

Global Jacquet–Langlands correspondence, multiplicity one and classification of automorphic representations^{*}

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Oblatum 28-IV-2007 & 16-XI-2007

Published online: 25 January 2008 – © Springer-Verlag 2008

Abstract. In this paper we generalize the local Jacquet–Langlands correspondence to all unitary irreducible representations. We prove the global Jacquet–Langlands correspondence in characteristic zero. As consequences we obtain the multiplicity one and strong multiplicity one theorems for inner forms of $GL(n)$ as well as a classification of the residual spectrum and automorphic representations in analogy with results proved by Mœglin–Waldspurger and Jacquet–Shalika for $GL(n)$.

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1. Introduction

The aim of this paper is to prove the global Jacquet–Langlands correspondence and its consequences for the theory of representations of the inner forms of GL_n over a global field of characteristic zero. In order to define the global Jacquet–Langlands correspondence, it is not sufficient to transfer

* with an appendix by Neven Grbac

only square integrable representations as in the classical local theory [JL], [Fl2], [Ro], [DKV]. It would be necessary to transfer at least the local components of global discrete series. This results are already necessary to the global correspondence with a division algebra (which can be locally any inner form). Here we prove, more generally, the transfer of all unitary representations. Then we prove the global Jacquet–Langlands correspondence, which is compatible with this local transfer. As consequences we obtain for inner forms of GL_n the multiplicity one theorem and strong multiplicity one theorem, as well as a classification of the residual spectrum à la Mœglin–Waldspurger and unicity of the cuspidal support à la Jacquet–Shalika. Using these classifications we give counterexamples showing that the global Jacquet–Langlands correspondence for discrete series does not extend well to all unitary automorphic representations.

We give here a list of the most important results, starting with the local study. We would like to point out that the local results in this paper have already been obtained by Tadić in [Ta6] in characteristic zero under the assumption that his conjecture U_0 holds. After we proved these results here independently of his conjecture (and some of them in any characteristic), Sécherre announced the proof of the conjecture U_0 ([Se]). The approach is completely different and we insist on the fact that we do not prove the conjecture U_0 here but more particular results which are enough to show the local transfer necessary for the global correspondence.

Let F be a local non-Archimedean field of characteristic zero and D a central division algebra over F of dimension d^2 . For $n \in \mathbb{N}^*$ set $G_n = GL_n(F)$ and $G'_n = GL_n(D)$. Let ν generically denote the character given by the absolute value of the reduced norm on groups like G_n or G'_n .

Let σ' be a square integrable representation of G'_n . If σ' is a cuspidal representation, then it corresponds by the local Jacquet–Langlands correspondence to a square integrable representation σ of G_{nd} . We set $s(\sigma') = k$, where k is the length of the Zelevinsky segment of σ . If σ' is not cuspidal, we set $s(\sigma') = s(\rho)$, where ρ is any cuspidal representation in the cuspidal support of σ' , and this does not depend on the choice. We set then $\nu_{\sigma'} = \nu^{s(\sigma')}$. For any $k \in \mathbb{N}^*$ we denote then by $u'(\sigma', k)$ the Langlands quotient of the induced representation from $\otimes_{i=0}^{k-1} (\nu_{\sigma'}^{\frac{k-1}{2}-i} \sigma')$, and if $\alpha \in]0, \frac{1}{2}[$, we denote $\pi'(u'(\sigma', k), \alpha)$ the induced representation from $\nu_{\sigma'}^\alpha u'(\sigma', k) \otimes \nu_{\sigma'}^{-\alpha} u'(\sigma', k)$. The representation $\pi'(u'(\sigma', k), \alpha)$ is irreducible ([Ta2]). Let \mathcal{U}' be the set of all representations of type $u'(\sigma', k)$ or $\pi'(u'(\sigma', k), \alpha)$ for all $G'_n, n \in \mathbb{N}^*$. Tadić conjectured in [Ta2] that

- (i) all the representations in \mathcal{U}' are unitary;
- (ii) an induced representation from a product of representations in \mathcal{U}' is always irreducible and unitary;
- (iii) every irreducible unitary representation of $G'_m, m \in \mathbb{N}^*$, is an induced representation from a product of representations in \mathcal{U}' .

The fact that the $u'(\sigma', k)$ are unitary has been proved in [BR1] if the characteristic of the base field is zero. In the third section of this paper

we complete the proof of the claim (i) (i.e. $\pi'(u'(\sigma', k), \alpha)$ are unitary; see Corollary 3.6) and prove (ii) (Proposition 3.9).

We also prove the Jacquet–Langlands transfer for all irreducible unitary representations of G_{nd} . More precisely, let us write $g' \leftrightarrow g$ if $g \in G_{nd}$, $g' \in G'_n$ and the characteristic polynomials of g and g' are equal and have distinct roots in an algebraic closure of F . Denote $G_{nd,d}$ the set of elements $g \in G_{nd}$ such that there exists $g' \in G'_n$ with $g' \leftrightarrow g$. We denote χ_π the function character of an admissible representation π . We say a representation π of G_{nd} is **d -compatible** if there exists $g \in G_{nd,d}$ such that $\chi_\pi(g) \neq 0$. We have (Proposition 3.9):

Theorem. *If u is a d -compatible irreducible unitary representation of G_{nd} , then there exists a unique irreducible unitary representation u' of G'_n and a unique sign $\varepsilon \in \{-1, 1\}$ such that*

$$\chi_u(g) = \varepsilon \chi_{u'}(g')$$

for all $g \in G_{nd,d}$ and $g' \leftrightarrow g$.

It is Tadić who first pointed out ([Ta6]) that this should hold if his conjecture U_0 were true. The sign ε and an explicit formula for u' may be computed. See for instance Subsect. 3.3.

The fifth section contains global results. Let us use the theorem above to define a map $|\mathbf{LJ}| : u \mapsto u'$ from the set of irreducible unitary d -compatible representations of G_{nd} to the set of irreducible unitary representations of G'_n .

Let now F be a global field of characteristic zero and D a central division algebra over F of dimension d^2 . Let $n \in \mathbb{N}^*$. Set $A = M_n(D)$. For each place v of F let F_v be the completion of F at v and set $A_v = A \otimes F_v$. For every place v of F , $A_v \simeq M_{r_v}(D_v)$ for some positive integer r_v and some central division algebra D_v of dimension d_v^2 over F_v such that $r_v d_v = nd$. We will fix once and for all an isomorphism and identify these two algebras. We say that $M_n(D)$ is split at a place v if $d_v = 1$. The set V of places where $M_n(D)$ is not split is finite. We assume in the sequel that V does not contain any infinite place.

Let $G_{nd}(\mathbb{A})$ be the group of adèles of $GL_{nd}(F)$, and $G'_n(\mathbb{A})$ the group of adèles of $GL_n(D)$. We identify $G_{nd}(\mathbb{A})$ with $M_{nd}(\mathbb{A})^\times$ and $G'_n(\mathbb{A})$ with $A(\mathbb{A})^\times$.

Let $Z(\mathbb{A})$ be the center of $G_{nd}(\mathbb{A})$. If ω is a smooth unitary character of $Z(\mathbb{A})$ trivial on $Z(F)$, let $L^2(Z(\mathbb{A})G_{nd}(F)\backslash G_{nd}(\mathbb{A}); \omega)$ be the space of classes of functions f defined on $G_{nd}(\mathbb{A})$ with values in \mathbb{C} such that f is left invariant under $G_{nd}(F)$, $f(zg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A})$ and almost all $g \in G_{nd}(\mathbb{A})$ and $|f|^2$ is integrable over $Z(\mathbb{A})G_{nd}(F)\backslash G_{nd}(\mathbb{A})$. The group $G_{nd}(\mathbb{A})$ acts by right translations on $L^2(Z(\mathbb{A})G_{nd}(F)\backslash G_{nd}(\mathbb{A}); \omega)$. We call a **discrete series** of $G_{nd}(\mathbb{A})$ an irreducible subrepresentation of such a representation (for any smooth unitary character ω of $Z(\mathbb{A})$ trivial on $Z(F)$). We adopt the analogous definition for the group $G'_n(\mathbb{A})$.

Denote DS_{nd} (resp. DS'_n) the set of discrete series of $G_{nd}(\mathbb{A})$ (resp. $G'_n(\mathbb{A})$). If π is a discrete series of $G_{nd}(\mathbb{A})$ or $G'_n(\mathbb{A})$, and v is a place of F ,

we denote π_v the local component of π at the place v . We will say that a discrete series π of $G_{nd}(\mathbb{A})$ is **D-compatible** if π_v is d_v -compatible for all places $v \in V$.

If $v \in V$, the Jacquet–Langlands correspondence between d_v -compatible unitary representations of $GL_{nd}(F_v)$ and $GL_{r_v}(D_v)$ will be denoted $|\mathbf{LJ}|_v$. Recall that if $v \notin V$, we have identified the groups $GL_{r_v}(D_v)$ and $GL_{nd}(F_v)$. We have the following (Theorem 5.1):

- Theorem.** (a) *There exists a unique injective map $\mathbf{G} : DS'_n \rightarrow DS_{nd}$ such that, for all $\pi' \in DS'_n$, we have $\mathbf{G}(\pi')_v = \pi'_v$ for every place $v \notin V$. For every $v \in V$, $\mathbf{G}(\pi')_v$ is d_v -compatible and we have $|\mathbf{LJ}|_v(\mathbf{G}(\pi')_v) = \pi'_v$. The image of \mathbf{G} is the set of D-compatible elements of DS_{nd} .*
- (b) *One has multiplicity one and strong multiplicity one theorems for the discrete spectrum of $G'_n(\mathbb{A})$.*

Since the original work of [JL] (see also [GeJ]), global correspondences with division algebras under some conditions (on the division algebra or on the representation to be transferred) have already been carried out (sometimes not explicitly stated) at least in [Fl2], [He], [Ro], [Vi], [DKV], [Fl1] and [Ba4]. They were using simple forms of the trace formula. For the general result obtained here these formulas are not sufficient. Our work is heavily based on the comparison of the general trace formulas for $G'_n(\mathbb{A})$ and $G_{nd}(\mathbb{A})$ carried out in [AC]. The reader should not be misled by the fact that here we use directly the simple formula Arthur and Clozel obtained in their over 200 pages long work. Their work overcomes big global difficulties and together with methods from [JL] and [DKV] reduces the global transfer of representations to local problems.

Let us explain now what are the main extra ingredients required for application of the spectral identity of [AC] in the proof of the theorem. The spectral identity as stated in [AC] is roughly speaking (and after using the multiplicity one theorem for $G_{nd}(\mathbb{A})$) of the type

$$\begin{aligned} \sum \text{tr}(\sigma_I)(f) + \sum \lambda_J \text{tr}(M_J \pi_J)(f) \\ = \sum m'_i \text{tr}(\sigma'_i)(f') + \sum \lambda'_j \text{tr}(M'_j \pi'_j)(f') \end{aligned}$$

where λ_J and λ'_j are certain coefficients, σ_I (resp. σ'_i) are discrete series of $G_{nd}(\mathbb{A})$ (resp. of $G'_n(\mathbb{A})$) of multiplicity m'_i , π_J (resp. π'_j) are representations of $G_{nd}(\mathbb{A})$ (resp. of $G'_n(\mathbb{A})$) which are induced from discrete series of proper Levi subgroups and M_J and M'_j are certain intertwining operators. As for f and f' , they are functions with matching orbital integrals.

The main step in proving the theorem is to choose a discrete series σ' of $G'_n(\mathbb{A})$ and to use the spectral identity to define $\mathbf{G}(\sigma')$. The crucial result is the local transfer of unitary representations (Proposition 3.9(c) of this paper) which allows to “globally” transfer the representations from the left side to the right side. This gives the correspondence when $n = 1$ as in [JL] or [Vi]. The trouble when $n > 1$ is that we do not know much about

the operators M'_j . We overcome this by induction over n . Then the Proposition 3.9(b) shows that π'_j are irreducible. This turns out to be enough to show that the contribution of σ' to the equality cannot be canceled by contributions from properly induced representations.

In the sequel of the fifth section we give a classification of representations of $G'_n(\mathbb{A})$. We define the notion of a basic cuspidal representation for groups of type $G'_k(\mathbb{A})$ (see Proposition 5.5 and the sequel). These basic cuspidal representations are all cuspidal. Neven Grbac will show in his appendix that these are actually the only cuspidal representations. Then the residual discrete series of $G'_n(\mathbb{A})$ are obtained from cuspidal representations in the same way the residual discrete series of $GL_n(\mathbb{A})$ are obtained from cuspidal representations in [MW2]. This classification is obtained directly by transfer from the Mœglin–Waldspurger classification for G_n .

Moreover, for any (irreducible) automorphic representation π' of G'_n , we know that ([La]) there exists a couple (P', ρ') where P' is a parabolic subgroup of G'_n containing the group of upper triangular matrices and ρ' is a cuspidal representation of the Levi factor L' of P' twisted by a real non-ramified character such that π' is a constituent (in the sense of [La]) of the induced representation from ρ' to G'_n with respect to P' . We prove (Proposition 5.7(c)) that this couple (ρ', L') is unique up to conjugation. This result is an analogue for G'_n of Theorem 4.4 of [JS].

The last section is devoted to the computation of L -functions, ϵ' -factors (in the sense of [GJ]) and their behavior under the local transfer of irreducible (especially unitary) representations. The behavior of the ϵ -factors then follows. These calculations are either well known or trivial, but we feel it is natural to give them explicitly here. The L -functions and ϵ' -factors in question are preserved under the correspondence for square integrable representations. In general, ϵ' -factors (but not L -functions) are preserved under the correspondence for irreducible unitary representations.

In the appendix Neven Grbac completes the classification of the discrete spectrum by showing that all the representations except the basic cuspidal ones are residual. His approach applies the Langlands spectral theory.

The essential part of this work has been done at the Institute for Advanced Study, Princeton, during the year 2004 and I would like to thank the Institute for the warm hospitality and support. They were expounded in a preprint from the beginning of 2006. The present paper contains exactly the same local results as that preprint. Two major improvements obtained in 2007 concern the global results. The first one is the proof of the fact that any discrete series of the inner form transfers (based on a better understanding of the trace formula from [AC]). The second is a complete classification of the residual spectrum thanks to the appendix of Neven Grbac.

The research at the IAS has been supported by the NSF fellowship no. DMS-0111298. I would like to thank Robert Langlands and James Arthur for useful discussions about global representations; Marko Tadić and David Renard for useful discussions on the local unitary dual; Abderrazak Bouaziz

who explained to me the intertwining operators. I would like to thank Guy Henniart and Colette Mœglin for the interest they showed for this work and their invaluable advices. I thank Neven Grbac for his appendix where he carries out the last and important step of the classification, and for his remarks on the manuscript. Discussions with Neven Grbac have been held during our stay at the Erwin Schrödinger Institute in Vienna and I would like to thank here Joachim Schwermer for his invitation.

2. Basic facts and notation (local)

In the sequel \mathbb{N} will denote the set of non-negative integers and \mathbb{N}^* the set of positive integers. A **multiset** is a set with finite repetitions. If $x \in \mathbb{R}$, then $[x]$ will denote the biggest integer inferior or equal to x .

Let F be a non-Archimedean local field and D a central division algebra of a finite dimension over F . Then the dimension of D over F is a square d^2 , $d \in \mathbb{N}^*$. If $n \in \mathbb{N}^*$, we set $G_n = GL_n(F)$ and $G'_n = GL_n(D)$. From now on we identify a smooth representation of finite length with its equivalence class, so we will consider two equivalent representations as being equal. By a **character** of G_n we mean a smooth representation of dimension one of G_n . In particular a character is not unitary unless we specify it. Let σ be an irreducible smooth representation of G_n . We say σ is **square integrable** if σ is unitary and has a non-zero matrix coefficient which is square integrable modulo the center of G_n . We say σ is **essentially square integrable** if σ is the twist of a square integrable representation by a character of G_n . We say σ is **cuspidal** if σ has a non-zero matrix coefficient which has compact support modulo the center of G_n . In particular a cuspidal representation is essentially square integrable.

For all $n \in \mathbb{N}^*$ let us fix the following notation:

Irr_n is the set of smooth irreducible representations of G_n ,

\mathcal{D}_n is the subset of essentially square integrable representations in Irr_n ,

\mathcal{C}_n is the subset of cuspidal representations in \mathcal{D}_n ,

Irr_n^u (resp. \mathcal{D}_n^u , \mathcal{C}_n^u) is the subset of unitary representations in Irr_n (resp. \mathcal{D}_n , \mathcal{C}_n),

\mathcal{R}_n is the Grothendieck group of admissible representations of finite length of G_n ,

v is the character of G_n defined by the absolute value of the determinant (notation independent of n – this will lighten the notation and cause no ambiguity in the sequel).

For any $\sigma \in \mathcal{D}_n$, there is a unique couple $(e(\sigma), \sigma^u)$ such that $e(\sigma) \in \mathbb{R}$, $\sigma^u \in \mathcal{D}_n^u$ and $\sigma = v^{e(\sigma)} \sigma^u$.

We will systematically identify $\pi \in Irr_n$ with its image in \mathcal{R}_n and consider Irr_n as a subset of \mathcal{R}_n . Then Irr_n is a \mathbb{Z} -basis of the \mathbb{Z} -module \mathcal{R}_n .

If $n \in \mathbb{N}^*$ and (n_1, n_2, \dots, n_k) is an ordered set of positive integers such that $n = \sum_{i=1}^k n_i$ then the subgroup L of G_n consisting of block diagonal matrices with blocks of sizes n_1, n_2, \dots, n_k in this order from the left upper corner to the right lower corner is called a **standard Levi subgroup** of G_n . The group L is canonically isomorphic with the product $\times_{i=1}^k G_{n_i}$, and we will identify these two groups. Then the notation $Irr(L), \mathcal{D}(L), \mathcal{C}(L), \mathcal{D}^u(L), \mathcal{C}^u(L), \mathcal{R}(L)$ extend in an obvious way to L . In particular $Irr(L)$ is canonically isomorphic to $\times_{i=1}^k Irr_{n_i}$ and so on.

We denote $ind_L^{G_n}$ the normalized parabolic induction functor where it is understood that we induce with respect to the parabolic subgroup of G_n containing L and the subgroup of upper triangular matrices. Then $ind_L^{G_n}$ extends to a group morphism $\mathbf{i}_L^{G_n} : \mathcal{R}(L) \rightarrow \mathcal{R}_n$. If $\pi_i \in \mathcal{R}_{n_i}$ for $i \in \{1, 2, \dots, k\}$ and $n = \sum_{i=1}^k n_i$, we denote $\pi_1 \times \pi_2 \times \dots \times \pi_k$ or abridged $\prod_{i=1}^k \pi_i$ the representation

$$ind_{\times_{i=1}^k G_{n_i}}^{G_n} \otimes_{i=1}^k \sigma_i$$

of G_n . Let π be a smooth representation of finite length of G_n . If distinction between quotient, subrepresentation and subquotient of π is not relevant, we consider π as an element of \mathcal{R}_n (identification with its class) with no extra explanation.

If $g \in G_n$ for some n , we say g is **regular semisimple** if the characteristic polynomial of g has distinct roots in an algebraic closure of F . If $\pi \in \mathcal{R}_n$, then we let χ_π denote the function character of π , as a locally constant map, stable under conjugation, defined on the set of regular semisimple elements of G_n .

We adopt the same notation adding a sign $'$ for $G'_n: Irr'_n, \mathcal{D}'_n, \mathcal{C}'_n, Irr'^u_n, \mathcal{D}'^u_n, \mathcal{C}'^u_n, \mathcal{R}'_n$.

There is a standard way of defining the determinant and the characteristic polynomial for elements of G'_n , in spite of D being non-commutative (see for example [Pi, Sect. 16]). If $g' \in G'_n$, then the characteristic polynomial of g' has coefficients in F , it is monic and has degree nd . The definition of a regular semisimple element of G'_n is then the same as for G_n . If $\pi \in \mathcal{R}'_n$, we let again χ_π be the function character of π . As for G_n , we will denote ν the character of G'_n given by the absolute value of the determinant (there will be no confusion with the one on G_n).

2.1. Classification of Irr_n (resp. Irr'_n) in terms of \mathcal{D}_l (resp. \mathcal{D}'_l), $l \leq n$.

Let $\pi \in Irr_n$. There exists a standard Levi subgroup $L = \times_{i=1}^k G_{n_i}$ of G_n and, for all $1 \leq i \leq k$, $\rho_i \in \mathcal{C}_{n_i}$, such that π is a subquotient of $\prod_{i=1}^k \rho_i$. The non-ordered multiset of cuspidal representations $\{\rho_1, \rho_2, \dots, \rho_k\}$ is determined by π and is called **the cuspidal support of π** .

We recall the Langlands classification which takes a particularly nice form on G_n . Let $L = \times_{i=1}^k G_{n_i}$ be a standard Levi subgroup of G_n and

$\sigma \in \mathcal{D}(L) = \times_{i=1}^k \mathcal{D}_{n_i}$. Let us write $\sigma = \otimes_{i=1}^k \sigma_i$ with $\sigma_i \in \mathcal{D}_{n_i}$. For each i , write $\sigma_i = v^{e_i} \sigma_i^u$, where $e_i \in \mathbb{R}$ and $\sigma_i^u \in \mathcal{D}_{n_i}^u$. Let p be a permutation of the set $\{1, 2, \dots, k\}$ such that the sequence $e_{p(i)}$ is decreasing. Let $L_p = \times_{i=1}^k G_{n_{p(i)}}$ and $\sigma_p = \otimes_{i=1}^k \sigma_{p(i)}$. Then $\text{ind}_{L_p}^{G_n} \sigma_p$ has a unique irreducible quotient π and π is independent of the choice of p under the condition that $(e_{p(i)})_{1 \leq i \leq k}$ is decreasing. So π is defined by the non-ordered multiset $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$. We write then $\pi = Lg(\sigma)$. Every $\pi \in \text{Irr}_n$ is obtained in this way. If $\pi \in \text{Irr}_n$ and $L = \times_{i=1}^k G_{n_i}$ and $L' = \times_{j=1}^{k'} G_{n'_j}$ are two standard Levi subgroups of G_n , if $\sigma = \otimes_{i=1}^k \sigma_i$, with $\sigma_i \in \mathcal{D}_{n_i}$, and $\sigma' = \otimes_{j=1}^{k'} \sigma'_j$, with $\sigma'_j \in \mathcal{D}_{n'_j}$, are such that $\pi = Lg(\sigma) = Lg(\sigma')$, then $k = k'$ and there exists a permutation p of $\{1, 2, \dots, k\}$ such that $n'_j = n_{p(i)}$ and $\sigma'_j = \sigma_{p(i)}$. So the non-ordered multiset $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$ is determined by π and it is called **the essentially square integrable support of π** which we abridge as **the esi-support of π** .

