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On the residual spectrum of split classical groups supported in the Siegel maximal parabolic subgroup

Neven Grbac

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Abstract For the split symplectic and special orthogonal groups over a number field, we decompose the part of the residual spectrum supported in the maximal parabolic subgroup with the Levi factor isomorphic to GL_n. The decomposition depends on the analytic properties of the symmetric and exterior square automorphic L-functions, but seems sufficient for the computation of the corresponding part of the Eisenstein cohomology. We also prove that if one assumed Arthur's conjectural description of the discrete spectrum for the considered groups, then one would be able to find the poles of the L-functions in question, and would make the decomposition more precise.

Keywords Spectral decomposition · Automorphic forms · Automorphic L-functions

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0 Introduction

In the spectral decomposition of the space of square-integrable automorphic forms on the adelic points of a reductive group defined over a number field the continuous part of the spectrum was described by Langlands in [12]. It remains to understand the discrete part of the spectrum which consists of the cuspidal and residual part. The Langlands spectral theory of [12] (see also [14]) describes the residual spectrum in terms of iterated residues at the poles of the Eisenstein series attached to cuspidal automorphic representations of Levi factors of proper parabolic subgroups. We recall briefly the strategy in Sect. 1 referring to [14] for more details. However, determining the poles of the Eisenstein series depends on the analytic properties of the automorphic

N. Grbac (✉)
Department of Mathematics, University of Rijeka, Omladinska 14, 51000 Rijeka, Croatia
e-mail: neven.grbac@math.uniri.hr

L-functions and possible cancellations among summands in the iterated residue of the constant term.

In his work, Arthur develops the trace formula in order to describe the discrete spectrum without distinguishing the cuspidal and residual part. Thus, he avoids the problems concerning the poles and zeros of the automorphic L-functions. In [1–3] he gives conjectures on the discrete spectrum for split connected classical groups over a number field. Here we use the statement of the conjectures given in Sect. 30 of [3]. Since all the required variants of the fundamental lemma are now at hand (proved by Ngô in [15] and Chaudouard and Laumon in [6]), it seems that the proof of Arthur's conjectures for split connected classical groups is now within reach. Based on Arthur's conjectural description, Mœglin in [13] explains a way to determine the residual spectrum for classical groups assuming further conjectures on the L-functions and the images of intertwining operators.

In this paper we compare the two approaches for the part of the residual spectrum supported in the Siegel maximal parabolic subgroup of split connected classical groups $G_n = Sp_{2n}, SO_{2n+1}, SO_{2n}$ over a number field k. The Siegel maximal parabolic subgroup is the parabolic subgroup corresponding to the set of simple roots with the last root removed (see Sect. 1). Along the way, we study the analytic properties of the symmetric and exterior square L-functions attached to a cuspidal automorphic representation of $GL_n(\mathbb{A})$. In fact, we relate the poles of those L-functions to the considered part of the residual spectrum.

In Sect. 1 the approach is that of Langlands, and we do not use Arthur's work. The obtained results are thus unconditional. The main result concerning the residual spectrum is Theorem 1.5, where the decomposition of the considered part of the residual spectrum is given in terms of the poles of the symmetric and exterior square L-functions. This is a consequence of Shahidi's work on the normalization of intertwining operators ([19], see also [23] and [7, Sect. 11]), and the analytic properties of the relative rank one Eisenstein series ([14, Sect. IV.3.12]).

Although depending on the analytic properties of automorphic L-functions, the decomposition of the part of the residual spectrum obtained in Theorem 1.5 is sufficient for the application in computing the corresponding part of the Eisenstein cohomology. This line of thought was pursued in a preprint [8] in the case of symplectic group Sp_n split over the field $\mathbb{Q}$ of rational numbers. It turns out that the non-vanishing conditions for the cohomology class rule out from the consideration all parts of the decomposition which depend on the unknown poles of the L-functions, and thus we obtain unconditional results for the Eisenstein cohomology (except for the non-vanishing condition for the principal L-function at $s = 1/2$).

