

$$\left[q_0(GL_n(\mathbb{R})) + \ell(w), q_0(GL_n(\mathbb{R})) + \ell_0(GL_n(\mathbb{R})) + \ell(w) + 1\right]. \tag{8.3}$$

In the case $L = GL_n(\mathbb{R})$, by [32, Section 3], we have $2q(L) = n^2 - 1 - \frac{n(n-1)}{2}$, $\ell_0(L) = n - 1 - \lfloor n/2 \rfloor$, and thus, for $n$ even, $q_0(L) = n^2/4$. By Proposition 7.9, for $w \in W^{P_n}$ such that $0 \le s_w < 1/2$, a cohomology class of type $(\tau, w)$ is trivial. In view of Corollary 7.4, since the length $\ell(w)$ increases as $s_w$ increases, the minimal possible length $\ell(w)$ is obtained for $s_w = 1/2$, and it equals $\ell(w) = \frac{n(n+2)}{4}$. Hence, we obtain

$$q_0(GL_n(\mathbb{R})) + \ell(w) = \frac{1}{4}n^2 + \frac{n(n+2)}{4} = \frac{1}{2}\dim X_{Sp_n(\mathbb{R})}.$$

as a lower bound in (8.3).

By the duality result [4, Section V, 1.5] regarding the relation between the cohomology (8.1) with $s_w = 1/2$, and the analogue (8.2), we obtain as the upper bound for the range outside of which the cohomology (8.2) vanishes the value

$$2q(Sp_n(\mathbb{R})) - \left(q_0(GL_n(\mathbb{R})) + \ell(w_{I_0,S_n\setminus I_0})\right) = \frac{1}{2}\dim X_{Sp_n(\mathbb{R})}.$$

Since the actual contribution of a class in (8.2), and (8.1) as well, is given by the image of $H^*(\mathfrak{l}_n, K_{\mathbb{R}} \cap L_n(\mathbb{R}); \tau_\infty \otimes F_{\mu_{w_{I_0,S_n\setminus I_0}}}) \otimes \Lambda^0\check{\mathfrak{a}}_{P_r,\mathbb{C}}$, the residual Eisenstein classes may occur only in degrees $j$ with $j \le \frac{1}{2}\dim X_{Sp_n(\mathbb{R})} - 1$. The lowest possible degree for these classes is $\frac{1}{2}n^2$,

i.e. $H^j(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{\mathbb{C},\{P_n\},\phi}) = (0)$ for $j \notin \left[\frac{1}{2}n^2, \frac{1}{2}n(n+1)-1\right]$. Since the lowest possible degree for a cohomology class in $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{A}_{\mathbb{C},\{P_r\},\phi})$ given by a regular value of an Eisenstein series in the given situation is $q_0(GL_n(\mathbb{R})) + \ell(w_{I_0,S_n\setminus I_0}) = \frac{1}{2}\dim X_{Sp_n(\mathbb{R})}$ our claim follows. □

*Remark* 8.4. Write

$$\tau_\infty \cong \mathrm{Ind}_{P_{(2,\dots,2)}(\mathbb{R})}^{GL_n(\mathbb{R})}(\delta_1 \otimes \dots \otimes \delta_{n/2}),$$

where $\delta_i$ are unitary discrete series representations of $GL_2(\mathbb{R})$ as in Theorem 8.2. Then the Archimedean component of the space $\mathcal{L}_{E,\{P_n\},\phi}^{s_0 \geq 1/2}$, which is by Corollary 5.10 spanned by the residues of the Eisenstein series attached to $\tau$ at the pole $s_0 = 1/2$, is isomorphic to the Langlands quotient of the standard module

$$\mathrm{Ind}_{P_{(2,\dots,2)}(\mathbb{R})}^{Sp_n(\mathbb{R})}(\delta_1|\det|^{1/2} \otimes \dots \otimes \delta_{n/2}|\det|^{1/2}).$$

This follows from Proposition 5.4.

8.2. **The case $P = P_r$ with $r < n$.** Now we turn our attention to the associate class $\{P_r\}$, $r < n$. In this case, for simplicity, we consider only the trivial coefficient system $E = \mathbb{C}$, and the corresponding cohomology spaces $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{\mathbb{C},\{P_r\},\phi,V_\pi})$ and $H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{A}_{\mathbb{C},\{P_r\},\phi})$. Here $V_\pi$ is the fixed realization of a cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$ in the space of cusp forms on $L_r(\mathbb{A})$, which is globally $\psi$–generic (for a fixed non–trivial additive character $\psi$ of $\mathbb{Q}\backslash\mathbb{A}$), i.e. there is a cusp form in $V_\pi$ whose $\psi$–Fourier coefficient is non–trivial. This assumption on $V_\pi$ is required for the computation of the poles of the Eisenstein series in Section 5.