An element $S = \mathbf{i}_L^{G_n} \sigma$ of \mathcal{R}_n , with $\sigma \in \mathcal{D}(L)$, is called a **standard representation** of G_n . We will often write $Lg(S)$ for $Lg(\sigma)$. The set \mathcal{B}_n of standard representations of G_n is a basis of \mathcal{R}_n and the map $S \mapsto Lg(S)$ is a bijection from \mathcal{B}_n onto Irr_n . All these results are consequences of the Langlands classification (see [Ze] and [Rod]). We also have the following result: if for all $\pi \in \text{Irr}_n$ we write $\pi = Lg(S)$ for some standard representation S and then for all $\pi' \in \text{Irr}_n \setminus \{\pi\}$ we set $\pi' < \pi$ if and only if π' is a subquotient of S , then we obtain a well defined partial order relation on Irr_n .

The same definitions and theory, including the order relation, hold for G'_n (see [Ta2]). The set of standard representations of G'_n is denoted here by \mathcal{B}'_n .

For G_n or G'_n we have the following proposition, where σ_1 and σ_2 are essentially square integrable representations:

- Proposition 2.1.** (a) *The representation $Lg(\sigma_1) \times Lg(\sigma_2)$ contains $Lg(\sigma_1 \times \sigma_2)$ as a subquotient with multiplicity 1.*
 (b) *If π is another irreducible subquotient of $Lg(\sigma_1) \times Lg(\sigma_2)$, then $\pi < Lg(\sigma_1 \times \sigma_2)$. In particular, if $Lg(\sigma_1) \times Lg(\sigma_2)$ is reducible, it has at least two different subquotients.*

For G_n , Assertion (a) is proven in its dual form in [Ze, Proposition 8.4]. It is proven in its present form in [Ta2, Proposition 2.3] for the more general case of G'_n . Assertion (b) is then obvious because of the definition (here) of the order relation, and since any irreducible subquotient of $Lg(\sigma_1) \times Lg(\sigma_2)$ is also an irreducible subquotient of $\sigma_1 \times \sigma_2$.

2.2. Classification of \mathcal{D}_n in terms of $\mathcal{C}_l, l|n$. Let k and l be two positive integers and set $n = kl$. Let $\rho \in \mathcal{C}_l$. Then the representation $\prod_{i=0}^{k-1} v^i \rho$ has a unique irreducible quotient σ . σ is an essentially square integrable representation of G_n . We write then $\sigma = Z(\rho, k)$. Every $\sigma \in \mathcal{D}_n$ is obtained in this way and l, k and ρ are determined by σ . This may be found in [Ze].

In general, a set $X = \{\rho, v\rho, v^2\rho, \dots, v^{a-1}\rho\}$, $\rho \in \mathcal{C}_b$, $a, b \in \mathbb{N}^*$, is called a **segment**, a is the **length** of the segment X and $v^{a-1}\rho$ is the **ending** of X .

2.3. Local Jacquet–Langlands correspondence. Let $n \in \mathbb{N}^*$. Let $g \in G_{nd}$ and $g' \in G'_n$. We say that g **corresponds** to g' if g and g' are regular semisimple and have the same characteristic polynomial. We shortly write then $g \leftrightarrow g'$.

Theorem 2.2. *There is a unique bijection $\mathbf{C} : \mathcal{D}_{nd} \rightarrow \mathcal{D}'_n$ such that for all $\pi \in \mathcal{D}_{nd}$ we have*

$$\chi_\pi(g) = (-1)^{nd-n} \chi_{\mathbf{C}(\pi)}(g')$$

for all $g \in G_{nd}$ and $g' \in G'_n$ such that $g \leftrightarrow g'$.

For the proof, see [DKV] if the characteristic of the base field F is zero and [Ba2] for the non-zero characteristic case. I should quote here also the particular cases [JL], [Fl2] and [Ro] which contain some germs of the general proof in [DKV].

We identify the centers of G_{nd} and G'_n via the canonical isomorphism. Then the correspondence \mathbf{C} preserves central characters so in particular $\sigma \in \mathcal{D}_{nd}^u$ if and only if $\mathbf{C}(\sigma) \in \mathcal{D}'^u_n$.

If $L' = \times_{i=1}^k G'_{n_i}$ is a standard Levi subgroup of G'_n we say that the standard Levi subgroup $L = \times_{i=1}^k G_{dn_i}$ of G_{nd} **corresponds** to L' . Then the Jacquet–Langlands correspondence extends in an obvious way to a bijective correspondence $\mathcal{D}(L)$ to $\mathcal{D}'(L')$ with the same properties. We will denote this correspondence by the same letter \mathbf{C} . A standard Levi subgroup L of G_n corresponds to a standard Levi subgroup of G'_r if and only if it is defined by a sequence (n_1, n_2, \dots, n_k) such that each n_i is divisible by d . We then say that L **transfers**.

2.4. Classification of \mathcal{D}'_n in terms of $\mathcal{C}'_l, l|n$. The invariant $s(\sigma')$. Let l be a positive integer and $\rho' \in \mathcal{C}'_l$. Then $\sigma = \mathbf{C}^{-1}(\rho')$ is an essentially square integrable representation of G_{ld} . We may write $\sigma = Z(\rho, p)$ for some $p \in \mathbb{N}^*$ and some $\rho \in \mathcal{C}'_{\frac{ld}{p}}$. Set then $s(\rho') = p$ and $v_{\rho'} = v^{s(\rho')}$.

Let k and l be two positive integers and set $n = kl$. Let $\rho' \in \mathcal{C}'_l$. Then the representation $\prod_{i=0}^{k-1} v_{\rho'}^i \rho'$ has a unique irreducible quotient σ' . σ' is an essentially square integrable representation of G'_n . We write then $\sigma' = T(\rho', k)$. Every $\sigma' \in \mathcal{D}'_n$ is obtained in this way and l, k and ρ' are determined by σ' . We set then $s(\sigma') = s(\rho')$. For this classification see [Ta2].

A set $S' = \{\rho', v_{\rho'}\rho', v_{\rho'}^2\rho', \dots, v_{\rho'}^{a-1}\rho'\}$, $\rho' \in \mathcal{C}'_b$, $a, b \in \mathbb{N}^*$, is called a **segment**, a is the **length** of S' and $v_{\rho'}^{a-1}\rho'$ is the **ending** of S' .

2.5. Multisegments, order relation, the function \mathbf{l} and rigid representations. Here we will give the definitions and results in terms of groups G_n ,

but one may replace G_n by G'_n . We have seen (Sects. 2.2 and 2.4) that to each $\sigma \in \mathcal{D}_n$ one may associate a segment. A multiset of segments is called a **multisegment**. If M is a multisegment, the multiset of endings of its elements (see Sects. 2.2 and 2.4 for the definition) is denoted $E(M)$.

If $\pi \in G_n$, the multiset of the segments of the elements of the esi-support of π is a multisegment; we will denote it by M_π . M_π determines π . The reunion with repetitions of the elements of M_π is the cuspidal support of π .

Two segments S_1 and S_2 are said to be **linked** if $S_1 \cup S_2$ is a segment different from S_1 and S_2 . If S_1 and S_2 are linked, we say they are **adjacent** if $S_1 \cap S_2 = \emptyset$.

Let M be a multisegment, and assume S_1 and S_2 are two linked segments in M . Let M' be the multisegment defined by

- $M' = (M \cup \{S_1 \cup S_2\} \cup \{S_1 \cap S_2\}) \setminus \{S_1, S_2\}$ if S_1 and S_2 are not adjacent (i.e. $S_1 \cap S_2 \neq \emptyset$), and
- $M' = (M \cup \{S_1 \cup S_2\}) \setminus \{S_1, S_2\}$ if S_1 and S_2 are adjacent (i.e. $S_1 \cap S_2 = \emptyset$).

We say that we made an **elementary operation** on M to get M' , or that M' was obtained from M by an elementary operation. We then say M' is inferior to M . It is easy to verify this extends by transitivity to a well defined partial order relation $<$ on the set of multisegments of G_n . The following proposition is a result of [Ze, Theorem 7.1] for G_n and [Ta2, Theorem 5.3] for G'_n .

Proposition 2.3. *If $\pi, \pi' \in Irr_n$, then $\pi < \pi'$ if and only if $M_\pi < M_{\pi'}$.*

If $\pi < \pi'$, then the cuspidal support of π equals the cuspidal support of π' .

Define a function **I** on the set of multisegments as follows: if M is a multisegment, then $\mathbf{I}(M)$ is the maximum of the lengths of the segments in M . If $\pi \in Irr_n$, set $\mathbf{I}(\pi) = \mathbf{I}(M_\pi)$. The following Lemma is obvious:

Lemma 2.4. *If M' is obtained from M by an elementary operation then $\mathbf{I}(M) \leq \mathbf{I}(M')$ and $E(M') \subseteq E(M)$. As a function on Irr_n , **I** is decreasing.*

The next important proposition is also a result from [Ze] and [Ta2]:

Proposition 2.5. *Let $\pi \in Irr_k$ and $\pi' \in Irr_l$. If for all $S \in M_\pi$ and $S' \in M_{\pi'}$ the segments S and S' are not linked, then $\pi \times \pi'$ is irreducible.*

There is an interesting consequence of this last proposition. Let $l \in \mathbb{N}^*$ and $\rho \in \mathcal{C}_l$. We will call the set $X = \{v^a \rho\}_{a \in \mathbb{Z}}$ a **line**, the line generated by ρ . Of course X is also the line generated by $v\rho$ for example. If $\pi \in Irr_n$, we say π is **rigid** if the set of elements of the cuspidal support of π is included in a single line. As a consequence of the previous proposition we have the

Corollary 2.6. *Let $\pi \in Irr_n$. Let X be the set of the elements of the cuspidal support of π . If $\{D_1, D_2, \dots, D_m\}$ is the set of all the lines with which X*

has a non-empty intersection, then one may write in the unique (up to permutation) way $\pi = \pi_1 \times \pi_2 \times \dots \times \pi_m$ with π_i rigid irreducible and the set of elements of the cuspidal support of π_i included in D_i , $1 \leq i \leq m$.

We will say $\pi = \pi_1 \times \pi_2 \times \dots \times \pi_m$ is the **standard decomposition** of π in a product of rigid representations (this is only *the shortest* decomposition of π in a product of rigid representations, but there might exist finer ones).

The same holds for G'_n .

2.6. The involution. Aubert defined in [Au] an involution (studied too by Schneider and Stuhler in [ScS]) of the Grothendieck group of smooth representations of finite length of a reductive group over a local non-Archimedean field. The involution sends an irreducible representation to an irreducible representation up to a sign. We specialize this involution to G_n , resp. G'_n , and denote it i_n , resp. i'_n . We will write i and i' when the index is not relevant or it is clearly understood. With this notation we have the relation $i(\pi_1) \times i(\pi_2) = i(\pi_1 \times \pi_2)$, i.e. “the involution commutes with the parabolic induction”. The same holds for i' . The reader may find all these facts in [Au].

If $\pi \in Irr_n$, then one and only one among $i(\pi)$ and $-i(\pi)$ is an irreducible representation. We denote it by $|i(\pi)|$. We denote $|i|$ the involution of Irr_n defined by $\pi \mapsto |i(\pi)|$. The same facts and definitions hold for i' .

The algorithm conjectured by Zelevinsky for computing the esi-support of $|i(\pi)|$ from the esi-support of π when π is rigid (and hence more generally for $\pi \in Irr_n$, cf. Corollary 2.6) is proven in [MW1]. The same facts and algorithm hold for $|i'|$ as explained in [BR2].

2.7. The extended correspondence. The correspondence \mathbf{C}^{-1} may be extended in a natural way to a correspondence **LJ** between the Grothendieck groups. Let $S' = \mathbf{i}_{L'}^{G'_n} \sigma' \in \mathcal{B}'_n$, where L' is a standard Levi subgroup of G'_n and σ' an essentially square integrable representation of L' . Set $M_n(S') = \mathbf{i}_L^{G_{nd}} \mathbf{C}^{-1}(\sigma')$, where L is the standard Levi subgroup of G_{nd} corresponding to L' . Then $M_n(S')$ is a standard representation of G_{nd} and M_n realizes an injective map from \mathcal{B}'_n into \mathcal{B}_{nd} . Define $Q_n : Irr'_n \rightarrow Irr_{nd}$ by $Q_n(Lg(S')) = Lg(M_n(S'))$. If $\pi'_1 < \pi'_2$, then $Q_n(\pi'_1) < Q_n(\pi'_2)$. So Q_n induces on $Irr(G'_n)$, by transfer from G_{nd} , an order relation \ll which is stronger than $<$.

Let $\mathbf{LJ}_n : \mathcal{R}_{nd} \rightarrow \mathcal{R}'_n$ be the \mathbb{Z} -morphism defined on \mathcal{B}_{nd} by setting $\mathbf{LJ}_n(M_n(S')) = S'$ and $\mathbf{LJ}_n(S) = 0$ if S is not in the image of M_n .

Theorem 2.7. (a) For all $n \in \mathbb{N}^*$, \mathbf{LJ}_n is the unique map from \mathcal{R}_{nd} to \mathcal{R}'_n such that for all $\pi \in \mathcal{R}_{nd}$ we have

$$\chi_\pi(g) = (-1)^{nd-n} \chi_{\mathbf{LJ}_n(\pi)}(g')$$

for all $g \leftrightarrow g'$.

(b) The map \mathbf{LJ}_n is a surjective group morphism.

(c) *One has*

$$\mathbf{LJ}_n(Q_n(\pi')) = \pi' + \sum_{\pi'_j \ll \pi'} b_j \pi'_j$$

where $b_j \in \mathbb{Z}$ and $\pi'_j \in \text{Irr}'_n$.

(d) *One has*

$$\mathbf{LJ}_n \circ i_{nd} = (-1)^{nd-n} i'_n \circ \mathbf{LJ}_n.$$

See [Ba4]. We will often drop the index and write only Q , M and \mathbf{LJ} . \mathbf{LJ} may be extended in an obvious way to standard Levi subgroups. For a standard Levi subgroup L' of G'_n which correspond to a standard Levi subgroup L of G_{nd} we have $\mathbf{LJ} \circ i_L^{G_{nd}} = i_{L'}^{G'_n} \circ \mathbf{LJ}$.

We will say that $\pi \in \mathcal{R}_{nd}$ is d -**compatible** if $\mathbf{LJ}_n(\pi) \neq 0$. This means that there exists a regular semisimple element g of G_{nd} which corresponds to an element of G'_n and such that $\chi_\pi(g) \neq 0$. A regular semisimple element of G_{nd} corresponds to an element of G'_n if and only if its characteristic polynomial decomposes into irreducible factors with the degrees divisible by d . So our definition depends only on d , not on D . A product of representations is d -compatible if and only if each factor is d -compatible.

2.8. Unitary representations of G_n . We are going to use the word **unitary** for **unitarizable**. Let k, l be positive integers and set $kl = n$.

Let $\rho \in \mathcal{C}_l$ and set $\sigma = Z(\rho, k)$. Then σ is unitary if and only if $v^{\frac{k-1}{2}} \rho$ is unitary. We set then $\rho^u = v^{\frac{k-1}{2}} \rho \in \mathcal{C}_l^u$ and we write $\sigma = Z^u(\rho^u, k)$. From now on, anytime we write $\sigma = Z^u(\rho, k)$, it is understood that σ and ρ are unitary.

Now, if $\sigma \in \mathcal{D}_l^u$, we set

$$u(\sigma, k) = Lg\left(\prod_{i=0}^{k-1} v^{\frac{k-1}{2}-i} \sigma\right).$$

The representation $u(\sigma, k)$ is an irreducible representation of G_n .

If $\alpha \in]0, \frac{1}{2}[$, we moreover set

$$\pi(u(\sigma, k), \alpha) = v^\alpha u(\sigma, k) \times v^{-\alpha} u(\sigma, k).$$

The representation $\pi(u(\sigma, k), \alpha)$ is an irreducible representation of G_{2n} (by Proposition 2.5).

Let us recall the Tadić classification of unitary representations in [Ta1].

Let \mathcal{U} be the set of all the representations $u(\sigma, k)$ and $\pi(u(\sigma, k), \alpha)$ where k, l range over \mathbb{N}^* , $\sigma \in \mathcal{C}_l$ and $\alpha \in]0, \frac{1}{2}[$. Then any product of elements of \mathcal{U} is irreducible and unitary. Every irreducible unitary representation π of some G_n , $n \in \mathbb{N}^*$, is such a product. The non-ordered multiset of the factors of the product are determined by π .

The fact that a product of irreducible unitary representations is irreducible is due to Bernstein ([Be]).

Tadić computed the decomposition of the representation $u(\sigma, k)$ in the basis \mathcal{B}_n of \mathcal{R}_n .

Proposition 2.8. ([Ta4]) *Let $\sigma = Z(\rho, l)$ and $k \in \mathbb{N}^*$. Let W_k^l be the set of permutations w of $\{1, 2, \dots, k\}$ such that $w(i) + l \geq i$ for all $i \in \{1, 2, \dots, k\}$. Then we have:*

$$u(\sigma, k) = v^{-\frac{k+l}{2}} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k Z(v^i \rho, w(i) + l - i) \right).$$

One can also compute the dual of $u(\sigma, k)$.

Proposition 2.9. *Let $\sigma = Z^u(\rho^u, l)$ and $k \in \mathbb{N}^*$. If $\tau = Z^u(\rho^u, k)$, then*

$$|i(u(\sigma, k))| = u(\tau, l).$$

This is the Theorem 7.1 iii) [Ta1], and also a consequence of [MW1].

2.9. Unitary representations of G'_n . Let $k, l \in \mathbb{N}^*$ and set $n = kl$. Let $\rho \in \mathcal{C}'_l$ and $\sigma' = T(\rho', k) \in \mathcal{D}'_n$. As for G_n , one has $\sigma' \in \mathcal{D}'^u_n$ if and only if $v^{\frac{k-1}{2}} \rho'$ is unitary; we set then $\rho'^u = v^{\frac{k-1}{2}} \rho'$ and write $\sigma' = T^u(\rho'^u, k)$.

If now $\sigma' \in \mathcal{D}'^u_l$, we set

$$u'(\sigma', k) = Lg \left(\prod_{i=0}^{k-1} v^{\frac{k-1}{2}-i} \sigma' \right)$$

and

$$\bar{u}(\sigma', k) = Lg \left(\prod_{i=0}^{k-1} v^{\frac{k-1}{2}-i} \sigma' \right).$$

The representations $u'(\sigma', k)$ and $\bar{u}(\sigma', k)$ are irreducible representations of G'_n .

If moreover $\alpha \in]0, \frac{1}{2}[$, we set

$$\pi(u'(\sigma', k), \alpha) = v_\sigma^\alpha u'(\sigma', k) \times v_\sigma^{-\alpha} u'(\sigma', k).$$

The representation $\pi(u'(\sigma', k), \alpha)$ is an irreducible representation of G'_{2n} (cf. [Ta2]; a consequence of the (restated) Proposition 2.5 here).

We have the formulas:

$$(2.1) \quad \bar{u}(\sigma', ks(\sigma')) = \prod_{i=1}^{s(\sigma')} v^{i-\frac{s(\sigma')+1}{2}} u'(\sigma', k);$$

and, for all integers $1 \leq b \leq s(\sigma') - 1$,

$$(2.2) \quad \bar{u}(\sigma', ks(\sigma') + b) = \left(\prod_{i=1}^b v^{i-\frac{b+1}{2}} u'(\sigma', k + 1) \right) \times \left(\prod_{j=1}^{s(\sigma')-b} v^{j-\frac{s(\sigma')-b+1}{2}} u'(\sigma', k) \right),$$

with the convention that we ignore the second product if $k = 0$.

The products are irreducible, by Proposition 2.5, because the segments appearing in the esi-support of two different factors are never linked. The fact that the product is indeed $\bar{u}(\sigma', ks(\sigma'))$ (and resp. $\bar{u}(\sigma', ks(\sigma') + b)$) is then clear by Proposition 2.1. This kind of formulas has been used (at least) in [BR1] and [Ta6].

The representations $u'(\sigma', k)$ and $\bar{u}(\sigma', k)$ are known to be unitary at least in zero characteristic ([Ba4] and [BR1]).

One has

Proposition 2.10. *Let $\sigma' = Z^u(\rho''', l)$ and $k \in \mathbb{N}^*$. If $\tau' = Z^u(\rho''', k)$, then (a) $|i'(u'(\sigma', k))| = u'(\tau', l)$ and (b) $|i'(\bar{u}(\sigma', ks(\sigma')))| = \bar{u}(\tau', ls(\sigma'))$.*

Proof. The claim (a) is a direct consequence of [BR2]. For the claim (b), it is enough to use the relation (2.1), the claim (a) here and the fact that i' commutes with parabolic induction. □

2.10. Hermitian representations and an irreducibility trick. If $\pi \in Irr'_n$, write $h(\pi)$ for the complex conjugated representation of the contragredient of π . A representation $\pi \in Irr'_n$ is called **hermitian** if $\pi = h(\pi)$ (we recall, to avoid confusion, that here we use “=” for the usual “equivalent”). A unitary representation is always hermitian. If $A = \{\sigma_i\}_{1 \leq i \leq k}$ is a multiset of essentially square integrable representations of some G'_{l_i} , we define the multiset $h(A)$ by $h(A) = \{h(\sigma_i)\}_{1 \leq i \leq k}$. If $\pi \in Irr'_n$ and $x \in \mathbb{R}$, then $h(v^x \pi) = v^{-x} h(\pi)$, so if $\sigma' \in \mathcal{D}'_l$ and we write $\sigma' = v^e \sigma'''$ with $e \in \mathbb{R}$ and $\sigma''' \in \mathcal{D}'_l$, then $h(\sigma') = v^{-e} \sigma''' \in \mathcal{D}'_l$. An easy consequence of Proposition 3.1.1 in [Ca] is the

Proposition 2.11. *If $\pi \in Irr'_n$, and A is the esi-support of π , then $h(A)$ is the esi-support of $h(\pi)$. In particular, π is hermitian if and only if the esi-support A of π satisfies $h(A) = A$.*

Let us give a lemma.