In Sect. 2 we briefly recall Arthur's conjectural description of the discrete spectrum for $G_n(\mathbb{A})$ following Sect. 30 of [3]. Then, in Sect. 3, we compare the conjectures with the results obtained using the approach of Langlands. The two consequences of Arthur's conjectures are Theorem 3.1, in which a more precise decomposition of the considered part of the residual spectrum is given, and its Corollary 3.2, in which we prove that, assuming the conjectures, the symmetric and exterior square L-functions are holomorphic inside the critical strip $0 < Re(s) < 1$. The result on the holomorphy of the L-functions inside $0 < Re(s) < 1$ is one of the possible consequences of Arthur's conjectures listed without proofs in [5].

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1 Decomposition of the residual spectrum

In order to fix the notation we recall very briefly the approach to the residual spectrum through the Langlands spectral theory as explained in [14] and [12]. See [14] for more details. The exposition is adjusted to the special case considered in this paper, i.e. the Siegel maximal parabolic subgroup of a split classical group.

Let k be an algebraic number field, k_v its completion at a place v and $\mathbb{A}$ the ring of adèles of k. For a positive integer n, let G_n denote either the k-split symplectic group Sp_{2n} given as the group of isometries of a $2n$-dimensional symplectic space over k, or one of the k-split special orthogonal groups SO_{2n} and SO_{2n+1} given by the group of determinant one isometries of a $2n$ and $2n+1$-dimensional orthogonal space over k, respectively. Throughout this paper we assume $n \geqslant 2$.

Let T be the maximal k-split torus of G_n, and we fix once for all the choice of the Borel subgroup B with the unipotent radical U. This choice determines the set of positive roots Σ^+ inside the set of roots Σ of G_n with respect to T. Let

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n\}$$

be the set of simple roots, where e_i is the projection of T to its ith coordinate and $\alpha_n = 2e_n$ if $G_n = Sp_{2n}$, $\alpha_n = e_{n-1}+e_n$ if $G_n = SO_{2n}$, and $\alpha_n = e_n$ if $G_n = SO_{2n+1}$. Let W be the Weyl group of G_n with respect to T.

Let P be the maximal standard parabolic k-subgroup of G_n corresponding to the subset $\Delta\backslash\{\alpha_n\}$ of the set of simple roots Δ. We call P the Siegel maximal parabolic subgroup of G_n. In its Levi decomposition $P = MN$, the Levi factor is $M \cong GL_n$, and N is the unipotent radical. By [20], P is self-associate, unless $G_n = SO_{2n}$ and n is odd. Let $W(M)$ be the set of Weyl group elements $w \in W$, of minimal length in their left coset modulo the Weyl group of M, such that wMw^{-1} is the Levi factor of a standard parabolic k-subgroup of G_n. Then, $W(M) = \{1, w_0\}$, where w_0 is the unique Weyl group element such $w_0(\Delta\backslash\{\alpha_n\}) \subset \Delta$.

Let $\mathfrak{a}^*_{M,\mathbb{C}}$ be the complexification of the $\mathbb{Z}$-module of k-rational characters of M. Since P is maximal, $\mathfrak{a}^*_{M,\mathbb{C}}$ is one-dimensional, and we identify $s \in \mathbb{C}$ with $\det \otimes s$. For $s \in \mathfrak{a}^*_{M,\mathbb{C}}$ and a cuspidal automorphic representation π of the Levi factor $M(\mathbb{A}) \cong GL_n(\mathbb{A})$, realized on a subspace V_π of the space of cusp forms on $M(\mathbb{A})$, we form a parabolically induced representation

$$\mathrm{I}(s, \pi) = \mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})} \left(\pi |\det|^s\right),$$

where $|\cdot|$ is the adèlic absolute value, and the induction is normalized.

We proceed as in Sect. II.1 of [14]. Choosing a section f_s of the induced representation, we define the Eisenstein series by the analytic continuation from the domain of convergence of the series

$$E(s, g; f_s, \pi) = \sum_{\gamma \in P(k)\backslash G_n(k)} f_s(\gamma g). \tag{1.1}$$

By Sect. IV.1 of [14], the Eisenstein series is a meromorphic function of s with finitely many poles in the region $Re(s) > 0$, and all other singularities in the region $Re(s) < 0$. The poles in the region $Re(s) > 0$ of the Eisenstein series attached to π determine the part of the residual spectrum of G_n supported in the cuspidal datum of a cuspidal automorphic representation π of $M(\mathbb{A})$.