**Theorem 8.5.** *Let $E = \mathbb{C}$ be the trivial representation of $Sp_n(\mathbb{C})$. Let $r < n$, and let $\{P_r\}$ be the associate class of the standard maximal proper parabolic $\mathbb{Q}$–subgroup $P_r$ of $Sp_n$, corresponding to the subset $\Delta \setminus \{\alpha_r\}$ of the set of simple roots, and with the Levi decomposition $P_r = L_r N_r$, where the Levi factor $L_r \cong GL_r \times Sp_{n-r}$. Let $\phi$ be the associate class of a cuspidal automorphic representation $\pi \cong \tau \otimes \sigma$ of $L_r(\mathbb{A})$ such that a fixed realization $V_\pi$ of $\pi$ in the space of cusp forms on $L_r(\mathbb{A})$ is globally $\psi$–generic (with respect to a fixed non–trivial additive character $\psi$ of $\mathbb{A}/\mathbb{Q}$).*

*Let*

$$\chi_{\pi_\infty} = \sum_{l=1}^{\lfloor r/2 \rfloor} (-x_l e_l + x_l e_{r+1-l}) - \sum_{l'=1}^{n-r} y_{l'} e_{r+l'}$$

*be the infinitesimal character of the Archimedean component $\pi_\infty$ of $\pi$, where $\lfloor x \rfloor$ denotes the greatest integer not greater than $x$. Then, the cohomology space*

$$H^*(\mathfrak{sp}_n, K_{\mathbb{R}}; \mathcal{L}_{\mathbb{C},\{P_r\},\phi,V_\pi})$$

*is trivial except possibly in the case where one of the following two sets of conditions is satisfied:*

**(A)** (a1) *a cuspidal automorphic representation $\tau$ is selfdual, $L(s,\tau,\wedge^2)$ has a pole at $s = 1$, and $L(1/2, \tau \times \Pi_j) \neq 0$ for all $\Pi_j$ appearing in the global functorial lift of $\sigma$,*

(a2) *$r$ is even,*

(a3) *the coefficients $x_l$ of the infinitesimal character $\chi_{\pi_\infty}$ belong to the set*

$$x_l \in \{3/2, 5/2, \dots, n-1/2\},$$

*and $|x_{l_1} - x_{l_2}| \neq 0, 1$ for $l_1 \neq l_2$,*

(a4) *the coefficients $y_{l'}$ of the infinitesimal character $\chi_{\pi_\infty}$ are uniquely determined (up to sign) by the coefficients $x_l$ through the formula*

$$y_{l'} = n + 1 - k_{l'},$$

*for $l' = 1, \dots, n-r$, where*

$$k_{l'} \in S_n \setminus \{n - x_l + 1/2, n - x_l + 3/2 : l = 1, \dots, r/2\}.$$

**(B)** (b1) *a cuspidal automorphic representation $\tau$ is isomorphic to one of $\Pi_j$ appearing in the global functorial lift of $\sigma$ (this implies that $\tau$ is selfdual, and $r \leq \frac{2n+1}{3}$),*

(b2) *the coefficients $x_l$ of the infinitesimal character $\chi_{\pi_\infty}$ belong to the set*

$$x_l \in \begin{cases} \{2,3,\dots,n\}, & \text{if } r \text{ is even,} \\ \{3,4,\dots,n\}, & \text{if } r \text{ is odd,} \end{cases}$$

*and $|x_{l_1} - x_{l_2}| \neq 0, 2$ for $l_1 \neq l_2$,*

(b3) *the coefficients $y_{l'}$ of the infinitesimal character $\chi_{\pi_\infty}$ are uniquely determined (up to sign) by the coefficients $x_l$ through the formula*

$$y_{l'} = n + 1 - k_{l'}$$

*for $l' = 1, \dots, n-r$, where*

$$k_{l'} \in \begin{cases} S_n \setminus \{n - x_l, n - x_l + 2 : l = 1, \dots, r/2\}, & \text{if } r \text{ is even,} \\ S_n \setminus \{n, n - x_l, n - x_l + 2 : l = 1, \dots, \lfloor r/2 \rfloor\}, & \text{if } r \text{ is odd.} \end{cases}$$

*Proof.* As in the case $r = n$, applying Lemma 8.1, the conditions (a1) and (b1) assure that the space $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi,V_\pi}$ is non–trivial (see Corollary 5.10). Already these two conditions are never both satisfied (see Remark 5.8). Moreover, the space $\mathcal{L}^{s_0 \geq 1/2}_{E,\{P_r\},\phi,V_\pi}$ is spanned by the residues of the Eisenstein series $E^{Sp_n}_{P_r}(f, \lambda_s)$ attached to the realization $V_\pi$ of $\pi$ at the pole $s_0 = 1/2$ if (a1) is satisfied, and at $s_0 = 1$ if (b1) is satisfied.