Lemma 2.12. *Let $\pi_1 \in Irr'_{n_1}$ and $\pi_2 \in Irr'_{n_2}$ and assume $h(\pi_1) \neq \pi_2$. Then there exists $\varepsilon > 0$ such that for all $x \in]0, \varepsilon[$ the representation $a_x = v^x \pi_1 \times v^{-x} \pi_2$ is irreducible, but not hermitian.*

Proof. For all $x \in \mathbb{R}$ let A_x be the esi-support of $v^x \pi_1$ and B_x be the esi-support of $v^{-x} \pi_2$. Then the set X of $x \in \mathbb{R}$ such that $A_x \cap h(A_x) \neq \emptyset$

or $B_x \cap h(B_x) \neq \emptyset$ is finite (it is enough to check the central character of the representations in these multisets). The set Y of $x \in \mathbb{R}$ such that the cuspidal supports of A_x and B_x have a non-empty intersection is finite too. Now, if $x \in \mathbb{R} \setminus Y$, a_x is irreducible by the Proposition 2.5. Assume moreover $x \notin X$. As a_x is irreducible, if it were hermitian one should have $h(A_x) \cup h(B_x) = A_x \cup B_x$ (where the reunions are to be taken with multiplicities, as reunions of multisets) by the Proposition 2.11. But if $A_x \cap h(A_x) = \emptyset$ and $B_x \cap h(B_x) = \emptyset$, then this would lead to $h(A_x) = B_x$, and hence to $h(\pi_1) = \pi_2$ which contradicts the hypothesis. \square

We now state our irreducibility trick.

Proposition 2.13. *Let $u'_i \in Irr_{n_i}^u$, $i \in \{1, 2, \dots, k\}$. If, for all $i \in \{1, 2, \dots, k\}$, $u'_i \times u'_i$ is irreducible, then $\prod_{i=1}^k u'_i$ is irreducible.*

Proof. There exists $\varepsilon > 0$ such that for all $i \in \{1, 2, \dots, k\}$ the cuspidal supports of $v^x u'_i$ and $v^{-x} u'_i$ are disjoint for all $x \in]0, \varepsilon[$. Then, for all $i \in \{1, 2, \dots, k\}$, for all $x \in]0, \varepsilon[$, the representation $v^x u'_i \times v^{-x} u'_i$ is irreducible. As, by hypothesis, $u'_i \times u'_i$ is irreducible and unitary, the representation $v^x u'_i \times v^{-x} u'_i$ is also unitary for all $x \in]0, \varepsilon[$ (see for example [Ta3, Sect. (b)]). So $\prod_{i=1}^k v^x u'_i \times v^{-x} u'_i$ is a sum of unitary representations. But we have (in the Grothendieck group)

$$\prod_{i=1}^k (v^x u'_i \times v^{-x} u'_i) = (v^x \prod_{i=1}^k u'_i) \times (v^{-x} \prod_{i=1}^k u'_i).$$

If $\prod_{i=1}^k u'_i$ were reducible, then it would contain at least two different unitary subrepresentations π_1 and π_2 (Proposition 2.1). But then, for some $x \in]0, \varepsilon[$, $(v^x \prod_{i=1}^k u'_i) \times (v^{-x} \prod_{i=1}^k u'_i)$ contains an irreducible, but not hermitian, subquotient of the form $v^x \pi_1 \times v^{-x} \pi_2$ (by Lemma 2.12). This subquotient would be non-unitary which contradicts our assumption. \square

3. Local results

3.1. First results. Let $\sigma' \in \mathcal{D}_n^u$ and set $\sigma = \mathbf{C}^{-1}(\sigma') \in \mathcal{D}_{nd}^u$. Write $\sigma' = T^u(\rho', l)$ for some $l \in \mathbb{N}^*$, $l|n$ and $\rho' \in \mathcal{C}_l^u$. As $\mathbf{C}^{-1}(\rho') \in \mathcal{D}_{\frac{n}{l}}^u$ we may write $\mathbf{C}^{-1}(\rho') = Z^u(\rho, s(\sigma'))$ for some $\rho \in \mathcal{C}_{\frac{n}{l}}^u$. We set $l' = ls(\sigma')$. Then we have $\sigma = Z^u(\rho, l')$ (means one can recover the cuspidal support of σ from the cuspidal support of σ' ; it is a consequence of the fact that the correspondence commutes with the Jacquet functor; the original proof for square integrable representations is [DKV, Theorem B.2.b]).

Let k be a positive integer and set $k' = ks(\sigma')$. Let H be the group of permutations w of $\{1, 2, \dots, k'\}$ such that $s(\sigma')|w(i) - i$ for all $i \in$

$\{1, 2, \dots, k'\}$. For the meaning of W_k^l and $W_{k'}^{l'}$ in the following, see Proposition 2.8.

This is Lemma 3.1 in [Ta5]:

Lemma 3.1. *If $w \in H$, then for each $j \in \{1, 2, \dots, s(\sigma')\}$, the set of elements of $\{1, 2, \dots, k'\}$ equal to $j \pmod{s(\sigma')}$ is stable under w , and w induces a permutation w_j of $\{1, 2, \dots, k\}$ defined by the fact that, if $w(as(\sigma') + j) = bs(\sigma') + j$ then $w_j(a + 1) = b + 1$. The map $w \mapsto (w_1, w_2, \dots, w_{s(\sigma')})$ is an isomorphism of groups from H to $(\mathfrak{S}_k)^{s(\sigma')}$. One has $w \in H \cap W_{k'}^{l'}$ if and only if for all j , $w_j \in W_k^l$. Moreover, $\text{sgn}(w) = \prod_{j=1}^{s(\sigma')} \text{sgn}(w_j)$.*

We have the following:

Theorem 3.2. (a) *One has*

$$\mathbf{LJ}(u(\sigma, k')) = \bar{u}(\sigma', k').$$

(b) *The induced representation $\bar{u}(\sigma', k') \times \bar{u}(\sigma', k')$ is irreducible.*

(c) *We have the character formula*

$$\bar{u}(\sigma', k') = v^{-\frac{k'+l'}{2} + \frac{s(\sigma')-1}{2}} \left(\sum_{w \in H \cap W_{k'}^{l'}} (-1)^{\text{sgn}(w)} \prod_{i=1}^{k'} T \left(v^i \rho', \frac{w(i) - i}{s(\sigma')} + l \right) \right).$$

Proof. (a) Let $\tau' = T^u(\rho', k)$ and set $\tau = \mathbf{C}^{-1}(\tau')$. For the same reasons as explained for σ , we have $\tau = Z^u(\rho, k')$.

We apply Theorem 2.7(c) to $\bar{u}(\sigma', k')$ and $\bar{u}(\tau', l')$. We get

$$(3.1) \quad \mathbf{LJ}(u(\sigma, k')) = \bar{u}(\sigma', k') + \sum_{\pi'_j \ll \bar{u}(\sigma', k')} b_j \pi'_j$$

and

$$(3.2) \quad \mathbf{LJ}(u(\tau, l')) = \bar{u}(\tau', l') + \sum_{\tau'_q \ll \bar{u}(\tau', l')} c_q \tau'_q.$$

We want to show that all the b_j vanish.

Let us write the dual equation to (3.1) (cf. Theorem 2.7(d)). As $|i(u(\sigma, k'))| = u(\tau, l')$ (Proposition 2.9) and $|i'(\bar{u}(\sigma', k'))| = \bar{u}(\tau', l')$ (Proposition 2.10), we obtain:

$$(3.3) \quad \mathbf{LJ}(u(\tau, l')) = \varepsilon_1 \bar{u}(\tau', l') + \varepsilon_2 \sum_{\pi'_j \ll \bar{u}(\sigma', k')} b_j i'(\pi'_j)$$

for some signs $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. The equations (3.2) and (3.3) imply then the equality:

$$(3.4) \quad \bar{u}(\tau', l') + \sum_{\tau'_q \ll \bar{u}(\tau', l')} c_q \tau'_q = \varepsilon_1 \bar{u}(\tau', l') + \varepsilon_2 \left(\sum_{\pi'_j \ll \bar{u}(\sigma', k')} b_j i'(\pi'_j) \right).$$

First, observe that since $\pi'_j \neq \bar{u}(\sigma', k')$ for all j , we also have $|i'(\pi'_j)| \neq \bar{u}(\tau', l')$ for all j . So by the linear independence of irreducible representations in the Grothendieck group, $\varepsilon_1 = 1$ and the term $\bar{u}(\tau', l')$ cancels.

We will now show that the remaining equality

$$\sum_{\tau'_q \ll \bar{u}(\tau', l')} c_q \tau'_q = \varepsilon_2 \left(\sum_{\pi'_j \ll \bar{u}(\sigma', k')} b_j i'(\pi'_j) \right)$$

implies that all the coefficients b_j vanish. The argument is the linear independence of irreducible representations and the lemma:

Lemma 3.3. *If $\pi'_j \ll \bar{u}(\sigma', k')$, it is impossible to have $|i'(\pi'_j)| \ll \bar{u}(\tau', l')$.*

Proof. The proof is complicated by the fact that we do not have in general equality $< = \ll$ between the order relations. But this does not really matter. Recall that $\pi'_j \ll \bar{u}(\sigma', k')$, means by definition $Q(\pi'_j) < Q(\bar{u}(\sigma', k'))$, i.e. there exists $\pi_j < u(\sigma, k')$ such that the esi-support of π'_j corresponds to the esi-support of π_j element by element by Jacquet–Langlands. This implies the only two properties we need:

- (*) the cuspidal support of π'_j equals the cuspidal support of $\bar{u}(\sigma', k')$ and
- (**) we have the inclusion relation $E(M_{\pi'_j}) \subset E(M_{\bar{u}(\sigma', k')})$ (Lemma 2.4).

The property (*) implies that, if

$$\pi'_j = a_1 \times a_2 \times \dots \times a_x$$

is a standard decomposition of π'_j in a product of rigid representations, then:

- $x = s(\sigma')$,
- we may assume that for $1 \leq t \leq s(\sigma')$ the line of a_t is generated by $v^t \rho'$ and
- the multisegment M_t of a_t has at most k elements.

So, if one uses the Zelevinsky–Mœglin–Waldspurger algorithm to compute the esi-support $M_t^\#$ of $|i'(a_t)|$ (cf. [BR2]), one finds that $\mathbf{I}(M_t^\#) \leq k$, since each segment in $M_t^\#$ is constructed by picking up at most one cuspidal representation from each segment in M_t . This implies that $\mathbf{I}(|i'(a_t)|) \leq k$. As

$$|i'(\pi'_j)| = |i'(a_1)| \times |i'(a_2)| \times \dots \times |i'(a_x)|$$

we eventually have $\mathbf{I}(|i'(\pi'_j)|) \leq k$.

Assume now $|i'(\pi'_j)| \ll \bar{u}(\tau', l')$. We will show that $\mathbf{I}(|i'(\pi'_j)|) > k$. Set $Q(|i'(\pi'_j)|) = \gamma$ and we know that $\gamma < u(\tau, l')$. We obviously have in our particular situation $\mathbf{I}(\gamma) = s(\sigma') \mathbf{I}(|i'(\pi'_j)|)$. So we want to prove $\mathbf{I}(\gamma) > k'$. The multisegment of γ is obtained by a sequence of elementary operation from the multisegment of $u(\tau, l')$: at the first elementary operation on the multisegment of $u(\tau, l')$ we get a multisegment M' such that $\mathbf{I}(M') > k'$ and then we apply Lemma 2.4. We get, indeed, $\mathbf{I}(\gamma) > k'$.

So our assumption leads to a contradiction. □

(b) The proof uses the claim (a) and is similar to its proof. Let τ and τ' be defined like in (a). By the part (a) we know now that

$$\mathbf{LJ}(u(\sigma, k')) = \bar{u}(\sigma', k') \quad \text{and} \quad \mathbf{LJ}(u(\tau, l')) = \bar{u}(\tau', l'),$$

so

$$\mathbf{LJ}(u(\sigma, k') \times u(\sigma, k')) = \bar{u}(\sigma', k') \times \bar{u}(\sigma', k')$$

and

$$\mathbf{LJ}(u(\tau, l') \times u(\tau, l')) = \bar{u}(\tau', l') \times \bar{u}(\tau', l').$$

Let us call K_1 the Langlands quotient of the esi-support of $\bar{u}(\sigma', k') \times \bar{u}(\sigma', k')$ and K_2 the Langlands quotient of the esi-support of $\bar{u}(\tau', l') \times \bar{u}(\tau', l')$. Using [BR2] it is easy to see that $|i'(K_1)| = K_2$. Then we may write, using Theorem 2.7(c) and Proposition 2.1:

$$(3.5) \quad \mathbf{LJ}(u(\sigma, k') \times u(\sigma, k')) = K_1 + \sum_{\pi_j \ll K_1} b_j \pi'_j$$

and

$$(3.6) \quad \mathbf{LJ}(u(\tau, l') \times u(\tau, l')) = K_2 + \sum_{\xi'_m \ll K_2} c_m \xi'_m.$$

We want to prove that all the b_j vanish. Let us take the dual in the equation (3.5) (cf. Proposition 2.7 (d)):

$$(3.7) \quad \mathbf{LJ}(i(u(\sigma, k') \times u(\sigma, k'))) = \pm(i'(K_1) + \sum_{\pi_j \ll K_1} b_j i'(\pi'_j)).$$

We know that $|i(u(\sigma, k') \times u(\sigma, k'))| = u(\tau, l') \times u(\tau, l')$ because i commutes with the induction functor and we have $|i(u(\sigma, k'))| = u(\tau, l')$ by Proposition 2.9. As $|i'(K_1)| = K_2$, we get from Equations (3.6) and (3.7) after cancellation of K_2 (as in the equation (3.4)):

$$\sum_{\pi_j \ll K_1} b_j i'(\pi'_j) = \pm \left(\sum_{\xi'_m \ll K_2} c_m \xi'_m \right).$$

To show that all the b_j vanish, it is enough, by the linear independence of irreducible representations, to show the following:

Lemma 3.4. *If $\pi' \ll K_1$ it is impossible to have $|i'(\pi')| \ll K_2$.*

Proof. The proof of Lemma 3.3 applies here with a minor modification. We write again

$$\pi' = a_1 \times a_2 \times \dots \times a_{s(\sigma')}$$

such that the line of a_t , $1 \leq t \leq s(\sigma')$, is generated by $v^t \rho'$. The difference here is that the multisegment M of a_t may have up to $2k$ elements. We will prove though, that in this case again:

Lemma 3.5. *The multisegment $m^\#$ of $|i'(a_t)|$ verifies $\mathbf{I}(m^\#) \leq k$.*

This implies that $\mathbf{I}(\pi') \leq k$ and the rest of the proof goes the same way as for (a).

Proof. Let us denote m the multisegment of a_t (m and $m^\#$ respect the notation in [MW1]). The multisegment $m^\#$ is obtained from m using the algorithm in [MW1] (cf. [BR2]). As $\pi' \ll K_1$, one has $E(m) \subset \{v_{\rho'}^{\frac{l-k}{2}+1} \rho', v_{\rho'}^{\frac{l-k}{2}+2} \rho', \dots, v_{\rho'}^{\frac{l+k}{2}} \rho'\}$ (it is the property (***) given at the beginning of the proof of Lemma 3.3). One constructs all the segments of $m^\#$ ending with $v_{\rho'}^{\frac{l+k}{2}} \rho'$ using only cuspidal representations in $E(m)$ (Remark II.2.2 in [MW1]). So the length of the constructed segments is at most k . Let m^- be the multisegment obtained from m after we dropped from each segment of m the cuspidal representations used in this construction. We obviously have then $E(m^-) \subset \{v_{\rho'}^{\frac{l-k}{2}} \rho', v_{\rho'}^{\frac{l-k}{2}+1} \rho', \dots, v_{\rho'}^{\frac{l+k}{2}-1} \rho'\}$ which has again k elements. So going through the algorithm we will find that all the segments of $m^\#$ have length at most k . □

(c) The claim (a) we have just proven allows us to transfer the formula of the Proposition 2.8 by **LJ**.

We have

$$\mathbf{LJ}(u(\sigma, k')) = v^{-\frac{k'+l'}{2}} \left(\sum_{w \in W_{k'}^{l'}} (-1)^{\text{sgn}(w)} \mathbf{LJ} \left(\prod_{i=1}^{k'} Z(v^i \rho, w(i) + l' - i) \right) \right).$$

The representations $\prod_{i=1}^{k'} Z(v^i \rho, w(i) + l' - i)$ are standard. If w is such that, for some i , $s(\sigma')$ does not divide $w(i) - i$, then $\mathbf{LJ}(\prod_{i=1}^{k'} Z(v^i \rho, w(i) + l' - i)) = 0$.

If w satisfies $s(\sigma') | w(i) - i$ for all i , i.e. $w \in H$, then

$$\mathbf{LJ} \left(\prod_{i=1}^{k'} Z(v^i \rho, w(i) + l' - i) \right) = \prod_{i=1}^{k'} T \left(v^{i+\frac{s(\sigma')-1}{2}} \rho', \frac{w(i) - i}{s(\sigma')} + l \right).$$

Hence the formula of (c). □

Corollary 3.6. *Let $n, k \in \mathbb{N}^*$ and $\sigma' \in \mathcal{D}_n^u$.*

- (a) $u'(\sigma', k) \times u'(\sigma', k)$ is irreducible. $\pi(u'(\sigma', k), \alpha)$ are unitary for $\alpha \in]0, \frac{1}{2}[$.
- (b) Write $\sigma' = T^u(\rho', l)$ for some unitary cuspidal representation ρ' . Let W_k^l be the set of permutation w of $\{1, 2, \dots, k\}$ such that $w(i) + l \geq i$ for all $i \in \{1, 2, \dots, k\}$. Then we have:

$$u'(\sigma', k) = v_{\sigma'}^{-\frac{k+l}{2}} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k T(v_{\sigma'}^i \rho', w(i) + l - i) \right)$$

Proof. (a) It is clear that $u'(\sigma', k) \times u'(\sigma', k)$ is irreducible from the part (b) of Theorem 3.2 and the formula (2.1). The fact that this implies that all the $\pi(u'(\sigma', k), \alpha)$ are unitary is explained in [Ta2].

(b) We want to show that

$$u'(\sigma', k) = v_{\sigma'}^{-\frac{k+l}{2}} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k T(v_{\sigma'}^i \rho', w(i) + l - i) \right).$$

We use the equality

$$\bar{u}(\sigma', ks(\sigma')) = \prod_{j=1}^{s(\sigma')} v^j - \frac{s(\sigma') + 1}{2} u'(\sigma', k)$$

and the character formula for $\bar{u}(\sigma', ks(\sigma'))$ obtained in Theorem 3.2(c).

Set

$$U = v_{\sigma'}^{-\frac{k+l}{2}} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k T(v_{\sigma'}^i \rho', w(i) + l - i) \right) \in \mathcal{R}'_n.$$

We have

$$\begin{aligned} & \prod_{j=1}^{s(\sigma')} v^j - \frac{s(\sigma') + 1}{2} U \\ &= v^{-\frac{k+l}{2} s(\sigma')} \prod_{j=1}^{s(\sigma')} v^j - \frac{s(\sigma') + 1}{2} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k T(v_{\sigma'}^i \rho', w(i) + l - i) \right) \\ &= v^{-\frac{k'+l'}{2} + \frac{s(\sigma') - 1}{2}} \prod_{j=1}^{s(\sigma')} \left(\sum_{w \in W_k^l} (-1)^{\text{sgn}(w)} \prod_{i=1}^k T(v^{(i-1)s(\sigma') + j} \rho', w(i) + l - i) \right) \\ &= v^{-\frac{k'+l'}{2} + \frac{s(\sigma') - 1}{2}} \\ & \quad \times \sum_{w_1, w_2, \dots, w_{s(\sigma')} \in W_k^l} \prod_{j=1}^{s(\sigma')} (-1)^{\text{sgn}(w_j)} \prod_{i=1}^k T(v^{(i-1)s(\sigma') + j} \rho', w_j(i) + l - i). \end{aligned}$$

Using Lemma 3.1 we find that this last formula is equal to the character formula of $\bar{u}(\sigma', ks(\sigma'))$ (Theorem 3.2(c)). As $\bar{u}(\sigma', ks(\sigma'))$ is irreducible, we will show that so is U .

The formula defining U is an alternated sum of $|W_k^l|$ terms which are distinct elements of \mathcal{B}'_n . The term $\prod_{i=1}^k v_{\sigma'}^{i - \frac{k+1}{2}} \sigma'$, corresponding to the trivial w , is maximal. To prove that, one may use Lemma 2.4 and the fact that one has $\mathbf{I}(\prod_{i=1}^k v_{\sigma'}^{i - \frac{k+1}{2}} \sigma') = l$, while $\mathbf{I}(t) > l$ for any other term t in

the sum. The Langlands quotient of this maximal term $\prod_{i=1}^k v_{\sigma'}^{i-\frac{k+1}{2}} \sigma'$ is $u'(\sigma', k)$ and appears then in the sum with coefficient 1. So we may write:

$$U = \pi'_0 + \sum_{t=1}^m b_t \pi'_t$$

where $\pi'_0 = u'(\sigma', k)$, b_t are non-zero integers, $\pi'_t \in Irr'_n$ and the π'_t , $0 \leq t \leq m$, are distinct, with the convention $m = 0$ if $U = u'(\sigma', k)$. The representations π'_t are rigid and supported on the same line L (generated by $v_{\rho'}^{-\frac{k+l}{2}} \rho'$). For different j in $\{1, 2, \dots, s(\sigma')\}$, the lines $v^{j-\frac{s(\sigma')+1}{2}} L$ are different. So, as the π'_t are distinct (and have distinct esi-support), $\prod_{j=1}^{s(\sigma')} v^{j-\frac{s(\sigma')+1}{2}} U$ is a linear combination of exactly $(m + 1)^{s(\sigma')}$ irreducible distinct representations each appearing with non-zero coefficient. As it is irreducible, we have $m = 0$. \square

3.2. Transfer of $u(\sigma, k)$. Let k, l, q be positive integers, set $n = klq$ and let $\rho \in \mathcal{C}_q^u$ and $\sigma = Z^u(\rho, l) \in \mathcal{D}_{lq}^u$, $\tau = Z^u(\rho, k) \in \mathcal{D}_{kq}^u$. Let s be the smallest positive integer such that $d|sq$. In the next proposition we give the general result of the transfer of $u(\sigma, k)$. The question has no meaning unless $d|n$ (i.e. $s|kl$) which we shall assume.