If P is self-associate, the poles coincide with the poles of its constant term

$$E_P(s, g; f_s, \pi) = \int_{N(k)\backslash N(\mathbb{A})} E(s, ng; f_s, \pi)dn \tag{1.2}$$

along the parabolic subgroup P. On the other hand, by Sect. II.1.7 of [14], the constant term equals

$$E_P(s, g; f_s, \pi) = f_s(g) + M(s, \pi, w_0) f_s(g), \tag{1.3}$$

where $M(s, \pi, w_0)$ is the standard intertwining operator defined in Sect. II.1.6 of [14], where the representative $\tilde{w}_0$ for w_0 in $G_n(k)$ used in the defining integral is chosen as in [17]. Away from its poles, it intertwines $\mathrm{I}(s, \pi)$ and $\mathrm{I}(w_0(s), w_0(\pi))$, where the action of w_0 on $\mathfrak{a}^*_{M,\mathbb{C}}$ and π is given by conjugation with $\tilde{w}_0$ on M. Note that $w_0(s) = -s$. In order to understand the singularities of the Eisenstein series, one has to study the singularities of the standard intertwining operators.

If P is not self-associate, i.e. $G_n = SO_{2n}$ and n is odd, the constant term of the Eisenstein series is non-trivial along P and Q, where Q is the standard parabolic subgroup with the Levi factor $M' = \tilde{w}_0 M \tilde{w}_0^{-1} \cong GL_n$. Observe that Q corresponds to $\Delta \backslash \{\alpha_{n-1}\}$. Those constant terms are given by

$$E_P(s, g; f_s, \pi) = f_s(g), \tag{1.4}$$

and

$$E_Q(s, g; f_s, \pi) = M(s, \pi, w_0) f_s(g), \tag{1.5}$$

by Sect. II.1.7 of [14].

Let $\pi \cong \otimes_v \pi_v$. At every place v, for the local unitary irreducible representation π_v, the local standard intertwining operator, denoted by $A(s, \pi_v, w_0)$, is defined by the analytic continuation of the local version of the integral defining the global ones. Then, by [12], one can define a scalar meromorphic normalizing factor $r(s, \pi_v, w_0)$ at all places v where π_v is unramified. It is given in terms of the local L-functions attached to the unramified representation π_v. The main property of the normalizing factor for places v where π_v is unramified is that

$$A(s, \pi_v, w_0) f^\circ_{s,v} = r(s, \pi_v, w_0) \widetilde{f}^\circ_{-s,v},$$

where $f^\circ_{s,v}$ and $\widetilde{f}^\circ_{-s,v}$ are the unique suitably normalized spherical vectors in the induced representations $I(s, \pi_v)$ and $I(-s, w_0(\pi_v))$.

Let $f_s = \otimes_v f_{s,v}$ be a decomposable section in $\mathrm{I}(s, \pi)$. Let S be a finite set of places containing all the archimedean places such that for all $v \notin S$ we have π_v is unramified and $f_{s,v} = f^\circ_{s,v}$ is the unique suitably normalized spherical vector in $I(s, \pi_v)$. Let

$$r^S(s, \pi, w_0) = \prod_{v \notin S} r(s, \pi_v, w_0).$$

It is given in terms of the partial L-functions attached to π. Then the global standard intertwining operator decomposes into

$$M(s, \pi, w_0) f_s(g) = \left[\otimes_{v\in S} A(s, \pi_v, w_0) f_{s,v}(g)\right] \otimes r^S(s, \pi, w_0) \left[\otimes_{v\notin S} \widetilde{f}^\circ_{-s,v}(g)\right]. \tag{1.6}$$