Again, as in the case $r = n$, in order to have a non–trivial cohomology class in $H^*(\mathfrak{sp}_n, K_\mathbb{R}; \mathcal{L}_{E,\{P_r\},\phi} \otimes_{\mathbb{C}} E)$, it is necessary to have a non–trivial cohomology class in $H^*(\mathfrak{l}_r, K_\mathbb{R} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes H^*(\mathfrak{n}_r, E))$ of type $(\pi, w_{I,J})$, with $w_{I,J} \in W^{P_r}$ such that $s_{w_{I,J}} = 1/2$ for (A), and $s_{w_{I,J}} = 1$ for (B).

The remaining conditions in (A) follow from Proposition 7.6 with $E = \mathbb{C}$, i.e. $\lambda_l = 0$ for all $l = 1, \dots, n$. Necessarily $r$ is even which is condition (a2). Furthermore, the formula for $\chi_{\pi_\infty}$ in that proposition shows that the subset $I$ of $S_n$ defining $w_{I,J}$ consists of all $i \in S_n$ which are solutions of one of the equations

$$n + 1/2 - i = x_l,$$

for $l = 1, \dots, r/2$. Every such equation has at most one solution $i \in S_n$. Hence, in order to have subset $I$ containing $r/2$ elements, the coefficients $x_l$ are necessarily distinct and every equation has a solution $i \in S_n$. In particular, $x_l$ is in $1/2 + \mathbb{Z}$, and $1/2 \leq x_l \leq n - 1/2$. However, the subset $J$ of $S_n$ defining $w_{I,J}$ is of the form $J = \{i + 1 : i \in I\}$ and disjoint with $I$. Thus, the subset $I$ should not contain neither $n$, nor a pair of consecutive integers. The former condition gives a lower bound $x_l \geq 3/2$. The latter shows that if $l_1 \neq l_2$, then

$$x_{l_1} - x_{l_2} = i_2 - i_1,$$

where $i_1$ and $i_2$ are solutions of the equations for $x_{l_1}$ and $x_{l_2}$, and thus $|x_{l_1} - x_{l_2}| \neq 1$, and (a3) is proved. Since (a3) also defines the subsets $I$ and $J$, (a4) follows from the formula for $\chi_{\pi_\infty}$ in Proposition 7.6.

The remaining conditions in (B) are obtained in a similar way using Proposition 7.11 with $E = \mathbb{C}$, i.e. $\lambda_l = 0$ for $l = 1, \dots, n$. For $E = \mathbb{C}$ in Proposition 7.11, we have $\epsilon_l = 2$ for all $l = 1, \dots, \lfloor r/2 \rfloor$. In this case the equations defining $I$ are

$$n - i = x_l,$$

for $l = 1, \dots, \lfloor r/2 \rfloor$. If $r$ is odd, besides the solutions of those equations, $I$ contains also $n$. Now, we have $x_l \in \mathbb{Z}$, all $x_l$ are distinct, and $0 \leq x_l \leq n - 1$. By Proposition 7.11, $J = \{i + 2 : i \in I \setminus \{n\}\}$.

Thus, the lower bound for $x_l$ is obtained from $i \leq n-2$ for $i \in I$ if $r$ is even, and from $i \leq n-3$ for $i \in I \setminus \{n\}$ if $r$ is odd. The condition on the difference again follows from

$$x_{l_1} - x_{l_2} = i_2 - i_1$$

for $l_1 \neq l_2$, where $i_1$ and $i_2$ are solutions of the above equations for $x_{l_1}$ and $x_{l_2}$. Thus, (b2) is proved, and the sets $I$ and $J$ are defined. Again, (b3) follows directly from Proposition 7.11. □

*Remark* 8.6. Theorem 8.5 is stated in such a way that given a globally $\psi$–generic cuspidal automorphic representation $\pi$ of $L_r(\mathbb{A})$ satisfying either (a1) or (b1), one can effectively check the conditions on the coefficients of the infinitesimal character $\chi_{\pi_\infty}$ of the Archimedean component $\pi_\infty$ of $\pi$.