Proposition 3.7. (a) *If $d|lq$ (i.e. $s|l$), then $\sigma' = \mathbf{C}(\sigma)$ is well defined; we have $s = s(\sigma')$ and*

$$\mathbf{LJ}(u(\sigma, k)) = \bar{u}(\sigma', k).$$

(b) *If $d|kq$ (i.e. $s|k$), then $\tau' = \mathbf{C}(\tau)$ is well defined; we have $s = s(\tau')$ and*

$$\mathbf{LJ}(u(\sigma, k)) = \varepsilon |i'(\bar{u}(\tau', l))|$$

where $\varepsilon = 1$ if s is odd and $\varepsilon = (-1)^{\frac{k}{s}}$ if s is even.

(c) *If d does not divide neither lq , nor kq (i.e. s does not divide neither l nor k), then $\mathbf{LJ}(u(\sigma, k)) = 0$.*

Proof. (a) We have the formula for the decomposition of $u(\sigma, k)$ in the standard basis \mathcal{B}_n (Proposition 2.8) so we may compute the formula for the decomposition of $\mathbf{LJ}(u(\sigma, k))$ in the standard basis \mathcal{B}'_n by transfer. On the other hand, we have the formula for the decomposition of $\bar{u}(\sigma, k)$ in the standard basis \mathcal{B}'_n using the formula (2.2) and the Corollary 3.6(b). The equality of the two decompositions in the basis \mathcal{B}'_n leads again to the combinatorial Lemma 3.1 in [Ta5].

(b) Up to the sign ε , this is a consequence of the claim (a) and the dual transform, Theorem 2.7(d), since $|i(u(\tau, l))| = u(\sigma, k)$. For the sign ε , see Proposition 4.1, b) in [Ba4].

(c) The proof is in [Ta6]. It is a consequence of Proposition 2.8 here, which is also due to Tadić, and the following lemma for which we give here a more straightforward proof.

Lemma 3.8. *Let $k, l, s \in \mathbb{N}^*$. Assume there is a permutation w of $\{1, 2, \dots, k\}$ such that for all $i \in \{1, 2, \dots, k\}$ one has $s|l + w(i) - i$. Then $s|k$ or $s|l$.*

Proof. Let $[x]$ denote the biggest integer less than or equal to x . If $y \in \mathbb{N}^*$, let \mathbb{N}_y denote the set $\{1, 2, \dots, y\}$.

Assume s does not divide l . Summing up all the k relations $s|l + w(i) - i$ we find that $s|kl$. So, if $(s, l) = 1$, then $s|k$. Assume $(s, l) = p$. Then for all $i \in \{1, 2, \dots, k\}$, $p|w(i) - i$. Let w_0 be the natural permutation of $\mathbb{N}_{[\frac{k}{p}]}$ induced by the restriction of w to $\{p, 2p, \dots, [\frac{k}{p}]p\}$ and w_1 the natural permutation of $\mathbb{N}_{[\frac{k-1}{p}]_{+1}}$ induced by the restriction of w to $\{1, p + 1, \dots, [\frac{k-1}{p}]p + 1\}$. Then for all $i \in \mathbb{N}_{[\frac{k}{p}]}$ one has $\frac{s}{p}|\frac{l}{p} + w_0(i) - i$, and for all $j \in \mathbb{N}_{[\frac{k-1}{p}]_{+1}}$ one has $\frac{s}{p}|\frac{l}{p} + w_1(j) - j$. As now $(\frac{s}{p}, \frac{l}{p}) = 1$ we have already seen that one has $\frac{s}{p}|\frac{k}{p}$ and $\frac{s}{p}|[\frac{k-1}{p}] + 1$. This implies $[\frac{k}{p}] = [\frac{k-1}{p}] + 1$ and so $p|k$. It follows $\frac{s}{p}|\frac{k}{p}$, i.e. $s|k$. \square

3.3. New formulas. The reader might have noticed that the dual of representations $u(\tau, l)$ and $u'(\tau', l)$ are of the same type, while the dual of representations $\bar{u}(\tau', l)$ are in general more complicated. This is why the claim (b) of Proposition 3.7 looks awkward. We could not express $i'(\bar{u}(\tau', l))$ in terms of $\sigma' = \mathbf{C}(\sigma)$, and for the good reason that $\mathbf{C}(\sigma)$ is not defined since the group on which σ lives does not have the appropriate size (divisible by d). Recall the hypothesis was $s(\sigma')|k$. We explain here that one can get a formula though, in terms of $u'(\sigma'_+, \frac{k}{s(\sigma')})$ and $u'(\sigma'_-, \frac{k}{s(\sigma')})$, where $\sigma'_+ = \mathbf{C}(\sigma_+)$ and $\sigma'_- = \mathbf{C}(\sigma_-)$, and the representations σ_+ and σ_- are obtained from σ by stretching and shortening it to get an appropriate size for the transfer. The formulas we will give here are required for the global proofs.

Let $\tau' \in \mathcal{D}'_n$ and $l = as(\tau') + b$ with $a, b \in \mathbb{N}$, $1 \leq b \leq s(\tau') - 1$. We start with the formula (2.2):

$$\bar{u}(\tau', l) = \prod_{i=1}^b v^{i-\frac{b+1}{2}} u'(\tau', a + 1) \times \prod_{j=1}^{s(\tau')-b} v^{j-\frac{s(\tau')-b+1}{2}} u'(\tau', a).$$

So one may compute the dual of $\bar{u}(\tau', l)$ using Proposition 2.9; if $\tau' = T^u(\rho', k)$, we set $\sigma'_+ = T^u(\rho', a + 1)$ and, if $a \neq 0$, $\sigma'_- = T^u(\rho', a)$; then

$$(3.8) \quad |i'(\bar{u}(\tau', l))| = \prod_{i=1}^b v^{i-\frac{b+1}{2}} u'(\sigma'_+, k) \times \prod_{j=1}^{s(\tau')-b} v^{j-\frac{s(\tau')-b+1}{2}} u'(\sigma'_-, k)$$

with the convention that if $a = 0$ we ignore the second product.

In particular the dual of a representation of type $\bar{u}(\sigma', k)$ is of the same type (i.e. some $\bar{u}(\gamma, p)$) if and only if $s(\sigma')|k$ or σ' is cuspidal and $k < s(\sigma')$.

One can see that comparing the formula (3.8) with the formula (2.1) and using the fact that a product of representations of the type $v^\alpha u'(\sigma', k)$ determines its factors up to permutation ([Ta2]).

This gives a formula for $\mathbf{LJ}(u(\sigma, k))$ when s divides k but s does not divide l (case (b) of Proposition 3.7). Let $|\mathbf{LJ}|(u(\sigma, k))$ stand for the irreducible representation among $\{\mathbf{LJ}(u(\sigma, k)), -\mathbf{LJ}(u(\sigma, k))\}$. Let $\rho \in \mathcal{C}_p^u$, $\sigma = Z^u(\rho, l) \in \mathcal{D}_{lp}^u$ and let s be the smallest positive integer such that $d|ps$. Assume $s \neq 1$ and $l = as + b$, $a, b \in \mathbb{N}$, $1 \leq b \leq s - 1$. Set $\sigma_+ = Z^u(\rho, (a + 1)s)$ and, if $a \neq 0$, $\sigma_- = Z^u(\rho, as)$. Let $\sigma'_+ = \mathbf{C}(\sigma_+)$ and, if $a \neq 0$, $\sigma'_- = \mathbf{C}(\sigma_-)$. If $s|k$ and $k = k's$, then

$$(3.9) \quad |\mathbf{LJ}|(u(\sigma, k)) = \prod_{i=1}^b v^{i-\frac{b+1}{2}} u'(\sigma'_+, k') \times \prod_{j=1}^{s(\sigma')-b} v^{j-\frac{s(\sigma')-b+1}{2}} u'(\sigma'_-, k'),$$

with the convention that if $a = 0$ we ignore the second product.

The following formula for the transfer is somehow artificial, but it has the advantage of being symmetric in k and l and adapted to the both cases (a) and (b) of Proposition 3.7. Let $\rho \in \mathcal{C}_p$ for some $p \in \mathbb{N}^*$, and let s be the smallest positive integer such that $d|ps$. Set $\rho' = \mathbf{C}(Z^u(\rho, s))$ (in particular ρ' is cuspidal and $s(\rho') = s$). Let $k, l \in \mathbb{N}^*$. Set $b = k - s[\frac{k}{s}] + l - s[\frac{l}{s}]$ and define a sign ε by $\varepsilon = 1$ if s is odd and $\varepsilon = (-1)^{\frac{k}{s}}$ if s is even. Make the convention that a product $\prod_{i=1}^0$ has to be ignored in a formula. The representation $u(Z^u(\rho, l), k)$ is d -compatible if and only if $s|k$ or $s|l$. In this case we have

$$(3.10) \quad \mathbf{LJ}(u(Z^u(\rho, l), k)) = \varepsilon \prod_{i=1}^b v^{i-\frac{b+1}{2}} u' \left(T^u \left(\rho', \left[\frac{l}{s} \right] \right), \left[\frac{k-1}{s} \right] + 1 \right) \\ \times \prod_{j=1}^{s-b} v^{j-\frac{s-b+1}{2}} u' \left(T^u \left(\rho', \left[\frac{l-1}{s} \right] + 1 \right), \left[\frac{k}{s} \right] \right),$$

with the convention that in this formula we ignore the first product if $[\frac{l}{s}] = 0$ and the second product if $[\frac{k}{s}] = 0$. (As s divides either l or k we cannot have $[\frac{l}{s}] = [\frac{k}{s}] = 0$.)

3.4. Transfer of unitary representations. Let \mathcal{U} be the set of all the representations $u'(\sigma', k)$ and $\pi(u'(\sigma', k), \alpha)$ where k, l range over \mathbb{N}^* , $\sigma' \in \mathcal{D}_l'$ and $\alpha \in]0, \frac{1}{2}[$. Here we will use the fact that the representations $u'(\sigma', k)$ are unitary so we will assume the characteristic of the base field F is zero. As Henniart pointed out to me it is not difficult to lift the result to the non-zero characteristic case by the Kazhdan’s close fields theory ([Ka]), but it has not been written yet.

The next proposition has been proven in [Ta6] under the assumption of the U_0 conjecture of Tadić. We prove it here without this assumption.

- Proposition 3.9.** (a) *All the representations in \mathcal{U}' are irreducible and unitary.*
 (b) *If $\pi'_i \in \mathcal{U}'$, $i \in \{1, 2, \dots, k\}$, then the product $\prod_{i=1}^k \pi'_i$ is irreducible and unitary.*
 (c) *If $u \in \text{Irr}_{nd}^u$, then $\mathbf{LJ}(u) = 0$ or $\mathbf{LJ}(u)$ is an irreducible unitary representation u' of G'_n up to a sign.*
 (d) *Let u' be an irreducible unitary representation of G'_n . If $u' \times u'$ is irreducible, then u' is a product of representations in \mathcal{U}' .*

Proof. The claim (a) is a part of the Tadić conjecture U2 in [Ta2]. It has already been solved for $s(\sigma') \geq 3$ in [BR1, Remark 4.3], which is actually a remark due to Tadić, not to the authors. The only problem, as explained in [Ta2], is to show that the product $u'(\sigma', k) \times u'(\sigma', k)$ is irreducible. This is just our Corollary 3.6(a).

(b) This follows from the irreducibility trick (Proposition 2.13) and the Corollary 3.6(a).

(c) This is a consequence of the Proposition 3.7, the formula (3.9) and of the parts (a) and (b) here.

(d) Assume $u' \times u'$ is irreducible. Then any product containing u' and representations in \mathcal{U}' is irreducible (by Proposition 2.13). As $u'(\sigma', k)$ are prime elements [Ta2, 6.2], the same proof as for $GL(n)$ (Tadić, [Ta1]) shows that u' is a product of representations in \mathcal{U}' . □

If u' is like in the second situation of the part (c) we write $u' = |\mathbf{LJ}^u|(u)$.

Let $\Pi\mathcal{U}'$ be the set of products of representations in \mathcal{U}' . Then $\Pi\mathcal{U}'$ is a set of irreducible unitary representations containing the $\bar{u}(\sigma', k)$ (2.2). We have:

- Proposition 3.10.** (a) *The set $\Pi\mathcal{U}'$ is stable under $|i'|$.*
 (b) *If π is a d -compatible unitary representation of G_{nd} , then $|\mathbf{LJ}^u|(\pi) \in \Pi\mathcal{U}'$.*

Proof. (a) is implied by Proposition 2.10(a).

(b) is implied by Proposition 3.7, the fact that $\bar{u}(\sigma', k) \in \Pi\mathcal{U}'$ and the part (a). □

So we have a map $|\mathbf{LJ}^u|$ from the set of unitary irreducible d -compatible representations of G_{nd} to the set $\Pi\mathcal{U}'$. We prove here a lemma we will need later to construct automorphic unitary representations of the inner form which do not transfer to the split form.

Lemma 3.11. *Assume $\dim_F D = 16$. Let St'_3 be the Steinberg representation of $GL_3(D)$ and St'_4 the Steinberg representation of $GL_4(D)$. Let*

$$\pi = v^{-\frac{3}{2}}u'(St'_3, 4) \times v^{-\frac{1}{2}}u'(St'_4, 3) \times v^{\frac{1}{2}}u'(St'_4, 3) \times v^{\frac{3}{2}}u'(St'_3, 4).$$

Then π is a representation of $GL_{48}(D)$. We have

- (i) π is unitary irreducible,
- (ii) we have $\pi < \bar{u}(St'_3, 16)$ and
- (iii) π is not in the image of $|\mathbf{LJ}^u|$.

Proof. (i) If 1_1 is the trivial representation of D^\times , we have $s(1_1) = 4$. So $s(St'_3) = s(St'_4) = 4$. By definition of $\Pi\mathcal{U}'$ it is clear then that $\pi \in \Pi\mathcal{U}'$.

(ii) By the formula (2.1) we get

$$\bar{u}(St'_3, 16) = v^{-\frac{3}{2}}u'(St'_3, 4) \times v^{-\frac{1}{2}}u'(St'_3, 4) \times v^{\frac{1}{2}}u'(St'_3, 4) \times v^{\frac{3}{2}}u'(St'_3, 4).$$

It is easy to prove that the esi-support of $u'(St'_4, 3)$ is obtained from the esi-support of $u'(St'_3, 4)$ by elementary operations. So $\pi < \bar{u}(St'_3, 16)$.

(iii) Any unitary representation of G_{nd} decomposes in the unique way up to permutation of factors in a product of representations of type $v^\alpha u(\sigma, k)$ and any unitary representation of G'_n decomposes in a unique way up to permutation of factors in a product of representations of type $v^\alpha u'(\sigma', k)$ ([Ta2]). The formula (3.10) implies that if $v^{-\frac{3}{2}}u'(St'_3, 4)$ appear in the decomposition of an element of the image of $|\mathbf{L}\mathbf{J}^u|$, then $v^{-\frac{1}{2}}u'(St'_3, 4)$ should appear too. So π is not in the image of $|\mathbf{L}\mathbf{J}^u|$. \square

It is natural to ask how big are the fibers of $|\mathbf{L}\mathbf{J}^u|$ over a given element $u' \in \Pi\mathcal{U}'$. A product of representations of type $\bar{u}(\sigma', k)$ and $|i'| \bar{u}(\sigma', k)$ may be equal to several different similar products and it does not seem to exist a manageable formula for the number of possibilities. They are of course finite since the cuspidal support is fixed.

3.5. Transfer of local components of global discrete series. Let $\gamma \in Irr_n^u$ be a generic representation. Then one may write

$$\gamma = \prod_{i=1}^m v^{e_i} \sigma_i$$

where σ_i are square integrable and $e_i \in]-\frac{1}{2}, \frac{1}{2}[$ ([Ze]). As it is explained in [Ba4, Sect. 4.1], for all $k \in \mathbb{N}^*$, the representation $\prod_{i=0}^{k-1} (v^{\frac{k-1}{2}-i} \gamma)$ is a standard representation and if we call $Lg(\gamma, k)$ its Langlands quotient, then we have

$$Lg(\gamma, k) = \prod_{i=1}^m v^{e_i} u(\sigma_i, k).$$

One may show that, as γ was unitary, $Lg(\gamma, k)$ is unitary. γ is tempered if and only if all e_i are zero. As the local component of global cuspidal representations are generic (see the next section), by the Mœglin–Waldspurger classification, all local components of the global discrete series of GL_n are of the type $Lg(\gamma, k)$. So it is important to know when do they transfer to a non-zero representation under $\mathbf{L}\mathbf{J}$.

Write $\sigma_i = Z^u(\rho_i, l_i)$, $\rho_i \in C_{p_i}^u$. Let J be the set of integers $j \in \{1, 2, \dots, m\}$ such that $d|p_j l_j$. Let $s_{\gamma, d}$ be the smallest positive integer s such that for all $i \in \{1, 2, \dots, m\} \setminus J$, $d|p_i s$. Then $\mathbf{L}\mathbf{J}(Lg(\gamma, k)) \neq 0$ if and only if for all $i \in \{1, 2, \dots, m\}$ we have $\mathbf{L}\mathbf{J}(u(\sigma_i, k)) \neq 0$ if and only if $s_{\gamma, d}|k$

(by Proposition 3.7). Then

$$\mathbf{LJ}(Lg(\gamma, k)) = \prod_{i=1}^m v^{e_i} \mathbf{LJ}(u(\sigma_i, k)).$$

4. Basic facts and notation (global)

Let F be a global field of characteristic zero and D a central division algebra over F of dimension d^2 . Let $n \in \mathbb{N}^*$. Set $A = M_n(D)$. For each place v of F let F_v be the completion of F at v and set $A_v = A \otimes F_v$. For every place v of F , $A_v \simeq M_{r_v}(D_v)$ for some positive integer r_v and some central division algebra D_v of dimension d_v^2 over F_v such that $r_v d_v = nd$. We will fix once and for all an isomorphism and identify these two algebras. We say that $M_n(D)$ is **split** at a place v if $d_v = 1$. The set V of places where $M_n(D)$ is not split is finite. We assume in the sequel V does not contain any infinite place. For each v , d_v divides d , and moreover d is the smallest common multiple of the d_v over all the places v .

Let $G'(F)$ be the group $A^\times = GL_n(D)$. For every place $v \in V$, set $G'_v = A_v^\times = GL_{r_v}(D_v)$. For every finite place v of F , we set $K_v = GL_{r_v}(O_v)$, where O_v is the ring of integers of D_v . We fix then a Haar measure dg_v on G'_v such that $vol(K_v) = 1$. For every infinite place v , we fix an arbitrary Haar measure dg_v on G'_v . Let \mathbb{A} be the ring of adèles of F . With these conventions, the group $G'(\mathbb{A})$ of adèles of $G'(F)$ is the restricted product of the G'_v with respect to the family of compact subgroups K_v . We consider the Haar measure dg on $G'(\mathbb{A})$ which is the restricted product of the measures dg_v (see [RV] for details). We consider $G'(F)$ as a subgroup of $G'(\mathbb{A})$ via the diagonal embedding.

4.1. Discrete series. Let $Z(F)$ be the center of $G'(F)$. For every place v , let Z_v be the center of G'_v . For every finite place v of F , let dz_v be a Haar measure on Z_v such that the volume of $Z_v \cap K_v$ is one. The center $Z(\mathbb{A})$ of $G'(\mathbb{A})$ is canonically isomorphic with the restricted product of the Z_v with respect to the $Z_v \cap K_v$. On $Z(\mathbb{A})$ we fix the Haar measure dz which is the restricted product of the measures dz_v . On $Z(\mathbb{A}) \backslash G'(\mathbb{A})$ we consider the quotient measure $dz \backslash dg$. As $G'(F) \cap Z(\mathbb{A}) \backslash G'(F)$ is a discrete subgroup of $Z(\mathbb{A}) \backslash G'(\mathbb{A})$, on the quotient space $Z(\mathbb{A}) G'(F) \backslash G'(\mathbb{A})$ we have a well defined measure coming from $dz \backslash dg$. The measure of the whole space $Z(\mathbb{A}) G'(F) \backslash G'(\mathbb{A})$ is finite.

Through all these identifications, $Z(F)$ is a subgroup of $Z(\mathbb{A})$. Fix a unitary smooth character ω of $Z(\mathbb{A})$, trivial on $Z(F)$.

Let $L^2(Z(\mathbb{A}) G'(F) \backslash G'(\mathbb{A}); \omega)$ be the space of classes of functions f defined on $G'(\mathbb{A})$ with values in \mathbb{C} such that

- (i) f is left invariant under $G'(F)$,
- (ii) f satisfy $f(zg) = \omega(z) f(g)$ for all $z \in Z(\mathbb{A})$ and almost all $g \in G'(\mathbb{A})$,
- (iii) $|f|^2$ is integrable over $Z(\mathbb{A}) G'(F) \backslash G'(\mathbb{A})$.

We consider the representation R'_ω of $G'(\mathbb{A})$ by right translations in the space $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)$. We call a **discrete series of $G'(\mathbb{A})$** any irreducible subrepresentation of any representation R'_ω for any unitary smooth character ω of $Z(\mathbb{A})$ trivial on $Z(F)$.

Every discrete series of $G'(\mathbb{A})$ with the central character ω appears in R'_ω with a finite multiplicity. Every discrete series π of $G'(\mathbb{A})$ is isomorphic with a restricted Hilbertian tensor product of (smooth) irreducible unitary representations π_v of the groups G'_v like in [F11]. Each representation π_v is determined by π up to isomorphism and is called the **local component of π at the place v** . For almost all finite places v , π_v has a non-zero fixed vector under K_v . We say then π_v is **spherical**. In general, an admissible irreducible representation σ of $G'(\mathbb{A})$ decomposes similarly into a restricted tensor product of smooth irreducible representations σ_v of G'_v and σ_v is spherical for almost all v (see [F11]).

Let $R'_{\omega, disc}$ be the subrepresentation of R'_ω generated by the discrete series. If π is a discrete series we call the **multiplicity of π in the discrete spectrum** the multiplicity with which π appears in $R'_{\omega, disc}$.

4.2. Cuspidal representations. Let $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)_c$ be the subspace of all the classes f in $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)$ satisfying

$$\int_{N(F)\backslash N(\mathbb{A})} f(ng)dn = 0$$

for almost all $g \in G'(\mathbb{A})$ and for all unipotent radicals N of a proper parabolic F -subgroup of $G'(F)$.

The space $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)_c$ is stable under R'_ω and decomposes discretely in a direct sum of irreducible representations. Such an irreducible subrepresentation is called **cuspidal**. It is automatically a discrete series.