Since every cuspidal automorphic representation of $GL_n(\mathbb{A})$ is generic, one can define as in [19] a scalar meromorphic normalizing factor $r(s, \pi_v, w_0)$ given in terms of the local L-functions attached to π_v at the remaining places. The precise formulas for the normalizing factors in our cases will be given later when required. Let $N(s, \pi_v, w_0)$ be the local normalized intertwining operator defined at all places by

$$A(s, \pi_v, w_0) = r(s, \pi_v, w_0) N(s, \pi_v, w_0). \tag{1.7}$$

Then, at places v where π_v is unramified, we have

$$N(s, \pi_v, w_0) f^\circ_{s,v} = \widetilde{f}^\circ_{-s,v}. \tag{1.8}$$

The global normalizing factor $r(s, \pi, w)$ is defined as the product of the local ones over all places. It is given in terms of the global L-functions attached to π. Then, the global intertwining operator (1.6) can be rewritten as

$$M(s, \pi, w_0) f_s(g) = r(s, \pi, w_0) \left[\left(\otimes_{v\in S} N(s, \pi_v, w_0) f_{s,v}(g)\right) \otimes \left(\otimes_{v\notin S} \widetilde{f}^\circ_{-s,v}(g)\right)\right]. \tag{1.9}$$

In view of (1.8), the square bracket is in fact

$$\otimes_v N(s, \pi_v, w_0) f_{s,v}(g),$$

and we denote by $N(s,\pi,w_0)$ the global normalized intertwining operator obtained in this way. In other words

$$N(s,\pi,w_0)f_s(g) = \left[\left(\otimes_{v\in S} N(s,\pi_v,w_0)f_{s,v}(g)\right)\otimes\left(\otimes_{v\notin S}\widetilde{f}^{\circ}_{-s,v}(g)\right)\right], \qquad (1.10)$$

and we usually write

$$M(s,\pi,w_0) = r(s,\pi,w_0)N(s,\pi,w_0).$$

By (1.10), the holomorphy and non-vanishing of the global normalized intertwining operators $N(s,\pi,w_0)$ in a certain region of $\mathfrak{a}^*_{M,\mathbb{C}}$ reduces to the holomorphy and non-vanishing of the local normalized operator at a finite number of places $v\in S$.

Theorem 1.1 *Let G_n and $P = MN$ be as above. Let π be a cuspidal automorphic representation of $M(\mathbb{A})$. Then, for the non-trivial element $w_0\in W(M)$, the Langlands–Shahidi normalizing factor $r(s,\pi,w_0)$ is given as*

$$r(s,\pi,w_0) = \begin{cases} \frac{L(s,\pi)}{L(1+s,\pi)\varepsilon(s,\pi)}\frac{L(2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)\varepsilon(2s,\pi,\wedge^2)}, & \text{for } G_n = Sp_{2n},\\ \frac{L(2s,\pi,Sym^2)}{L(1+2s,\pi,Sym^2)\varepsilon(2s,\pi,Sym^2)}, & \text{for } G_n = SO_{2n+1},\\ \frac{L(2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)\varepsilon(2s,\pi,\wedge^2)}, & \text{for } G_n = SO_{2n}, \end{cases}$$

where the global L-functions and ε-factors are the products of the local ones which are defined in [19]. The L-functions $L(s,\pi,\wedge^2)$ and $L(s,\pi,Sym^2)$ are the exterior and symmetric square L-functions, respectively, and the L-function $L(s,\pi)$ is the principal L-function.

The normalized intertwining operator $N(s,\pi,w_0)$ obtained using the Langlands–Shahidi normalizing factor $r(s,\pi,w_0)$ is holomorphic and non-vanishing for $Re(s)\geq 0$. Hence, the poles inside $Re(s)\geq 0$ of the Eisenstein series attached to π coincide with the poles inside $Re(s)\geq 0$ of the normalizing factor $r(s,\pi,w_0)$.

Proof The formula for $r(s,\pi,w_0)$ is given in [20]. The holomorphy and non-vanishing of $N(s,\pi,w_0)$ for $Re(s)\geq 0$ follows from the tempered case in [23] using the classification of the generic unitary dual of $GL_n(k_v)$ [21,22]. Therefore, the poles inside $Re(s)\geq 0$ of the standard intertwining operator $M(s,\pi,w_0)$, and thus of the Eisenstein series attached to π (cf. [14, Sect. IV.3.10]), coincide with the poles of the normalizing factor $r(s,\pi,w_0)$. □

On the other hand, Sect. IV.3.12 of [14] provides a necessary condition for the existence of a pole in the region $Re(s)\geq 0$ for the Eisenstein series attached to a cuspidal automorphic representation of the Levi factor of a maximal parabolic subgroup. We recall this condition in the following Lemma. For the proof see [14, Sect. IV.3.12].