*Remark* 8.7. The relative Lie algebra cohomology $H^*(\mathfrak{l}_r, K_\mathbb{R} \cap L_r(\mathbb{R}); V_{\pi_\infty} \otimes F_{\mu_w})$ (with $\pi_\infty \cong \tau_\infty \otimes \pi_\infty$ the Archimedean component of $\pi \cong \tau \otimes \sigma$, and $w \in W^{P_r}$, as above subject to the given conditions), attached to the Levi factor $L_r \cong GL_r \times Sp_{n-r}$, $r < n$, obeys the Künneth rule [4, Section 1.3]. Thus (non)–vanishing results for this cohomology rely on the corresponding results for the two factors. Since $\tau_\infty$ is a unitary tempered representation of $GL_r(\mathbb{R})$, the cohomology $H^*(\mathfrak{gl}_r, K_\mathbb{R} \cap GL_r(\mathbb{R}); V_{\tau_\infty} \otimes F) = (0)$ for

$$j \notin [q_0(GL_r(\mathbb{R})), q_0(GL_r(\mathbb{R})) + \ell_0(GL_r(\mathbb{R}))]$$

with

$$q_0(GL_r(\mathbb{R})) = \begin{cases} \frac{r^2}{4}, & \text{if } r \text{ is even,} \\ \frac{r^2-1}{4}, & \text{if } r \text{ is odd,} \end{cases}$$

and $\ell_0(GL_r(\mathbb{R})) = r - 1 - \lfloor r/2 \rfloor$ ([32, Section 3]). With regard to the other factor, one has a vanishing result for the cuspidal cohomology $H^j(\mathfrak{sp}_{n-r}, K_\mathbb{R} \cap Sp_{n-r}(\mathbb{R}); V_{\sigma_\infty} \otimes F')$ in degrees

$$j < \kappa_r := \left\{ \frac{1}{4}(n-r)(n-r+2) \right\},$$

where $\{x\}$ denotes the smallest integer larger or equal to the rational number $x$ ([25, Section 7.2]). The value $\kappa_r$ is larger than $\frac{1}{4}\dim X_{Sp_{n-r}(\mathbb{R})}$ but quite close to it. Following the line of argument as given in the case $P = P_n$, and using the "critical" values for $\ell(w) = \ell(w_{I,J})$, one can determine a range of degrees outside of which the Eisenstein cohomology classes cannot occur at all.

*Remark* 8.8. Write

$$\tau_\infty \cong \begin{cases} \mathrm{Ind}_{P_{(2,\ldots,2)}(\mathbb{R})}^{GL_r(\mathbb{R})}(\delta_1 \otimes \ldots \otimes \delta_{r/2}), & \text{if } r \text{ is even,} \\ \mathrm{Ind}_{P_{(2,\ldots,2,1)}(\mathbb{R})}^{GL_r(\mathbb{R})}(\delta_1 \otimes \ldots \otimes \delta_{r/2} \otimes \chi), & \text{if } r \text{ is odd.} \end{cases}$$

where $\delta_i$ are unitary discrete series of $GL_2(\mathbb{R})$ of lowest $O(2)$–type $2x_l + 1$, and $\chi$ is a unitary character of $\mathbb{R}^\times$, as in Theorem 8.5. At infinity $\sigma_\infty$ is the local component of a globally $\psi$–generic cuspidal automorphic representation $\sigma$, and hence a fully induced representation of the form

$$\sigma_\infty \cong \mathrm{Ind}_{P_{(r'_1,\ldots,r'_m)}(\mathbb{R})}^{Sp_{n-r}(\mathbb{R})}(\delta'_1 |\det|^{x_1} \otimes \ldots \otimes \delta'_m |\det|^{x_m} \otimes \sigma_t)$$

with $\delta'_i$ unitary discrete series representation of $GL_{r'_i}(\mathbb{R})$, $\sigma_t$ a tempered representation of $Sp_{r'_0}(\mathbb{R})$, where $r'_0 = n - r - \sum r'_j$, and $1/2 > x_1 \geq x_2 \geq \ldots \geq x_m > 0$. Then, the Archimedean component of the space spanned by the residues of the Eisenstein series attached to $\pi$ at $s_0 = 1/2$ in case (A), is isomorphic to the Langlands quotient of the standard module

$$Ind_{P_{(2,\ldots,2,r'_1,\ldots,r'_m)}(\mathbb{R})}^{Sp_n(\mathbb{R})}(\delta_1 |\det|^{1/2} \otimes \ldots \otimes \delta_{r/2} |\det|^{1/2} \otimes \delta'_1 |\det|^{x_1} \otimes \ldots \otimes \delta'_m |\det|^{x_m} \otimes \sigma_t).$$

In the case (B), i.e. $s_0 = 1$, the exponents $1/2$ are just replaced by 1.

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