We let now n vary. For all $n \in \mathbb{N}^*$ let G'_n be the group of adèles of $GL_n(D)$ and $G'_{n,v}$ the local component of G'_n at a place v . Let DS'_n be the set of classes of discrete series of G'_n .

If (n_1, n_2, \dots, n_k) is an ordered set of positive integers such that $n_1 + n_2 + \dots + n_k = n$, we call a **standard Levi subgroup** of $G'_n(F)$ a subgroup formed by block diagonal matrices with blocks of given sizes n_1, n_2, \dots, n_k in this order.

A **standard Levi subgroup** of $G'_n(\mathbb{A})$ will be by definition a subgroup defined by the adèle group $L(\mathbb{A})$ of a standard Levi subgroup L of $G'_n(F)$. Let L be like in the previous paragraph. For every place v of F , one has $d_v|n_i d$ for all $1 \leq i \leq k$. If L_v is the subgroup of G'_v formed by block diagonal matrices with k blocks of sizes $n_1 d/d_v, n_2 d/d_v, \dots, n_k d/d_v$ in this order, then $L(\mathbb{A})$ is the restricted product of the L_v with respect to $L_v \cap K_v$. We naturally identify L with the ordered product $\times_{i=1}^k G'_{n_i}$.

Let ν denote here the character $|\det|_F$ on G'_n , product of local characters $\nu_v = |\det|_v$ where $|\cdot|_v$ is the normalized absolute value on F_v .

4.3. Automorphic representations. Let us recall some facts from [La]. Let $L = \times_{i=1}^k G'_{n_i}$ be a standard Levi subgroup of G'_n . For $1 \leq i \leq k$, let ρ_i be a cuspidal representation of $G'_{n_i}(\mathbb{A})$ and e_i a real number. Set $\rho = \otimes_{i=1}^k v^{e_i} \rho_i$.

Then for each place v , the induced representation $\Pi_v = \text{ind}_{L_v}^{G'_v} \rho_v$ is of finite length. For every place v where all the $\rho_{i,v}$ are spherical, Π_v has a unique subquotient π_v which is a spherical representation. An irreducible subquotient of $\text{ind}_{L(\mathbb{A})}^{G'_n(\mathbb{A})} \rho$ is said to be a **constituent** of $\text{ind}_{L(\mathbb{A})}^{G'_n(\mathbb{A})} \rho$. Then an irreducible admissible representation σ of G'_n is a constituent of $\text{ind}_{L(\mathbb{A})}^{G'_n(\mathbb{A})} \rho$ if and only if for all v , σ_v is an irreducible subquotient of Π_v and for almost all v , $\sigma_v = \pi_v$. The notion of a cuspidal representation differs between [La] and here: here we allow only what would be in the [La] language a *unitary* cuspidal representation. Using the Proposition 2 in [La], an **automorphic** representation \mathcal{A} of G'_n will be here by definition a constituent of $\text{ind}_{L(\mathbb{A})}^{G'_n(\mathbb{A})} \rho$ for some ρ as above. One would like to prove then that the couples (ρ_i, e_i) are all determined by \mathcal{A} up to permutation. This has been shown in [JS] in the case where $D = F$, and in the present paper we will show it for general D . For the case $D = F$, we will then call the non-ordered multiset $\{v^{e_1} \rho_1, v^{e_2} \rho_2, \dots, v^{e_k} \rho_k\}$ the cuspidal support of \mathcal{A} . For the classical definition of automorphic representations we refer to [BJ]; here we used an equivalent one, cf. [La, Proposition 2]. Let us point out that a discrete series is always a (unitary) automorphic representation.

In the particular case $D = F$ some other facts are known. However, we make the following convention: for the case of a general division algebra D we keep the notation adopted above, while for the particular case $D = F$ we consider another class of groups $G_n = GL_n(F)$. All the definitions adapt then to G_n by setting $D = F$ and we will write DS_n for the set of discrete series of $G_n(\mathbb{A})$.

4.4. Multiplicity one theorems for G_n . We recall in this subsection three facts about G_n . There is the *multiplicity one theorem*: every discrete series of $G_n(\mathbb{A})$ appears with multiplicity one in the discrete spectrum. And the *strong multiplicity one theorem*: if π and π' are two discrete series of G_n such that $\pi_v = \pi'_v$ for almost all place v , then $\pi = \pi'$. This results may be found in [Sh] and [P-S] (when $D = F$). We will prove them in this paper for general G'_n .

One also knows that the local component of a cuspidal representation of G_n at any place is generic and unitary, hence an irreducible product $\prod_{i=1}^m v^{e_i} \sigma_i$ where σ_i are square integrable representations and $e_i \in]-\frac{1}{2}, \frac{1}{2}[$ (see [Sh] and [Ze]).

4.5. The residual spectrum of G_n . We recall now the Mœglin–Waldspurger classification of the discrete series for groups $G_n(\mathbb{A})$. Let $m \in \mathbb{N}^*$ and $\rho \in DS_m$ be a cuspidal representation. If $k \in \mathbb{N}^*$, then the induced

representation $\prod_{i=0}^{k-1} (v^{\frac{k-1}{2}-i} \rho)$ has a unique constituent π which is a discrete series (i.e. $\pi \in DS_{mk}$). One has $\pi_v = Lg(\rho_v, k)$ for all place v (we used the definition of $Lg(\rho_v, k)$ of Sect. 3.5 since ρ_v is generic). We set then $\pi = MW(\rho, k)$. Discrete series π of groups $G_n(\mathbb{A})$, $n \in \mathbb{N}^*$, are all of this type, k and ρ are determined by π and π is cuspidal if and only if $k = 1$. These are the results of [MW2]. We will prove in the sequel the same classification holds for $G'_n(\mathbb{A})$

Let us prove, for future purposes, a proposition generalizing the strong multiplicity one theorem.

Proposition 4.1. *Let $\sigma_i \in DS_{n_i}$, $i \in \{1, 2, \dots, k_1\}$, $\sum_{i=1}^{k_1} n_i = n$ and $\tau_j \in DS_{m_j}$, $j \in \{1, 2, \dots, k_2\}$, $\sum_{j=1}^{k_2} m_j = n$. Assume that for almost all finite places v the local components of the (irreducible) products $\sigma = \prod_{i=1}^{k_1} \sigma_i$ and $\tau = \prod_{j=1}^{k_2} \tau_j$ at the place v are equal. Then $(\sigma_1, \sigma_2, \dots, \sigma_{k_1})$ equals up to a permutation $(\tau_1, \tau_2, \dots, \tau_{k_2})$.*

Proof. By the Theorem 4.4 in [JS], the cuspidal supports of the automorphic representations σ and τ are equal. We call a **line** the set of representations $\{v^k \rho\}_{k \in \mathbb{Z}}$, where ρ is a cuspidal representation of some $G_m(\mathbb{A})$. We call a **shifted line** the set of representations $\{v^{k+\frac{1}{2}} \rho\}_{k \in \mathbb{Z}}$, where ρ is a cuspidal representation of some $G_m(\mathbb{A})$. Thanks to the Mœglin–Waldspurger classification we know that the set of the elements of the cuspidal support of a given σ_i or τ_j is either included in a line, or in a shifted line. So we may then “separate the supports” and reduce the problem to the case where there exists a line or a shifted line T such that the set of elements of the cuspidal supports of all the σ_i and all the τ_j are included in T . Then there exists a cuspidal representation ρ such that $\sigma_i = MW(\rho, p_i)$ for all i and $\tau_j = MW(\rho, q_j)$ for all j . And moreover the p_i and the q_j are either all odd, or all even. Let X be the cuspidal support of σ and τ in this case. We show that X determines the σ_i up to permutation.

If the p_i are all odd, the result is a consequence of the following combinatorial lemma:

Lemma 4.2. *Let A be a multiset of integers which may be written as a re-union with multiplicities of sets of the form $B = \{-k, -k + 1, -k + 2, \dots, k - 2, k - 1, k\}$. Then the sets B are determined by A .*

Proof. Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be the multiplicity map: $f(a)$ is the multiplicity of a in A . The number $f(a)$ is also the number of sets B containing a . If $a \geq 1$ and a set B contains a , then it contains also $a - 1$. So f is decreasing on \mathbb{N} and for all $p \in \mathbb{N}$, the number of sets $\{-p, -p + 1, -p + 2, \dots, p - 2, p - 1, p\}$ in A is exactly $f(p) - f(p + 1)$. □

If the p_i are even, the proof is essentially the same. This finishes the proof of the Proposition 4.1. □

4.6. Transfer of functions. For each finite place v let $H(G'_{n,v})$ be the Hecke algebra of locally constant functions with compact support on $G'_{n,v}$. Let $H(G'_n)$ be the set of functions $f : G'_n(\mathbb{A}) \rightarrow \mathbb{C}$ such that f is a product $f = \prod_v f_v$ over all places of F , where f_v is C^∞ with compact support when v is infinite, $f_v \in H(G'_{n,v})$ when v is finite and, for almost all finite places v , f_v is the characteristic function of K_v . We write then $f = (f_v)_v$. As the local components of an automorphic representation π are almost all spherical, the product of traces $\prod_v \text{tr} \pi_v(f_v)$ has a meaning for all $f = (f_v)_v \in H(G'_n)$ and we may set $\text{tr}(\pi(f)) = \prod_v \text{tr} \pi_v(f_v)$. We adopt similar notation and definitions for the groups G_n .

Let $v \in V$. We fix measures on the maximal tori of $G_{nd,v}$ and $G'_{n,v}$ in a compatible way and define the orbital integrals Φ on $G_{nd,v}$ and Φ' on $G'_{n,v}$ for regular semisimple elements with respect to these choices (see Sect. 2 of [Ba1] for example). If $f_v \in H(G_{nd,v})$ and $f'_v \in H(G'_{n,v})$ we say that f_v and f'_v **correspond** to each-other, and write $f_v \leftrightarrow f'_v$, if:

- f_v and f'_v are supported in the set of regular semisimple elements, and
- for all $g \leftrightarrow g'$ we have $\Phi(f_v, g) = \Phi'(f'_v, g')$, and
- for all regular semisimple $g \in G_{nd,v}$ which does not correspond to any $g' \in G'_{n,v}$ we have $\Phi(f_v, g) = 0$.

It is known that for every $f'_v \in H(G'_{n,v})$ supported on the regular semisimple set there exists $f_v \in H(G_{nd,v})$ such that $f_v \leftrightarrow f'_v$. Also, if $f_v \leftrightarrow f'_v$ then $\text{tr}(\pi(f_v)) = 0$ for all representation π induced from a Levi subgroup of $G_{nd,v}$ which does not transfer (Sect. 2 of [Ba1] for example).

For $f = (f_v)_v \in H(G_{nd})$ and $f' = (f'_v)_v \in H(G'_n)$ we write $f \leftrightarrow f'$ and say that f and f' **correspond** to each other if

- (i) $\forall v \notin V$ we have $f_v = f'_v$ and
- (ii) $\forall v \in V$ we have $f_v \leftrightarrow f'_v$.

For every $f' = (f'_v)_v \in H(G'_n)$ such that for all $v \in V$ the support of f'_v is included in the set of regular semisimple elements of G'_v there exists $f \in H(G_n)$ such that $f \leftrightarrow f'$. If $f \in H(G_{nd})$, we say f **transfers** if there exists $f' \in H(G'_n)$ such that $f \leftrightarrow f'$.

5. Global results

5.1. Global Jacquet–Langlands, multiplicity one and strong multiplicity one for inner forms. For all $v \in V$, denote \mathbf{LJ}_v (resp. $|\mathbf{LJ}|_v$) the correspondence \mathbf{LJ} (resp. $|\mathbf{LJ}|$), as defined in Subsect. 2.7, applied to $G_{nd,v}$ and $G'_{n,v}$.

If $\pi \in DS_{nd}$ we say π is **D-compatible** if, for all $v \in V$, π_v is d_v -compatible. Then $\mathbf{LJ}(\pi_v) \neq 0$ and $|\mathbf{LJ}|_v(\pi_v)$ is an irreducible representation of G'_n (Proposition 3.2(c)). \square

- Theorem 5.1.** (a) *There exists a unique map $\mathbf{G} : DS'_n \rightarrow DS_{nd}$ such that for all $\pi' \in DS'_n$, if $\pi = \mathbf{G}(\pi')$, one has $|\mathbf{L}\mathbf{J}|_v(\pi_v) = \pi'_v$ for all places $v \in V$, and $\pi_v = \pi'_v$ for all places $v \notin V$. The map \mathbf{G} is injective. The image of \mathbf{G} is the set DS_{nd}^D of D -compatible discrete series of $G_{nd}(\mathbb{A})$.*
- (b) *We have the multiplicity one theorem for discrete series of $G'_n(\mathbb{A})$: if $\pi' \in DS'_n$, then the multiplicity of π' in the discrete spectrum is one.*
- (c) *We have the strong multiplicity one theorem for discrete series of $G'_n(\mathbb{A})$: if $\pi', \pi'' \in DS'_n$, and if $\pi'_v = \pi''_v$ for almost all v , then $\pi'_v = \pi''_v$ for all v .*
- (d) *For all $\pi' \in DS'_n$, for all places $v \in V$, $\pi'_v \in \Pi\mathcal{U}'$ (see Sect. 3.4).*

Proof. We will use the results of [AC]. The authors compare the trace formulas of G_{nd} and G'_n . We will restate the result here.

Let F_∞^* be the product $\times_i F_i^*$ where i runs over the set of infinite places of F . Let μ be a unitary character of F_∞^* . We use the embedding of F_∞^* in \mathbb{A}^\times trivial at finite places to realize it as a subgroup of the center $Z(\mathbb{A})$.

Let $\mathcal{L}(G_{nd})$ be the set of F -Levi subgroups of G_{nd} which contain the maximal diagonal torus.

Let

$$I_{disc,t,\mu,G_{nd}}(f) = \sum_{L \in \mathcal{L}(G_{nd})} |W_0^L| |W_0^{G_{nd}}|^{-1} \sum_{s \in W(\mathfrak{a}_L)_{reg}} |\det(s - 1)_{\mathfrak{a}_L^{G_{nd}}}|^{-1} \text{tr}(M_L^{G_{nd}}(s, 0) \rho_{L,t}(0, f))$$

where, in the order of the appearance:

- $t \in \mathbb{R}_+$;
- $|W_0^L|$ is the cardinality of the Weyl group of L ;
- $|W_0^{G_{nd}}|$ is the cardinality of the Weyl group of G_{nd} ;
- \mathfrak{a}_L is the real space $\text{Hom}(X(L)_F, \mathbb{R})$ where $X(L)_F$ is the lattice of rational characters of L ; $W(\mathfrak{a}_L)$ is the Weyl group of \mathfrak{a}_L of L ; $\mathfrak{a}_L^{G_{nd}}$ is the quotient of \mathfrak{a}_L by $\mathfrak{a}_{G_{nd}}$; $W(\mathfrak{a}_L)_{reg}$ is the set of $s \in W(\mathfrak{a}_L)$ such that $\det(s - 1)_{\mathfrak{a}_L^{G_{nd}}} \neq 0$;
- $M_L^{G_{nd}}(s, 0)$ is the intertwining operator associated to s at the point 0; it intertwines representations $\text{ind}_L^{G_{nd}} \sigma$ and $\text{ind}_{sL}^{G_{nd}} s\sigma$, where σ is a representation of L ;
- $\rho_{L,t}$ is the induced representation with respect to any parabolic subgroup with Levi factor L from the direct sum of discrete series π of L such that π is μ -equivariant and the imaginary part of the Archimedean infinitesimal character of π has norm t [AC, pp. 131–132];
- f is an element of $H(G_{nd})$.

For this definition see p. 198, and the formula (4.1) p. 203, in [AC]. It is the “ μ formula”, and not the original definition-equality (9.2) p. 132, which does not contain any μ .

Now let us compute the terms. It turns out that $W(\mathfrak{a}_L)_{reg}$ is empty unless L is conjugated to a Levi subgroup given by block diagonal matrices with blocks of equal size. Let L be the Levi subgroup given by block diagonal matrices with l blocks of size m , $lm = nd$. If we identify $W(\mathfrak{a}_L)$ with \mathfrak{S}_l , then $W(\mathfrak{a}_L)_{reg}$ is the set of l -cycles. So the cardinality of $W(\mathfrak{a}_L)_{reg}$ is $(l - 1)!$ and for any $s \in W(\mathfrak{a}_L)_{reg}$, $|\det(s - 1)_{\mathfrak{a}_L}| = l$. We also have $|W_0^L| = (m!)^l$ and $|W_0^{G_{nd}}| = (nd)!$. So the coefficient of the character attached to L in the linear combination over $\mathcal{L}(G_{nd})$ is $\frac{(m!)^l}{(nd)!l}$. Now, if L' is conjugated with L , the contribution of L' to the sum is the same as that of L [AC, p. 207]. Let us compute the number of Levi subgroups L' conjugated to L , and containing the diagonal torus. The diagonal torus is then a maximal torus of L' , and so the center of L' is contained in the diagonal torus. As L' is the centralizer of its center there will be exactly as many L' as the non-ordered partitions of $\{1, 2, \dots, nd\}$ in l subsets of cardinality m . This number is $l!^{-1} \binom{nd-m}{m} \binom{nd-2m}{m} \binom{nd-3m}{m} \dots \binom{m}{m}$ (product of binomial coefficients), which is $\frac{(nd)!}{l!(m!)^l}$ (for a more theoretical formula for the same result see [AC, p. 207]).

So we may rewrite the formula: for all $l|nd$, if L_l is the Levi subgroup of G_{nd} given by block diagonal matrices with l blocks of equal size $\frac{nd}{l}$, then

$$I_{disc,t,\mu,G_{nd}}(f) = \sum_{l|nd} \frac{1}{l!l} \sum_{s \in W(\mathfrak{a}_L)_{reg}} \text{tr}(M_{L_l}^{G_{nd}}(s, 0)\rho_{L_l,t}(0, f)).$$

In [AC] it is shown moreover, pp. 207–208, that for any L_l , the $(l - 1)!$ elements $s \in W(\mathfrak{a}_L)_{reg}$ give all the same contribution to the sum. So, in the end, if s_0 is the cycle $(1, 2, \dots, l)$, the definition of $I_{disc,t,\mu,G_{nd}}(f)$ turns out to be simply:

$$\sum_{l|nd} \frac{1}{l^2} \text{tr}(M_{L_l}^{G_{nd}}(s_0, 0)\rho_{L_l,t}(0, f)).$$

Let us turn now to the operator $M_{L_l}^{G_{nd}}(s_0, 0)\rho_{L_l,t}(0, f)$. A discrete series ρ of L_l is an ordered product $\otimes_{i=1}^l \rho_i$, where each ρ_i is a discrete series of $G_{\frac{nd}{l}}$. Let $Stab_\rho$ be the subgroup of \mathfrak{S}_l which stabilizes the ordered multiset $(\rho_1, \rho_2, \dots, \rho_l)$ for the obvious action. Let X_ρ be a set of representatives of $\mathfrak{S}_l/Stab_\rho$ in \mathfrak{S}_l . Let V_ρ be the subspace $\oplus_{x \in X_\rho} \times_{i=1}^l \rho_{x(i)}$ of $\rho_{L_l,t}$. Then V_ρ is stable under the operator $M_{L_l}^{G_{nd}}(s_0, 0)$. But, if the ρ_i are not all equal, $M_{L_l}^{G_{nd}}(s_0, 0)$ permutes without fixed point the subspaces $\times_{i=1}^l \rho_{x(i)}$. So the trace of the operator induced by $M_{L_l}^{G_{nd}}(s_0, 0)$ on V_ρ is zero. Then in the formula only the contributions from representations $\rho = \otimes_{i=1}^l \rho_i$ of L_l such that all the ρ_i are equal remain.

So

$$(5.1) \quad I_{disc,t,\mu,G_{nd}}(f) = \sum_{\rho \in DS_{nd,t,\mu}} \text{tr}(\rho(f)) + \sum_{l|nd, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{t}{l}, \mu_l}} \text{tr}(M_{L_l}^{G_{nd}}(s_0, 0) \rho^l(0, f)),$$

where $DS_{k, \frac{t}{l}, \mu_l}$ is the set of discrete series ρ of $G_k(\mathbb{A})$ such that ρ is μ' -equivariant for some character μ' of F_∞^* such that $\mu'^l = \mu$ and the norm of the imaginary part of its infinitesimal character is $\frac{t}{l}$, and ρ^l is the induced representation $\rho \times \rho \times \dots \times \rho$ from L_l . In the last formula we used the multiplicity one theorem for G_k , $k|nd$. The representation ρ being unitary, the representation ρ^l is irreducible and hence $M_{L_l}^{G_{nd}}(s_0, 0)$ acts as a scalar on ρ^l . As it is also a unitary operator, the scalar is some complex number λ_ρ of module 1.

The analogous definition of $I_{disc,t,\mu,G'_n}(f')$ is given in [AC] for the groups G'_n and $f' \in H(G'_n)$:

$$I_{disc,t,\mu,G'_n}(f') = \sum_{L \in \mathcal{L}(G'_n)} |W_0^L| |W_0^{G'_n}|^{-1} \sum_{s \in W(\mathfrak{a}_L)_{reg}} |\det(s - 1)_{\mathfrak{a}_L^{G'_n}}|^{-1} \text{tr}(M_L^{G'_n}(s, 0) \rho_{L,t}(0, f'))$$

where the symbols have the same definition as for $I_{disc,t,\mu,G_{nd}}(f)$ when replacing G_{nd} by G'_n and f by f' . All the computation made for G_{nd} to simplify the formula, up to Formula (5.1) itself, are combinatorial and work exactly the same for G'_n (replacing nd with n). We get then an analogous formula to (5.1), taking in account we do not have multiplicity one (yet) for $G'_n(\mathbb{A})$:

$$(5.2) \quad I_{disc,t,\mu,G'_n}(f') = \sum_{\rho' \in DS'_{n,t,\mu}} m_{\rho'} \text{tr}(\rho'(f')) + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho' \in DS'_{\frac{n}{l}, \frac{t}{l}, \mu_l}} m_{\rho'}^l \text{tr}(M_{L_l}^{G'_n}(s_0, 0) \rho'^l(0, f')),$$

where $m_{\rho'}$ is the multiplicity of ρ' in the discrete spectrum ($m_{\rho'}^l$ is the power l of the positive integer $m_{\rho'}$) and the other symbols are defined as for G_{nd} in the formula (5.1).

One of the main results of [AC] is the fundamental equality (Equation (17.8), p. 198):

$$(5.3) \quad I_{disc,t,\mu,G_{nd}}(f) = I_{disc,t,\mu,G'_n}(f')$$

for any $f \leftrightarrow f'$.

We have an easy lemma.

Lemma 5.2. *Let $l|nd$ and $\rho \in DS_{\frac{nd}{l}}$. Let $f' \in H(G'_n)$ and $f \in H(G_{nd})$ such that $f \leftrightarrow f'$. If l does not divide n , or if $l|n$ and ρ is not D -compatible, then $\text{tr}(M(s_0, 0)\rho^l(f)) = 0$.*

Proof. Assume l does not divide n . Then d does not divide $\frac{nd}{l}$. By the class field theory the smallest common multiple of the integers d_v is d , so there exists a place w such that d_w does not divide $\frac{nd}{l}$. Then ρ_w^l is not d_w -compatible. The same, if ρ is not D -compatible, there exists a place w such that ρ_w is not d_w -compatible and hence ρ_w^l is not d_w -compatible.

In both cases we have then $\text{tr}\rho_w^l(f_w) = 0$ and as the operator $M(s_0, 0)$ acts as a scalar, the result follows. \square

Another lemma:

Lemma 5.3. *Assume the multiplicity one theorem is true for all G'_k , $k < n$. Then*

(a)

$$I_{disc,t,\mu,G'_n}(f') = \sum_{\rho' \in DS'_{n,t,\mu}} m_{\rho'} \text{tr}\rho'(f') + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho' \in DS'_{\frac{n}{l}, \frac{t}{l}, \mu_l}} \text{tr}(M_{L'_l}^{G'_n}(s_0, 0) \rho'^l(0, f')),$$

where $m_{\rho'}$ is the multiplicity of ρ' in the discrete spectrum.

(b) For all $f \leftrightarrow f'$, one has

$$(5.4) \quad \sum_{\rho \in DS_{nd,t,\mu}^D} \text{tr}\rho(f) + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{t}{l}, \mu_l}^D} \text{tr}(M_{L_l}^{G_{nd}}(s_0, 0) \rho^l(0, f)) = \sum_{\rho' \in DS'_{n,t,\mu}} m_{\rho'} \text{tr}\rho'(f') + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho' \in DS'_{\frac{n}{l}, \frac{t}{l}, \mu_l}} \text{tr}(M_{L'_l}^{G'_n}(s_0, 0) \rho'^l(0, f')),$$

where DS_{γ}^D is by definition the subset of D -compatible representations in DS_{γ} .

Proof. (a) Comes straight from the formula (5.2).

(b) We used (a) and the equality (5.3). But the G_{nd} side has been modified due to Lemma 5.2. Lemma 5.2 allows also the replacement of DS_{γ} by DS_{γ}^D . \square

Let us finish now the proof of Theorem 5.1 by induction on n . So, among other things, we will use the formula (5.4). Let us point out, not to recall it

all the time, that the correspondence \mathbf{G} , once assumed or proven, preserves the quantities t and μ .

First assume $n = 1$. Then we get from the relation (5.4):

$$(5.5) \quad \sum_{\rho \in DS_{d,t,\mu}^D} \text{tr} \rho(f) = \sum_{\rho' \in DS'_{1,t,\mu}} m_{\rho'} \text{tr} \rho'(f')$$

for all $f \leftrightarrow f'$, where $m_{\rho'}$ is the multiplicity of ρ' in the discrete spectrum.

Let us fix a representation $\sigma' \in DS'_1$. Then we have $\sigma' \in DS'_{1,t,\mu}$ for some t and μ . We will show there exists $\sigma \in DS_{d,t,\mu}^D$ such that $|\mathbf{LJ}|_v(\sigma_v) = \sigma'_v$ for all $v \in V$ and $\sigma_v = \sigma'_v$ for all $v \notin V$, and also that $m_{\sigma'} = 1$. Let S be a finite set of places of F containing all the places in V , all the infinite places and all the places v such that σ'_v is not a spherical representation. For any $\pi \in DS_{d,t,\mu}^D$ or $\pi \in DS'_{1,t,\mu}$ write π_S for the tensor product $\otimes_{v \in S} \pi_v$ and π^S for the restricted tensor product $\otimes_{v \notin S} \pi_v$. Let $DS_{d,t,\mu,\sigma'}^D$ (resp. $DS'_{1,t,\mu,\sigma'}$) be the set of $\pi \in DS_{d,t,\mu}^D$ (resp. $\pi \in DS'_{1,t,\mu}$) such that $\pi^S = \sigma'^S$. Then we have for all $f \leftrightarrow f'$:

$$\sum_{\rho \in DS_{d,t,\mu,\sigma'}^D} \text{tr} \rho(f) = \sum_{\rho' \in DS'_{1,t,\mu,\sigma'}}$$

This statement is inferred from the equation (5.5) by a standard argument one may find well expounded in [Fl2]. According to the strong multiplicity one theorem applied to G_d , the cardinality of $DS_{d,t,\mu,\sigma'}^D$ is either zero or 1. The cardinality of $DS'_{1,t,\mu,\sigma'}$ is finite by [BB]. As $f_v = f'_v$ for $v \notin S$, we may cancel in this equality $\prod_{v \notin S} \text{tr} \sigma'_v(f'_v)$, by choosing f'_v such that this product is not zero. We get

$$\sum_{\rho \in DS_{d,t,\mu,\sigma'}^D} \prod_{v \in S} \text{tr} \rho_v(f_v) = \sum_{\rho' \in DS'_{1,t,\mu,\sigma'}}$$

for functions such that $f_v \leftrightarrow f'_v$ for all $v \in V$ and $f_v = f'_v$ for all $v \in S \setminus V$. On the right side we have a finite non-empty set (containing at least σ') of distinct characters on a finite product of groups. The linear independence of characters on these groups implies the linear independence of characters on the product, and so there exist functions $f'_v \in H'(G'_{1,v})$ for $v \in S$, supported on the set of regular semisimple elements, such that the right side of the equality does not vanish on $(f'_v)_{v \in S}$. Then $DS_{d,t,\mu,\sigma'}^D$ is not empty and hence contains one element. Let us call this element σ . As σ is D -compatible, for every $v \in V$ we have that $|\mathbf{LJ}|_v(\sigma_v)$ is an irreducible unitary representation u'_v of $G'_{1,v}$ such that $\text{tr}(\sigma_v(f_v)) = \text{tr}(u'_v(f'_v))$ for all $f_v \leftrightarrow f'_v$. So by the linear independence of characters on the group $\times_{v \in S} G'_{1,v}$ we must have $u'_v = \sigma'_v$ for all $v \in V$ and $\sigma_v = \sigma'_v$ for all $v \in S \setminus V$.

This obviously implies $m_{\sigma'} = 1$ which is the claim (b). Now $\mathbf{G}(\sigma')$ is defined. If $\mathbf{G}(\sigma') = \mathbf{G}(\sigma'') = \sigma$ then we have $\sigma'_v = \sigma''_v = \sigma_v$ for all $v \notin V$ and $\sigma'_v = \sigma''_v = |\mathbf{L}\mathbf{J}|_v(\sigma_v)$ for all $v \in V$, which shows that \mathbf{G} is injective.

Let us show the surjectivity of \mathbf{G} onto DS_d^D . We start again from the equality (5.5)

$$\sum_{\rho \in DS_{d,t,\mu}^D} \text{tr} \rho(f) = \sum_{\rho' \in DS'_{1,t,\mu}} \text{tr} \rho'(f')$$

for all $f \leftrightarrow f'$ (the multiplicities on the left side have disappeared). Consider $\sigma \in DS_{d,t,\mu}^D$ and let S be a finite set of places containing all the places in V , all the infinite places and all the places v such that σ_v is not spherical. Let $DS_{d,t,\mu,\sigma}^D$ (resp. $DS'_{1,t,\mu,\sigma}$) be the set of $\pi \in DS_{d,t,\mu}^D$ (resp. $\pi \in DS'_{1,t,\mu}$) such that $\pi^S = \sigma^S$. By the same arguments as before (simplification of the trace formula as expounded in [F12]), we have for all $f \leftrightarrow f'$:

$$\sum_{\rho \in DS_{d,t,\mu,\sigma}^D} \text{tr} \rho(f) = \sum_{\rho' \in DS'_{1,t,\mu,\sigma}} \text{tr} \rho'(f').$$

But by strong multiplicity one theorem on G_d , $DS_{d,t,\mu,\sigma}^D$ contains the unique element σ . As σ is D -compatible, there exist $f \leftrightarrow f'$ such that $\text{tr} \sigma(f) \neq 0$. So $DS'_{1,t,\mu,\sigma}$ is not empty. Consider $\sigma' \in DS'_{1,t,\mu,\sigma}$. Then $\mathbf{G}(\sigma')$ is defined. By multiplicity one theorem on G_d applied to places outside S , $\mathbf{G}(\sigma')$ has to be σ .

We have seen that $\sigma'_v = \mathbf{G}(\sigma')_v$ for all $v \notin V$. The strong multiplicity one theorem for G_d implies then the strong multiplicity one theorem for G'_1 ((c)). The claim (d) is obtained now by transfer under \mathbf{G}^{-1} and Proposition 3.10(b).

Thus, we finished the proof of the theorem for $n = 1$.

Let us now assume the theorem has been proven for all $k < n$ and call \mathbf{G}_k the transfer map at level k . This hypothesis enables us to apply Lemma 5.3 and implies the relation (5.4) which we recall:

$$\begin{aligned} (5.6) \quad & \sum_{\rho \in DS_{nd,t,\mu}^D} \text{tr} \rho(f) + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{t}{l}, \mu_l}^D} \text{tr}(M_{L_l}^{G_{nd}}(s_0, 0) \rho^l(0, f)) \\ & = \sum_{\rho' \in DS_{n,t,\mu}} m_{\rho'} \text{tr} \rho'(f') + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho' \in DS'_{\frac{n}{l}, \frac{t}{l}, \mu_l}} \text{tr}(M_{L'_l}^{G'_n}(s_0, 0) \rho'^l(0, f')). \end{aligned}$$

Moreover, using the part (d) of the theorem for \mathbf{G}_k , $k < n$, the induction hypothesis implies that the representations ρ'^l are irreducible (Proposition 3.9(b)). So $M_{L'_l}^{G'_n}(s_0, 0)$ is again a scalar and as it is unitary the scalar is

a complex number $\lambda_{\rho'}$ of module 1. So the equation is actually, using again the induction to transfer the representations in $DS'_{\frac{n}{l}, \frac{l}{l}, \mu_l}$:

$$\begin{aligned}
 (5.7) \quad & \sum_{\rho \in DS_{nd,t,\mu}^D} \text{tr} \rho(f) + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{l}{l}, \mu_l}^D} \lambda_{\rho} \text{tr}(\rho^l(0, f)) \\
 &= \sum_{\rho' \in DS'_{n,t,\mu}} m_{\rho'} \text{tr} \rho'(f') + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{l}{l}, \mu_l}^D} \lambda_{\mathbf{G}_{\frac{n}{l}}^{-1}(\rho)} \text{tr}(\rho^l(0, f))
 \end{aligned}$$

for $f \leftrightarrow f'$.

Now the proof goes as for the case $n = 1$ with a minor modification in the end. Choose a representation $\sigma' \in DS'_{n,t,\mu}$. Fix a finite set S of places of F which contains all the places in V , all the infinite places and all the places v for which σ'_v is not spherical. By the theorem of multiplicity one for G_{nd} the set A of $\sigma \in DS_{nd,t,\mu}^D$ such that $\sigma^S = \sigma'^S$ is empty or contains only one element. If we apply Proposition 4.1 to the representations ρ^l and all the places out of S , then we conclude that the set B of representations $\gamma = \rho^l$ (where $l|n$ and $l \neq 1$) such that $\gamma^S = \sigma'^S$ is empty or contains one element. Let $DS'_{n,t,\mu,\sigma'}$ be the set of $\tau' \in DS'_{n,t,\mu}$ such that $\tau'^S = \sigma'^S$. Then $DS'_{n,t,\mu,\sigma'}$ is not empty (contains σ') and finite ([Ba3]; we do not quote [BB] again since the representations may not be cuspidal).

By the same argument in [Fl2], already quoted for the case $n = 1$, we obtain then

$$\sum_{\sigma \in A} \prod_{v \in S} \text{tr} \sigma_v(f_v) + \sum_{\gamma \in B} \frac{\lambda_{\gamma} - \lambda_{\mathbf{G}_{\frac{n}{l}}^{-1}(\gamma)}}{l^2} \prod_{v \in S} \text{tr} \gamma_v(f_v) = \sum_{\rho' \in DS'_{n,t,\mu,\sigma'}} m_{\rho'} \prod_{v \in S} \text{tr} \rho'_v(f'_v)$$

if $f_v \leftrightarrow f'_v$ for all $v \in V$ and $f_v \leftrightarrow f'_v$ for all $v \in S \setminus V$.

If A is not empty and σ is the unique element of A , then the local components of σ are unitary and we can transfer them. If B is not empty and γ is the unique element of B , then the local components of γ are unitary and we can transfer them. In any possible case we do so. But *the coefficient* $\frac{\lambda_{\gamma} - \lambda_{\mathbf{G}_{\frac{n}{l}}^{-1}(\gamma)}}{l^2}$ *cannot be a non-zero integer because its module is less than* $\frac{1}{2}$. So the linear independence of characters on the group $\times_{v \in S} G'_v$ implies that B was empty, A was not empty, on the right side there is only σ' and $m_{\sigma'} = 1$. The injectivity is proven like for $n = 1$.

Let us prove the surjectivity of \mathbf{G} . Fix $\sigma \in DS_{nd,t,\mu}^D$ and let S be a set of places of F which contains all the places in V , all the infinite places and all the places v for which σ_v is not spherical. We start again with the

relation (5.7):

$$\begin{aligned} & \sum_{\rho \in DS_{nd,t,\mu}^D} \text{tr} \rho(f) + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{t}{l}, \mu_l}^D} \lambda_\rho \text{tr}(\rho^l(0, f)) \\ &= \sum_{\rho' \in DS_{n,t,\mu}^{S'}} m_{\rho'} \text{tr} \rho'(f') + \sum_{l|n, l \neq 1} \frac{1}{l^2} \sum_{\rho \in DS_{\frac{nd}{l}, \frac{t}{l}, \mu_l}^D} \lambda_{\mathbf{G}_{\frac{1}{l}}^{-1}(\rho)} \text{tr}(\rho^l(0, f)) \end{aligned}$$

for $f \leftrightarrow f'$. As before, we may restrict this relation to representations which have the same local component as σ outside S .

By strong multiplicity one theorem for G_{nd} , the set of $\pi \in DS_{n,t,\mu}^D$ such that $\pi^S = \sigma^S$ contains the unique element σ . By the Proposition 4.1, no representation $\gamma = \rho^l$ (where $l|n$ and $l \neq 1$) can verify $\gamma^S = \sigma^S$. The relation becomes then

$$\text{tr} \sigma(f) = \sum_{\rho' \in DS_{n,t,\mu,\sigma}^{S'}} m_{\rho'} \text{tr} \rho'(f')$$

where $DS_{n,t,\mu,\sigma}^{S'}$ is the set of $\rho' \in DS_{n,t,\mu}^{S'}$ such that $\rho'^S = \sigma^S$. As σ is D -compatible, there exist $f \leftrightarrow f'$ such that $\text{tr} \sigma(f) \neq 0$, and so there exists at least one representation $\sigma' \in DS_{n,t,\mu,\sigma}^{S'}$. Then $\mathbf{G}(\sigma')$ is defined and, by strong multiplicity one theorem on $G_{nd}(\mathbb{A})$, $\mathbf{G}(\sigma')$ must be σ . This proves the surjectivity.

Claims (c) and (d) may be proven like for $n = 1$. □

Corollary 5.4. *The intertwining operators $M_{L'_1}^{G_{nd}}(s_0, 0)$ and $M_{L'_1}^{G'_n}(s_0, 0)$ are given by the same scalar. In particular, the computations in [KS] transfer to $G'_n(\mathbb{A})$.*

Proof. This is the consequence of $\lambda_\gamma - \lambda_{\mathbf{G}_{\frac{1}{l}}^{-1}(\gamma)} = 0$ implied by the end of the proof of the theorem. □

5.2. A classification of discrete series and automorphic representations of G'_n . If $L = \times_{i=1}^k G'_{n_i}$ is a standard Levi subgroup of G'_n , we call **essentially square integrable** (resp. **essentially cuspidal**) representation of L a representation $\pi' = \otimes_{i=1}^k v^{a_i} \rho'_i$ where, for each i , ρ'_i is a discrete series (resp. cuspidal representation) of G'_{n_i} and a_i is a real number.

We adopt the same definitions for representations $\pi = \otimes_{i=1}^k v^{a_i} \rho_i$ of standard Levi subgroups of G_{nd} . Such a representation π is said to be **D-compatible** if all the ρ_i are D -compatible.

Proposition 5.5. *Let $\rho \in DS_m$ be a cuspidal representation. Let $s_{\rho,D}$ be the smallest common multiple of s_{ρ_v, d_v} , $v \in V$ (cf. Sect. 3.5). Then*

- (a) $MW(\rho, k)$ is D -compatible if and only if $s_{\rho, D} | k$.
- (b) $\mathbf{G}^{-1}(MW(\rho, s_{\rho, D})) = \rho' \in DS'_{\frac{ms_{\rho, D}}{d}}$ is cuspidal (in particular \mathbf{G}^{-1} sends cuspidal to cuspidal).

Proof. (a) This is an easy consequence of the discussion in Sect. 3.5 and the definition of $s_{\rho, D}$.

(b) Assume ρ' is not cuspidal. Then there exists an essentially cuspidal representation τ' of a proper standard Levi subgroup L' of G'_n such that π' is a constituent of the induced representation to $G'_{\frac{ms_{\rho, D}}{d}}$ from τ' . Set $\tau = \mathbf{G}(\tau')$.

So τ is a D -compatible essentially square integrable representation of $L(\mathbb{A})$ where L is a proper standard Levi subgroup of $G_{ms_{\rho, D}}$ corresponding to L' . By the Theorem 4.4 of [JS], τ has the same cuspidal support as $MW(\rho, s_{\rho, D})$. As it is a D -compatible essentially square integrable representation and lives on a smaller subgroup, this contradicts the minimality of $s_{\rho, D}$. \square

Remark 5.6. It will be proved in the appendix that all the cuspidal representations of $G'_n(\mathbb{A})$ are obtained like in Proposition 5.5. But at this point this proof cannot be made, so for now we will call these representations basic cuspidal. Later, using the next proposition, Grbac will prove in the appendix that basic cuspidal and cuspidal is the same thing. Therefore, the reader may drop the word “basic” in the next proposition and have a clean classification.

Let us call **basic cuspidal** a cuspidal representation obtained as $\rho' = \mathbf{G}^{-1}(MW(\rho, s_{\rho, D}))$ in the part (b) of the Proposition. We then set $s(\rho') = s_{\rho, D}$ and $v_{\rho'} = v^{s_{\rho, D}}$. If $L = \times_{i=1}^k G'_{n_i}$ is a standard Levi subgroup of G'_n , we call **basic essentially cuspidal** representation of L a representation $\otimes_{i=1}^k v^{a_i} \rho'_i$ where, for each i , ρ'_i is a basic cuspidal representation of G'_{n_i} and a_i is a real number.

We now give a classification of discrete series of groups G'_n . The part (a) generalizes [MW2] and the part (b) generalizes the Theorem 4.4 in [JS].

Proposition 5.7. (a) *Let $\rho' \in DS'_m$ be a basic cuspidal representation.*

Let $k \in \mathbb{N}^$. The induced representation $\prod_{i=0}^{k-1} (v_{\rho'}^{\frac{k-1}{2}-i} \rho')$ has the unique constituent π' which is a discrete series. We write then $\pi' = MW'(\rho', k)$. Every discrete series π' of a group G'_n , $n \in \mathbb{N}^*$, is of this type, and k and ρ' are determined by π' . The discrete series π' is basic cuspidal if and only if $k = 1$. If $\pi' = MW'(\rho', k)$, then $\mathbf{G}(\rho') = MW(\rho, s_{\rho, D})$ if and only if $\mathbf{G}(\pi') = MW(\rho, ks_{\rho, D})$.*

- (b) *Let (L_i, ρ'_i) , $i = 1, 2$, be such that L_i is a standard Levi subgroup of G'_n and ρ'_i is a basic essentially cuspidal representation of $L_i(\mathbb{A})$ for $i = 1, 2$. Fix any finite set of places V' containing the infinite places and all the finite places where ρ'_1 or ρ'_2 is not spherical. If, for all places $v \notin V'$, the spherical subquotients of the induced representations from $\rho'_{i, v}$ to G'_n are equal, then the couples (L_i, ρ'_i) are conjugated.*

(c) If π' is an automorphic representation of G'_n , then there exists a couple (L, ρ') where L is a standard Levi subgroup of G'_n and ρ' is a basic essentially cuspidal representation of $L(\mathbb{A})$ such that π' is a constituent of the induced representation from ρ' to $G'_n(\mathbb{A})$. The couple (L, ρ') is unique up to conjugation.

Proof. (a) Let $\mathbf{G}(\rho') = MW(\rho, s_{\rho,D})$. The discrete series $MW(\rho, ks_{\rho,D})$ is D -compatible (Proposition 5.5(a)). We will show directly that $\mathbf{G}^{-1}(MW(\rho, ks_{\rho,D}))$ is a constituent of $\prod_{i=0}^{k-1} (v_{\rho'}^{\frac{k-1}{2}-i} \rho')$.

It is enough to show that, for every place $v \in V$, $|\mathbf{LJ}|_v(MW(\rho, ks_{\rho,D})_v)$ is a subquotient of the local representation $\prod_{i=0}^{k-1} (v_{\rho'}^{\frac{k-1}{2}-i} \rho'_v)$. By Proposition 2.1, it is enough to show that the esi-support of $|\mathbf{LJ}|_v(MW(\rho, ks_{\rho,D})_v)$ is the reunion of the esi-supports of representations $v_{\rho'}^{\frac{k-1}{2}-i} \rho'_v$. As in Sect. 3.5, we may write the generic representation ρ_v as a product of essentially square integrable representations $\prod_{j=1}^m v^{e_j} \sigma_j$ and we have seen then that

$$\rho'_v = |\mathbf{LJ}|_v(Lg(\rho_v, s_{\rho,D})) = \prod_{j=1}^m v^{e_j} |\mathbf{LJ}|_v(u(\sigma_j, s_{\rho,D}))$$

and

$$|\mathbf{LJ}|_v(Lg(\rho_v, ks_{\rho,D})) = \prod_{j=1}^m v^{e_j} |\mathbf{LJ}|_v(u(\sigma_j, ks_{\rho,D})).$$

Fix an index j . If σ_j transfers to σ'_j (case (a) of the Proposition 3.2), we know that $|\mathbf{LJ}|_v(u(\sigma_j, s_{\rho,D})) = \bar{u}(\sigma'_j, s_{\rho,D})$ and $|\mathbf{LJ}|_v(u(\sigma_j, ks_{\rho,D})) = \bar{u}(\sigma'_j, ks_{\rho,D})$. One may easily verify that the esi-support of $\bar{u}(\sigma'_j, ks_{\rho,D})$ is the reunion of the esi-supports of $v^{(\frac{k-1}{2}-i)s_{\rho,D}} \bar{u}(\sigma'_j, s_{\rho,D})$ for $i \in \{0, \dots, k-1\}$. If σ_j does not transfer (case (b) of the Proposition 3.2), one has to use the formula (3.9) in Sect. 3.5 involving σ'_{j+} and σ'_{j-} , but then the proof goes exactly the same as for the case when σ_j transfers.

So $\prod_{i=0}^{k-1} (v_{\rho'}^{\frac{k-1}{2}-i} \rho')$ has a constituent π' which is a discrete series. The strong multiplicity one theorem for discrete series of G'_n (Proposition 5.1(c)) implies this induced representation has no other constituent which is a discrete series.

Let $\pi' \in DS'_n$ be a discrete series and let us show it is obtained in this way. Set $\mathbf{G}(\pi') = MW(\rho, p)$. We have $s_{\rho,D} | p$ since $MW(\rho, p)$ is D -compatible (Proposition 5.5(a)). So, if we set $\rho' = \mathbf{G}^{-1}(MW(\rho, s_{\rho,D}))$, ρ' is a basic cuspidal, and we have $\pi' = MW'(\rho', \frac{p}{s_{\rho,D}})$. The strong multiplicity one theorem for G_{nd} implies p and ρ are determined by π' , so $k = \frac{p}{s_{\rho,D}}$ and ρ' are determined by π' . It is clear that π' is basic cuspidal if and only if $p = s_{\rho,D}$, if and only if $k = 1$.

(b) $\mathbf{G}(\rho'_1) = \rho_1$ is a tensor product of the form $\otimes_{i=1}^{p_1} v^{\alpha_i} MW(\xi_i, s_{\xi_i, D})$ and $\mathbf{G}(\rho'_2) = \rho_2$ is a tensor product of the form $\otimes_{j=1}^{p_2} v^{\beta_j} MW(\tau_j, s_{\tau_j, D})$, where ξ_i and τ_j are cuspidal. As the induced representations to G_{nd} from ρ_1 and ρ_2 have equal spherical subquotient at all finite places which are not in $V \cup V'$, we know that the essentially cuspidal supports of ρ_1 and ρ_2 are equal (Theorem 4.4 in [JS]). As ξ_i and τ_j are cuspidal, it follows from the formulas for ρ_1 and ρ_2 that the multisets $\{(\alpha_i, \xi_i)\}$ and $\{(\beta_j, \tau_j)\}$ are equal and so the tensor products are the same up to permutation.

(c) The existence is proven in (a). The unicity in (b). □

5.3. Further comments. The question whether the transfer of discrete series could be extended to unitary automorphic representations or not seems natural. Let us extend in an obvious way the notion of D -compatible from discrete series to unitary automorphic representations of $G_{nd}(\mathbb{A})$. Let us formulate two questions.

Question 1. Given a unitary automorphic representation a' of $G'_n(\mathbb{A})$, is it possible to find a unitary automorphic representation a of $G_{nd}(\mathbb{A})$ such that $a_v = a'_v$ for all $v \notin V$ and $|\mathbf{LJ}|_v(a_v) = a'_v$ for all $v \in V$?

Question 2. Given a D -compatible unitary automorphic representation a of $G_{nd}(\mathbb{A})$, is it possible to find a unitary automorphic representation a' of $G'_n(\mathbb{A})$ such that $a_v = a'_v$ for all $v \notin V$ and $|\mathbf{LJ}|_v(a_v) = a'_v$ for all $v \in V$?

These questions are independent and the answer is in general “no” for both.

Consider the first question. Roughly speaking the counterexample comes from the fact that there exist unitary irreducible representations of an inner form of GL_n over a local field which do not correspond to a unitary representation of GL_n . The problem is to realize such a representation as a local component of a unitary automorphic representation. Here is the construction, based on Lemma 3.11.

Let $\dim_F D = 16$. Let $G' = GL_3(D)$. Assume there is a finite place v_0 of F such that the local component of $G'(\mathbb{A})$ at the place v_0 is $G'_{v_0} \simeq GL_3(D_{v_0})$ with $\dim_{F_{v_0}} D_{v_0} = 16$. It is possible to choose such a D by the global class field theory. Let ρ' be a cuspidal representation of $G'(\mathbb{A})$ such that ρ'_{v_0} is the Steinberg representation of G'_{v_0} . Then $\mathbf{G}(\rho')$ is cuspidal. Indeed, its local component at the place v_0 has to be the Steinberg representation of $GL_{12}(F_{v_0})$ (the only unitary irreducible elliptic representations being the trivial representation and the Steinberg representation). In particular $s_{\rho'} = 1$.

Let $\tau' = MW'(\rho', 16)$. Let St'_3 be the Steinberg representation of $GL_3(D_{v_0})$ and St'_4 the Steinberg representation of $GL_4(D_{v_0})$. Then $\tau'_{v_0} = u'(St'_3, 16)$.

Let τ'' be the global representation defined by: $\tau''_v = \tau'_v$ for all $v \neq v_0$ and $\tau''_{v_0} = v^{-\frac{3}{2}}u'(St'_3, 4) \times v^{-\frac{1}{2}}u'(St'_4, 3) \times v^{\frac{1}{2}}u'(St'_4, 3) \times v^{\frac{3}{2}}u'(St'_3, 4)$. Let us show that τ'' is an automorphic representation. We have $\tau''_{v_0} < \tau'_{v_0}$ by Lemma 3.11(ii). So τ''_{v_0} is a subquotient of $\times_{i=1}^{16} v^{\frac{17}{2}-i} St'_3$. So τ'' is a constituent of $\times_{i=1}^{16} v^{\frac{17}{2}-i} \rho'$. As ρ' is cuspidal, τ'' is automorphic. All the local components of τ'' are unitary. It is true by definition for τ''_v , $v \neq v_0$, and by Lemma 3.11(i) for τ''_{v_0} . So τ'' is a unitary automorphic representation. It cannot correspond to a unitary automorphic representation of $GL_{192}(\mathbb{A})$ because by Lemma 3.11(iii) there is a transfer problem at the place v_0 .

Consider now the second question. Let $\dim_F D = d^2 = 4$. Let $G' = GL_3(D)$. Assume there is a finite place v_0 of F such that the local component of $G'(\mathbb{A})$ at the place v_0 is $G'_{v_0} \simeq GL_3(D_{v_0})$ with $\dim_{F_{v_0}} D_{v_0} = 4$. For all $i \in \mathbb{N}^*$, write St_i for the Steinberg representation of $GL_i(F_{v_0})$ and St'_i for the Steinberg representation of $GL_i(D_{v_0})$. Let ρ be a cuspidal representation of $GL_3(\mathbb{A})$ such that $\rho_{v_0} = St_3$. Set $\tau = MW(\rho, 2)$. We have $s_{\rho, D} = 2$ (since $s_{\rho, D}$ always divides d and here $d = 2$ and $s_{\rho, D} \neq 1$). So τ is D -compatible and $\tau' = \mathbf{G}^{-1}(\tau)$ is a cuspidal representation. We have $\tau_{v_0} = u(St_3, 2)$. Let π be the representation $St_4 \times St_2$ of $GL_6(F_{v_0})$. Then π is tempered. We also have $\pi < \tau_{v_0}$, so π is a subquotient of $v^{\frac{1}{2}}St_3 \times v^{-\frac{1}{2}}St_3$. So the representation ξ defined by $\xi_v = \tau_v$ if $v \neq v_0$ and $\xi_{v_0} = \pi$ is a constituent of $v^{\frac{1}{2}}\rho \times v^{-\frac{1}{2}}\rho$, hence an automorphic representation. All its local components are unitary. It is a D -compatible representation because π is 2-compatible. Let us show that the representation ξ' defined by $\xi'_v = |\mathbf{LJ}|_v(\xi_v)$ for all places v of F is not automorphic. For every place $v \neq v_0$, we have $\xi'_v = \tau'_v$. As τ' is cuspidal, it is enough to show that $\xi' \neq \tau'$ by Theorem 5.7(b) applied to τ' and the cuspidal support of ξ' . So this comes to show that $|\mathbf{LJ}_{v_0}|(u(St_3, 2)) \neq |\mathbf{LJ}_{v_0}|(\pi)$. Using the formulas we have for the transfer (Proposition 3.7) we find $|\mathbf{LJ}_{v_0}|(u(St_3, 2)) = u(St'_1, 3)$ and $|\mathbf{LJ}_{v_0}|(\pi) = St'_2 \times St'_1$. If 1_2 is the trivial representation of $GL_2(D_{v_0})$, we have $u(St'_1, 3) = 1_2 \times St'_1$ hence $\xi'_{v_0} \neq \tau'_{v_0}$.

6. L -functions and ϵ' -factors

In this section we examine the local transfer of L -functions and ϵ' -factors. The results are simple computations using [GJ] and [Ja] included here for the completeness.

Let F be again the non-Archimedean local field (of any characteristic) and D a division algebra of dimension d^2 over F . For all n , recall that $G_n = GL_n(F)$ and $G'_n = GL_n(D)$.

Suppose the characteristic of the residual field of F is p and its cardinality is q . Let O_F be the ring of integers of F and π_F be a uniformizer of F . Fix an additive character ψ of F trivial on O and non-trivial on $\pi_F^{-1}O$. For

irreducible representations π of G_n or G'_n , we adopt the notation $L(s, \pi)$ and $\epsilon'(s, \pi, \psi)$ for the L -function and the ϵ' -factor, as defined in [GJ].

In this section we will specify ν , because confusion may appear. For all $n \in \mathbb{N}^*$, ν_n (resp. ν'_n) will denote the absolute value of the determinant on G_n (resp. G'_n); 1_n (resp. $1'_n$) will denote the trivial representation of G_n (resp. G'_n); let $St_n = Z^u(1_1, n)$ (resp. $St'_n = T^u(1'_1, n)$) be the **Steinberg representation** of G_n (resp. G'_n). One has $St_n = |i(1_n)|$ and $St'_n = |i'(1'_n)|$. The character of the Steinberg representation is constant on the set of elliptic elements, equal to $(-1)^{n-1}$. In particular, we have $\mathbf{C}(St_d) = 1'_1$. This implies that $s(1'_1) = d$ (here $s(1'_1)$ is the invariant defined in Sect. 2.4, nothing to do with the complex variable s). For all $n \in \mathbb{N}^*$, one has $\mathbf{C}(St_{nd}) = St'_n$.

We bring together facts from [GJ] in the following theorem:

Theorem 6.1. (a) We have $L(s, 1'_1) = (1 - q^{-s - \frac{d-1}{2}})^{-1}$,

$$L(s, 1'_n) = \prod_{j=0}^{n-1} L\left(s + d\frac{n-1}{2} - dj, 1'_1\right) = \prod_{j=0}^{n-1} (1 - q^{-s+dj - \frac{dn-1}{2}})^{-1}$$

and

$$\begin{aligned} \epsilon'(s, 1'_n, \psi) &= \prod_{j=0}^{n-1} \epsilon'\left(s + d\frac{n-1}{2} - dj, 1'_1, \psi\right) \\ &= \prod_{j=0}^{dn-1} \epsilon'\left(s + \frac{dn-1}{2} - j, 1_1, \psi\right). \end{aligned}$$

(b) We have $L(St'_n) = L(s + d\frac{n-1}{2}, 1'_1) = (1 - q^{-s - \frac{dn-1}{2}})^{-1}$ and

$$\begin{aligned} \epsilon'(s, St'_n, \psi) &= \prod_{j=0}^{n-1} \epsilon'\left(s + d\frac{n-1}{2} - dj, 1'_1, \psi\right) \\ &= \prod_{j=0}^{dn-1} \epsilon'\left(s + \frac{dn-1}{2} - j, 1_1, \psi\right). \end{aligned}$$

(c) If ρ' is a cuspidal representation of G'_x , then $L(s, \rho') = 1$ unless $x = 1$ and ρ' is an unramified character of D^\times . If $x = 1$ and ρ' is an unramified character of D^\times , then $\rho' = \nu_1^t$ for some $t \in \mathbb{C}$ and we have $L(s, \rho') = (1 - q^{-s-t - \frac{d-1}{2}})^{-1}$.

(d) Let $\sigma' = T(\rho', k)$ be an essentially square integrable representation of G'_{xk} where ρ' is a cuspidal representation of G'_x . Then $L(s, \sigma') = L(s, \rho')$.

In particular, $L(s, \sigma') = 1$ unless $x = 1$ and ρ' is an unramified character of D^\times . If $x = 1$ and ρ' is an unramified character of D^\times then $\rho' = \nu_1^t$ for some $t \in \mathbb{C}$ and then $\sigma' = \nu_n^{t+d\frac{n-1}{2}} St'_n$. We have $L(s, \sigma') = (1 - q^{-s-t - \frac{d-1}{2}})^{-1}$ in this case.

We have, in general,

$$\epsilon'(s, \sigma', \psi) = \prod_{j=0}^{k-1} \epsilon'(s + js(\sigma'), \rho', \psi)$$

(in this formula, $s(\sigma')$ is the invariant defined in Sect. 2.4).

- (e) Let $\sigma'_i \in \mathcal{D}'_{n_i}$, $i \in \{1, 2, \dots, k\}$, $\sum_{i=1}^k n_i = n$. Let $a_1 \geq a_2 \geq \dots \geq a_k$ be real numbers. Set $S' = \times_{i=1}^k v'^{a_i}_{n_i} \sigma'_i$ and $\pi' = \text{Lg}(S')$.

Then

$$L(s, \pi') = \prod_{i=1}^k L(s, \sigma'_i)$$

and

$$\epsilon'(s, \pi', \psi) = \prod_{i=1}^k \epsilon'(s, \sigma'_i, \psi).$$

In particular, if $\rho'_1, \rho'_2, \dots, \rho'_p$ is the cuspidal support of π' , then

$$\epsilon'(s, \pi', \psi) = \prod_{i=1}^p \epsilon'(s, \rho'_i, \psi).$$

Proof. (a) This is shown in the Proposition 6.11 in [GJ], where the formula is slightly wrong. The reader may verify that the good formula for the L -function in [GJ, Proposition 6.9], is with $(d - 1)$ instead of $(n - 1)$, as indicated by the proof of this proposition. Then this typo error is propagated to [GJ, Proposition 6.9], where the reader may easily verify that the right formula obtained, after correcting the Proposition 6.9, is our formula. For the ϵ' -factor our formula fits the [GJ] one.

(b) The ϵ' -factor of S'_n equals the ϵ' -factor of $1'_n$ as they are both subquotients of the same induced representation ([GJ], Corollary 3.6).

Let us check the L -function. For the particular case $D = F$, the computation of the L -function is Theorem 7.11 (4), [GJ]. Let us give a general (different) proof by induction on n .

For $n = 1$ we have $S'_n = S'_1 = 1'_1$ and the result is implied by (a).

For any $n > 1$, the representation S'_n is a subquotient of the induced representation from $v_1^{-\frac{d(n-1)}{2}} 1'_1 \otimes v_{n-1}^{\frac{d}{2}} S'_{n-1}$. We know that

$$L(v_1^{\frac{d(n-1)}{2}} 1'_1) = (1 - q^{-s - \frac{d-1}{2} + \frac{d(n-1)}{2}})^{-1}$$

and, by the induction hypothesis, we have

$$L(s, v_{n-1}^{\frac{d}{2}} S'_{n-1}) = (1 - q^{-s - \frac{dn-1}{2}})^{-1}.$$

By [GJ], Corollary 3.6, $L(s, St'_n)$ is equal to one of these two functions or to their product. But, by [GJ], Proposition 1.3 and Theorem 3.3 (1) and (2), the poles of $L(s, St'_n)$ cannot be greater than $\frac{d(n-1)}{2} - \frac{dn-1}{2} = -\frac{d-1}{2}$, so there is no positive pole (this trick comes from the original proof: an L -function of a square integrable representation cannot have a pole with a positive real part). So $L(s, St'_n) = L(s, v'^{\frac{d}{2}}_{n-1} St'_{n-1}) = (1 - q^{-s - \frac{dn-1}{2}})^{-1}$.

(c) The first assertion is a consequence of Lemma 4.1, Proposition 4.4 and Proposition 5.11 of [GJ] (Proposition 5.11 is not enough, since the authors assume $m > 1$ at the beginning of the Sect. 5). The second assertion is a direct consequence of the part (a) of the present theorem.

(d) For the particular case of G_n this is explained after Proposition 3.1.3 of [Ja]. The same proof applies to G'_n , using the calculation for St'_1 , i.e. the part (b).

(e) This is proven in [Ja] for G_n , but the same proof applies to G'_n . \square

Theorem 6.2. *Let \mathbf{C} be the local Jacquet–Langlands correspondence between G_{nd} and G'_n . Then, for all $\sigma \in \mathcal{D}^u_{nd}$, we have $L(s, \sigma) = L(s, \mathbf{C}(\sigma))$ and $\epsilon'(s, \sigma, \psi) = \epsilon'(s, \mathbf{C}(\sigma), \psi)$.*

Proof. Let us show it first for the Steinberg representation and its twists. We have $\mathbf{C}(St_{nd}) = St'_n$. Theorem 6.1(a) and (b) implies the statement in this case. This implies then the statement for all the twist of St_{nd} with characters.

Lemma 6.3. *For all $\sigma \in \mathcal{D}^u_{nd}$, we have $\epsilon'(s, \sigma, \psi) = \epsilon'(s, \mathbf{C}(\sigma), \psi)$.*

Proof. The proof is standard, using an easy global correspondence (true in all characteristics) and the previous calculus for the Steinberg representations. See for example [Ba2, p. 741] : *Les facteurs ϵ'* . \square

Let us complete the proof of the theorem with the calculation of L -functions. If $\sigma \in \mathcal{D}^u_{nd}$ or \mathcal{D}^u_n which is not a twist of the Steinberg representation, then Theorem 6.1(d) implies that its L -function is trivial and so its ϵ' -factor is equal to its ϵ -factor. As $\mathbf{C}(\sigma)$ is a twist of the Steinberg representation if and only if σ itself is a twist of the Steinberg representation, the statement has been now proven for all $\sigma \in \mathcal{D}^u_{nd}$. \square

Corollary 6.4. *Let $\sigma'_i \in \mathcal{D}^u_{n_i}$, $i \in \{1, 2, \dots, k\}$, $\sum_{i=1}^k n_i = n$. Let $a_1 \geq a_2 \geq \dots \geq a_k$ be real numbers. Set $S' = \times_{i=1}^k v'^{a_i}_{n_i} \sigma'_i$. Let $\mathbf{C}^{-1}(\sigma'_i) = \sigma_i \in \mathcal{D}^u_{n_i}$ and set $S = \times_{i=1}^k v^{\alpha_i}_{n_{id}} \sigma_i$. Then $L(s, Lg(S')) = L(s, Lg(S))$ and $\epsilon'(s, Lg(S'), \psi) = \epsilon'(s, Lg(S), \psi)$.*

Proof. This is implied by the previous theorem and the part (e) of Theorem 6.1. \square

Corollary 6.5. *Assume the characteristic of F is zero. If $u \in \text{Irr}^u_{nd}$ is such that $\mathbf{LJ}_n(u) \neq 0$. Then $\epsilon'(s, u, \psi) = \epsilon'(s, |\mathbf{LJ}|_n(u), \psi)$.*

Proof. It is enough to prove it for $u = u(\sigma, k)$, $\sigma \in \mathcal{D}_p^u$, $k, p \in \mathbb{N}^*$, such that $|\mathbf{L}_{pk}(u) = u' \neq 0$. If we are in the case (a) of the Proposition 3.2, then u and u' are like in the Corollary 6.4. In particular, their L functions are equal too. If we are in the case (b) of the Proposition 3.2, then $|i(u)|$ and $|i'(u')|$ are like in the Corollary 6.4. Now, the ϵ' -factor depends only on the cuspidal support (Theorem 6.1(e)). So the ϵ' -factor is the same for an irreducible representation and its dual. But in general we do not get equality for the L -functions in this case. \square

Appendix A. The residual spectrum of GL_n over a division algebra¹

A.1. Introduction. In this appendix the residual spectrum of GL_n over a division algebra is decomposed. The approach is the Langlands spectral theory as explained in [MW3] and [La2]. However, the results in the paper, obtained using the Arthur trace formula of [AC], classify the entire discrete spectrum of GL_n over a division algebra. Hence, the problem reduces to distinguishing the residual representations in the discrete spectrum. This simplifies the application of the Langlands spectral theory since it reduces the region of the possible poles of the Eisenstein series to a cone well inside the positive Weyl chamber. Having in mind the classification of the discrete spectrum and the multiplicity one theorem, we obtain the classification of the cuspidal spectrum as a consequence of the decomposition of the residual spectrum. In fact, it turns out that the only cuspidal representations are the basic cuspidal ones.

The idea of writing this appendix was born during our stay at the Erwin Schrödinger Institute, Vienna in December 2006 and February 2007. I would like to thank Joachim Schwermer for his kind invitation. My gratitude goes to Goran Muić for many useful conversations and constant help. I am grateful to Colette Mœglin for sharing her insight and advices on the normalization of the standard intertwining operators. Also, I would like to thank Marko Tadić for the support and interest in my work. I thank Ioan Badulescu for explaining his results and including this appendix to the paper. And finally, I would like to thank my wife Tiki for bringing so much joy into my life.

A.2. Normalization of intertwining operators. Let F be an algebraic number field (a global field of characteristic zero) and D a central division algebra of dimension d^2 over F . Let F_v denote the completion of F at a place v and \mathbb{A} the ring of adèles of F . We use the global notation of Sects. 4 and 5. Let G'_r be the inner form, defined via D , of the split general linear group $G_{rd} = GL_{rd}$. Let V be the finite set of places where D is non-split. As in the paper, we assume that D splits at all infinite places, i.e. V consists only of finite places.

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Recall from Sect. 5.2 the description of the basic cuspidal representations of $G'_r(\mathbb{A})$. Let ρ be a cuspidal representation of $G_q(\mathbb{A})$ and $s_{\rho,D}$ the smallest positive integer such that the discrete spectrum representation $\sigma \cong MW(\rho, s_{\rho,D})$ of $G_{qs_{\rho,D}}(\mathbb{A})$ is compatible at every place. Then,

$$\sigma' \cong \mathbf{G}^{-1}(\sigma) \cong \otimes_v |\mathbf{LJ}|_v(\sigma_v)$$

is a basic cuspidal representation of $G'_r(\mathbb{A})$. Observe that $\sigma'_v \cong \sigma_v$ at all places $v \notin V$. The goal of this appendix is to show that all cuspidal representations of $G'_r(\mathbb{A})$ are obtained in this way. In fact, we show that all the remaining representations in the discrete spectrum belong to the residual spectrum and apply the multiplicity one theorem.

In the sequel we always assume that the cuspidal representations are such that the poles of the attached Eisenstein series and L-functions are real. There is no loss in generality since this can be achieved simply twisting by the imaginary power of the absolute value of the determinant. Hence, our assumption is just a convenient choice of the coordinates. Furthermore, as in the paper, along with the notation \times for the parabolic induction, we use the notation ind_M^G when we want to point out the Levi factor M of the standard parabolic subgroup in G .

Consider first a cuspidal representation $\sigma' \otimes \sigma'$ of the Levi factor $L'(\mathbb{A}) \cong G'_r(\mathbb{A}) \times G'_r(\mathbb{A})$ of a maximal proper standard parabolic subgroup in $G'_{2r}(\mathbb{A})$, where σ' is basic cuspidal as above. Let $\underline{s} = (s_1, s_2) \in \mathfrak{a}_{L',\mathbb{C}}$ and w the unique nontrivial Weyl group element such that $wL'w^{-1} = L'$.

Lemma A.1. *Let $v \notin V$ be a split place. The normalizing factor for the standard intertwining operator*

$$A((s_1, s_2), \sigma_v \otimes \sigma_v, w)$$

acting on the induced representation

$$ind_{G_{rd}(F_v) \times G_{rd}(F_v)}^{G_{2rd}(F_v)}(v^{s_1}\sigma_v \otimes v^{s_2}\sigma_v)$$

is given by

$$\begin{aligned} \text{(A.1)} \quad r((s_1, s_2), \sigma_v \otimes \sigma_v, w) &= \frac{\prod_{j=1}^{s_{\rho,D}} L(s_1 - s_2 - s_{\rho,D} + j, \rho_v \times \tilde{\rho}_v)}{\prod_{j=1}^{s_{\rho,D}} L(s_1 - s_2 + j, \rho_v \times \tilde{\rho}_v) \cdot \varepsilon(s_1 - s_2, \sigma_v \times \tilde{\sigma}_v, \psi_v)}, \end{aligned}$$

where the L-functions and ε -factors are the local Rankin–Selberg ones of pairs. Then, the normalized intertwining operator $N((s_1, s_2), \sigma_v \otimes \sigma_v, w)$, defined by

$$A((s_1, s_2), \sigma_v \otimes \sigma_v, w) = r((s_1, s_2), \sigma_v \otimes \sigma_v, w)N((s_1, s_2), \sigma_v \otimes \sigma_v, w),$$

is holomorphic and non-vanishing for $Re(s_1 - s_2) \geq s_{\rho,D}$.

Proof. This lemma is a weaker form of Lemma I.10 of [MW2] where the holomorphy and non-vanishing is proved in a certain region slightly bigger than the closure of the positive Weyl chamber for any unitary representation. We just show that the normalizing factor defined in [MW2] is the same as here.

By [MW2],

$$(A.2) \quad r((s_1, s_2), \sigma_v \otimes \sigma_v, w) = \frac{L(s_1 - s_2, \sigma_v \times \tilde{\sigma}_v)}{L(1 + s_1 - s_2, \sigma_v \times \tilde{\sigma}_v)\varepsilon(s_1 - s_2, \sigma_v \times \tilde{\sigma}_v, \psi_v)}.$$

But, σ_v is a quotient of the induced representation

$$v^{\frac{s_{\rho,D}-1}{2}} \rho_v \times v^{\frac{s_{\rho,D}-3}{2}} \rho_v \times \dots \times v^{-\frac{s_{\rho,D}-1}{2}} \rho_v,$$

where ρ_v , being unitary and generic as the local component at v of a cuspidal representation ρ , is a fully induced representation of the form

$$v^{e_{1,v}} \delta_{1,v} \times v^{e_{2,v}} \delta_{2,v} \times \dots \times v^{e_{m_v,v}} \delta_{m_v,v}$$

with $e_{i,v}$ real, $|e_{i,v}| < 1/2$ and $\delta_{i,v} \in \mathcal{D}^u$. We may arrange the indices in such a way that $e_{1,v} \geq e_{2,v} \geq \dots \geq e_{m_v,v}$.

This shows that σ_v is the Langlands quotient and we can apply the formulas for the Rankin–Selberg L-function and ε -factor of the Langlands quotient. Having in mind that ρ_v is fully induced, we obtain

$$(A.3) \quad L(s, \sigma_v \times \tilde{\sigma}_v) = L(s, \rho_v \times \tilde{\rho}_v)^{s_{\rho,D}} \\ \times \prod_{j=1}^{s_{\rho,D}-1} L(s + s_{\rho,D} - j, \rho_v \times \tilde{\rho}_v)^j L(s - s_{\rho,D} + j, \rho_v \times \tilde{\rho}_v)^j$$

and the ε -factor is of the same form, but since it has no zeroes nor poles we do not need to refine its form. Inserting the formula for the L-function into Equation (A.2) gives after cancellation the normalizing factor (A.1). \square

Lemma A.2. *Let $v \in V$ be a non-split place. Then the standard intertwining operator*

$$A((s_1, s_2), \sigma'_v \otimes \sigma'_v, w)$$

is holomorphic and non-vanishing for $Re(s_1 - s_2) \geq s_{\rho,D}$.

Proof. Sections 3.2, 3.3 and 3.5 give rather precise form of the local component σ'_v of a basic cuspidal representation of $GL'_r(\mathbb{A})$. By Sect. 3.5, it is a fully induced representation of the form

$$\sigma'_v \cong v^{e_{1,v}} |\mathbf{LJ}|_v(u(\delta_{1,v}, s_{\rho,D})) \times \dots \times v^{e_{m_v,v}} |\mathbf{LJ}|_v(u(\delta_{m_v,v}, s_{\rho,D})),$$

where $e_{i,v}$ are real, $|e_{i,v}| < 1/2$ and $\delta_{i,v} \in \mathcal{D}^u$. More precisely, $e_{i,v}$ and $\delta_{i,v}$ are defined by

$$\rho_v \cong v^{e_{1,v}} \delta_{1,v} \times \dots \times v^{e_{m_v,v}} \delta_{m_v,v}.$$

The precise formula for $|\mathbf{LJ}|_v(u(\delta_{i,v}, s_{\rho,D}))$ is given in Proposition 3.7 and Equation (3.9). If $\delta_{i,v}$ is compatible, then

$$|\mathbf{LJ}|_v(u(\delta_{i,v}, s_{\rho,D})) = \bar{u}(\delta'_{i,v}, s_{\rho,D}),$$

and the highest exponent of v appearing in the corresponding standard module is $\frac{s_{\rho,D}-1}{2}$. If $\delta_{i,v}$ is not compatible, then, by the choice of $s_{\rho,D}$, we have

$$\begin{aligned} |\mathbf{LJ}|_v(u(\delta_{i,v}, s_{\rho,D})) &= \prod_{i=1}^b v^{i-\frac{b+1}{2}} u'(\delta'_{i+,v}, s_{\rho,D}/s(\delta_{i,v})) \\ &\quad \times \prod_{j=1}^{s(\delta_{i,v})-b} v^{j-\frac{s(\delta_{i,v})-b+1}{2}} u'(\delta'_{i-,v}, s_{\rho,D}/s(\delta_{i,v})), \end{aligned}$$

where $\delta'_{i\pm,v} \in \mathcal{D}^u$ are certain unitary discrete series representations. See Sect. 3.3 for the unexplained notation. In this case the highest exponent of v appearing among the standard modules is either

$$\frac{b-1}{2} + s(\delta_{i,v}) \frac{s_{\rho,D}/s(\delta_{i,v}) - 1}{2} < \frac{s_{\rho,D} - 1}{2}$$

or

$$\frac{s(\delta_{i,v}) - b - 1}{2} + s(\delta_{i,v}) \frac{s_{\rho,D}/s(\delta_{i,v}) - 1}{2} \leq \frac{s_{\rho,D} - 1}{2},$$

where the upper bounds are obtained using the fact that $0 \leq b < s(\delta_{i,v})$ (see Sect. 3.3).

The description of σ'_v shows that the induced representation

$$v^{s_1} \sigma'_v \times v^{s_2} \sigma'_v$$

is a product of possibly twisted representations of the form $\bar{u}(\cdot)$ and $u'(\cdot)$ which are the Langlands quotients of the standard module induced from a discrete series representation. In other words there is a unitary discrete series representation δ'_v of the appropriate Levi factor $L'_0(F_v)$ of $G'_{2r}(F_v)$ and $\underline{s} \in \mathfrak{a}_{L'_0, \mathbb{C}}$ such that, by the Langlands classification, the standard intertwining operator

$$A(\underline{s}, \delta'_v, w_0) : \text{ind}_{L'_0(k_v)}^{G'_{2r}(k_v)}(\underline{s}, \delta'_v) \rightarrow \text{ind}_{w_0(L'_0(k_v))}^{G'_{2r}(k_v)}(w_0(\underline{s}), w_0(\delta'_v))$$

is holomorphic and its image is the induced representation $\nu^{s_1}\sigma'_\nu \times \nu^{s_2}\sigma'_\nu$. Therefore, by the decomposition property of the intertwining operators according to the reduced decomposition of the Weyl group element ww_0 , the standard intertwining operator $A((s_1, s_2), \sigma'_\nu \otimes \sigma'_\nu, w)$ fits into the commutative diagram

$$\begin{array}{ccc}
 \text{ind}_{L'_0(k_\nu)}^{G'_{2r}(k_\nu)}(\underline{s}, \delta'_\nu) & \xrightarrow{A(\underline{s}, \delta'_\nu, w_0)} & \nu^{s_1}\sigma'_\nu \times \nu^{s_2}\sigma'_\nu \\
 \downarrow A(\underline{s}, \delta'_\nu, ww_0) & & \downarrow A((s_1, s_2), \sigma'_\nu \otimes \sigma'_\nu, w) \\
 \text{ind}_{ww_0(L'_0(k_\nu))}^{G'_{2r}(k_\nu)}(ww_0(\underline{s}), ww_0(\delta'_\nu)) & \leftrightarrow & \nu^{s_2}\sigma'_\nu \times \nu^{s_1}\sigma'_\nu,
 \end{array}$$

where the upper horizontal arrow is surjective. Observe that the right vertical arrow is in fact just the restriction of the intertwining operator $A(w_0(\underline{s}), w_0(\delta'_\nu), w)$ to the subrepresentation $\nu^{s_1}\sigma'_\nu \times \nu^{s_2}\sigma'_\nu$. This follows by the analytic continuation from the fact that the integrals defining the two intertwining operators are over the same unipotent subgroups and hence agree in the domain of convergence. The diagram implies the lemma if we prove that, for $Re(s_1 - s_2) \geq s_{\rho, D}$, the left vertical arrow is holomorphic and non-vanishing.

By the Langlands classification it suffices to check that the real parts of all the differences between exponents of ν appearing in the parts of $I(\underline{s}, \delta'_\nu)$ corresponding to $\nu^{s_1}\sigma'_\nu$ and $\nu^{s_2}\sigma'_\nu$ are strictly positive. However, we already checked that the highest exponent appearing among the standard modules in the expressions for $|\mathbf{LJ}|_\nu(u(\delta_{i,v}, s_{\rho, D}))$ is at most $\frac{s_{\rho, D}-1}{2}$. Therefore, in the worst case we obtain the difference

$$Re(s_1 - s_2) + e_{i,v} - e_{j,v} - 2 \cdot \frac{s_{\rho, D} - 1}{2} > 0$$

since $e_{i,v} - e_{j,v} > -1$. □

Remark A.3. The proof of the previous lemma follows the idea of the proof of Lemma I.8 of [MW2]. Since the results of this paper based on the trace formula reduce the question of determining the residual spectrum to the point $Re(s_1 - s_2) = s_{\rho, D}$ and give bounds on the exponents of the local component at a non-split place of a cuspidal representation of an inner form, we do not require the full power of Lemma I.8, and hence the proof becomes simpler. However, its analogue for inner forms could have been obtained using first the transfer of the Plancherel measure for discrete series representations (see [MS]) to define the normalization using L-functions for the split group. For the classical hermitian quaternionic groups we used this technique to obtain the parts of the residual spectra in [Gr1], [Gr2], [Gr3], [Gr4].

Corollary A.4. *The normalizing factor for the global standard intertwining operator*

$$A((s_1, s_2), \sigma' \otimes \sigma', w)$$

acting on the induced representation

$$\text{ind}_{L'(\mathbb{A})}^{G'_{2r}(\mathbb{A})} (v^{s_1} \sigma' \otimes v^{s_2} \sigma')$$

is given by

$$(A.4) \quad r((s_1, s_2), \sigma' \otimes \sigma', w) = \frac{\prod_{j=1}^{s_{\rho,D}} L^V(s_1 - s_2 - s_{\rho,D} + j, \rho \times \tilde{\rho})}{\prod_{j=1}^{s_{\rho,D}} L^V(s_1 - s_2 + j, \rho \times \tilde{\rho}) \cdot \varepsilon^V(s_1 - s_2, \sigma' \times \tilde{\sigma}')} ,$$

where the L -functions and ε -factors are the partial Rankin–Selberg ones with respect to the finite set V of non-split places of D . Then, the normalized intertwining operator $N((s_1, s_2), \sigma' \otimes \sigma', w)$ defined by

$$A((s_1, s_2), \sigma' \otimes \sigma', w) = r((s_1, s_2), \sigma' \otimes \sigma', w) N((s_1, s_2), \sigma' \otimes \sigma', w)$$

is holomorphic and non-vanishing for $\text{Re}(s_1 - s_2) \geq s_{\rho,D}$. Moreover, the only pole of the standard intertwining operator $A((s_1, s_2), \sigma' \otimes \sigma', w)$ in the region $\text{Re}(s_1 - s_2) \geq s_{\rho,D}$ is at $s_1 - s_2 = s_{\rho,D}$ and it is simple.

Proof. The global normalizing factor is obtained as a product over all places of the local ones. Note that, for our purposes, at a non-split places the normalizing factor is taken to be trivial. Then the holomorphy and non-vanishing of the normalized intertwining operator in the region $\text{Re}(s_1 - s_2) \geq s_{\rho,D}$ follows from the local results of the previous two lemmas.

The analytic properties of the Rankin–Selberg L -functions are well-known. The global Rankin–Selberg L -function $L(z, \rho \times \tilde{\rho})$ has the only poles at $z = 0$ and $z = 1$ and they are both simple. It has no zeroes for $\text{Re}(z) \geq 1$. Writing ρ_v at a non-split place $v \in V$ as a fully induced representation from the discrete series representation as in the proof of the previous lemma shows that the local Rankin–Selberg L -function equals

$$L(z, \rho_v \times \tilde{\rho}_v) = \prod_{i,j=1}^{m_v} L(z + e_{i,v} - e_{j,v}, \delta_{i,v} \times \tilde{\delta}_{j,v}).$$

Since the local L -functions attached to unitary discrete series representations are holomorphic in the strict right half-plane, and $e_{i,v} - e_{j,v} > -1$, the L -function $L(z, \rho_v \times \tilde{\rho}_v)$ is holomorphic for $\text{Re}(z) \geq 1$. Local L -functions have no zeroes.

Therefore, the partial L -function $L^V(z, \rho \times \tilde{\rho})$ is holomorphic for $\text{Re}(z) \geq 1$ except for a simple pole at $z = 1$. It has no zeroes for $\text{Re}(z) \geq 1$. The ε -factor has neither zeroes nor poles. Since for $\text{Re}(s_1 - s_2) \geq s_{\rho,D}$ real

parts of all the arguments of the L-functions in the global normalizing factor (A.4), except $Re(s_1 - s_2 - s_{\rho,D} + 1) \geq 1$, are strictly greater than one, it has no zeroes and the only pole occurs for $s_1 - s_2 = s_{\rho,D}$. Since the normalized intertwining operator is holomorphic and non-vanishing for $Re(s_1 - s_2) \geq s_{\rho,D}$, it turns out that the only pole in the region $Re(s_1 - s_2) \geq s_{\rho,D}$ of the global standard intertwining operator is at $s_1 - s_2 = s_{\rho,D}$ and it is simple. \square

A.3. Poles of Eisenstein series. Let σ' be as above and $k > 1$ an integer. Let $\pi' \cong \sigma' \otimes \dots \otimes \sigma'$ be a cuspidal representation of the Levi factor $M'(\mathbb{A}) \cong G'_r(\mathbb{A}) \times \dots \times G'_r(\mathbb{A})$ of a standard parabolic subgroup of $G'_{kr}(\mathbb{A})$, with k copies of $G'_r(\mathbb{A})$ and σ' in the products. We fix an isomorphism $\mathfrak{a}^*_{M',\mathbb{C}} \cong \mathbb{C}^k$ using the absolute value of the reduced norm of the determinant at each copy of G'_r and denote its elements by $\underline{s} = (s_1, s_2, \dots, s_k) \in \mathfrak{a}^*_{M',\mathbb{C}}$. By the results of the paper, the study of the residual spectrum is reduced to the point

$$\underline{s}_0 = \left(\frac{s_{\rho,D}(k-1)}{2}, \frac{s_{\rho,D}(k-3)}{2}, \dots, -\frac{s_{\rho,D}(k-1)}{2} \right),$$

i.e. we have to prove that the unique discrete series constituent of the induced representation

$$\text{ind}_{M'(\mathbb{A})}^{G'_{kr}(\mathbb{A})}(\underline{s}_0, \pi') = v^{\frac{s_{\rho,D}(k-1)}{2}} \sigma' \times v^{\frac{s_{\rho,D}(k-3)}{2}} \sigma' \times \dots \times v^{-\frac{s_{\rho,D}(k-1)}{2}} \sigma',$$

which is denoted in the paper by $MW'(\sigma', k)$, is in the residual spectrum. Of course, the case $k = 1$ is excluded since it gives just the (basic) cuspidal representation σ' .

Lemma A.5. *Let*

$$E(\underline{s}, g; \pi', f_{\underline{s}})$$

be the Eisenstein series attached to a ‘good’ (in the sense of Sects. II.1.1 and II.1.2 of [MW3]) section $f_{\underline{s}}$ of the above induced representation from a cuspidal representation π' . Then, its only pole in the region $Re(s_i - s_{i+1}) \geq s_{\rho,D}$, for $i = 1, \dots, k - 1$, is at \underline{s}_0 and it is simple. The constant term map gives rise to an isomorphism between the space of automorphic forms $\mathcal{A}(\sigma', k)$ spanned by the iterated residue at \underline{s}_0 of the Eisenstein series and the irreducible image $MW'(\sigma', k)$ of the normalized intertwining operator

$$N(\underline{s}_0, \pi', w_l),$$

where w_l is the longest among Weyl group elements w such that $wM'w^{-1} \cong M'$.

Proof. By the general theory of the Eisenstein series explained in Sect. V.3.16 of [MW3], its poles coincide with the poles of its constant

term along the standard parabolic subgroup with the Levi factor M' which equals the sum of the standard intertwining operators

$$E_0(\underline{s}, g; \pi', f_{\underline{s}}) = \sum_{w \in W(M')} A(\underline{s}, \pi', w) f_{\underline{s}}(g),$$

where $W(M')$ is the set of the Weyl group elements such that $wM'w^{-1} \cong M'$. Hence, the poles of the Eisenstein series are determined by the poles of the standard intertwining operators.

By Corollary A.4, in the region $Re(s_i - s_{i+1}) \geq s_{\rho, D}$, for $i = 1, \dots, k - 1$, the only possibility for the pole is at \underline{s}_0 . However, it indeed occurs only for the intertwining operators corresponding to the Weyl group element inverting the order of any two successive indices, i.e. the longest element w_l in $W(M')$. Since the iterated pole is simple in every iteration, the iterated residue of the constant term, up to a non-zero constant, equals the normalized intertwining operator

$$N(\underline{s}_0, \pi', w_l),$$

as claimed.

The irreducibility of its image follows from the uniqueness of the discrete series constituent in the considered induced representation obtained in Proposition 5.6 (a). The square integrability follows from the Langlands criterion (Sect. I.4.11 of [MW3]). \square

Remark A.6. The proof of the lemma shows that $MW'(\sigma', k)$, for $k > 1$, is at every place an irreducible quotient of the corresponding induced representation.

Theorem A.7. *The residual spectrum $L^2_{\text{res}}(G'_n)$ of an inner form $G'_n(\mathbb{A})$ of the split general linear group decomposes into a Hilbert space direct sum*

$$L^2_{\text{res}}(G'_n) \cong \bigoplus_{\substack{r|n \\ 1 \leq r < n}} \bigoplus_{\substack{\sigma' \in DS'_r \\ \text{(basic) cuspidal}}} \mathcal{A}(\sigma', n/r),$$

where $\mathcal{A}(\sigma', n/r) \cong MW'(\sigma', n/r)$ are the spaces of automorphic forms obtained in the previous lemma.

Proof. The results of Sect. 5 classify the discrete spectrum DS'_n of the inner form $G'_n(\mathbb{A})$ using the trace formula. The basic cuspidal representations are proved to be cuspidal. Hence, it remains to show that the representations of the form $MW'(\sigma', k)$, for $k > 1$ and a basic cuspidal representation σ' , are in the residual spectrum. However, this is precisely the content of the previous Lemma A.5. \square

Corollary A.8. *The cuspidal spectrum of an inner form $G'_n(\mathbb{A})$ consists of the basic cuspidal representations.*

Proof. Theorem A.7 shows that in the discrete spectrum DS'_n of an inner form $G'_n(\mathbb{A})$ obtained in Sect. 5 all the representations not being basic cuspidal belong to the residual spectrum. Hence, the multiplicity one of Theorem 5.1 for inner forms implies the corollary. \square

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