Lemma 1.2 *Let G_n and $P = MN$ be as above. Let π be a cuspidal automorphic representation of $M(\mathbb{A})$. If*

- *either P is not self-associate,*
- *or P is self-associate, but $w_0(\pi)$ and π do not belong to the same cuspidal datum,*

then the Eisenstein series attached to π, and thus the standard intertwining operator $M(s, \pi, w_0)$, has no poles in the region $Re(s) \geq 0$.

In the following theorem we recall the well-known analytic properties of the symmetric and exterior L-functions $L(s, \pi, Sym^2)$ and $L(s, \pi, \wedge^2)$ attached to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$. Except for the behavior inside the strip $0 < Re(s) < 1$ in part (3) of the theorem, which follows from Lemma 1.2, these properties are independent of the spectral decomposition. Note that for π as in part (2) of the theorem the possible poles of the symmetric and exterior L-functions inside $0 < Re(s) < 1$ are not known. In Sect. 3 we prove that if one assumed Arthur's conjectures, then it would follow that those L-functions are holomorphic inside that strip.

Theorem 1.3 *Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A})$.*

(1) *The L-functions $L(s, \pi, Sym^2)$ and $L(s, \pi, \wedge^2)$, as well as the principal L-function $L(s, \pi)$, are non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$. In particular, the denominators of the normalizing factors $r(s, \pi, w_0)$ given in Theorem 1.1 are non-zero for $Re(s) \geq 0$.*
(2) *If $\widetilde{\pi} \cong \pi \otimes |\det|^{it}$, with $i = \sqrt{-1}$ and $t \in \mathbb{R}$ (such t is unique if it exists), then exactly one of the L-functions $L(s, \pi, Sym^2)$ and $L(s, \pi, \wedge^2)$ is holomorphic in the region $Re(s) \geq 1$ and $Re(s) \leq 0$, while the only poles in that region of the other L-function are simple poles at $s = 1 + it$ and $s = it$.*
(3) *If there is no $t \in \mathbb{R}$ such that $\widetilde{\pi} \cong \pi \otimes |\det|^{it}$, with $i = \sqrt{-1}$, then the L-functions $L(s, \pi, Sym^2)$ and $L(s, \pi, \wedge^2)$ are both entire.*

Proof We give a sketch of proof for each claim separately.

(1) This is the non-vanishing result of [18]. The second claim follows, since the arguments of the L-functions in denominators are of the form $1 + s$ or $1 + 2s$.
(2) Let $\widetilde{\pi} \cong \pi \otimes |\det|^{it}$, with $t \in \mathbb{R}$. Write the Rankin–Selberg L-function as

$$L(s, \pi \times \pi) = L(s, \pi, \wedge^2)L(s, \pi, Sym^2).$$

The analytic properties of $L(s, \pi \times \pi)$ follow from the integral representation for the Rankin–Selberg L-functions developed in [9–11]. It has simple poles at $s = 1 + it$ and $s = it$, and is holomorphic elsewhere. Since by (1), i.e. [18], both L-functions on the right hand side are non-zero for $Re(s) \geq 1$ and $Re(s) \leq 0$, they are both holomorphic in that region except at $s = 1 + it$ and $s = it$ where exactly one of them has a simple pole.
(3) Since in this case the Rankin–Selberg L-function $L(s, \pi \times \pi)$ is entire, the same argument as in the proof of (2) gives holomorphy in the region $Re(s) \geq 1$ and $Re(s) \leq 0$. However, for the strip $0 < Re(s) < 1$, one needs Lemma 1.2. Since there is no $t \in \mathbb{R}$ such that $\widetilde{\pi} \cong \pi \otimes |\det|^{it}$, and $w_0(\pi) \cong \widetilde{\pi}$, it follows that $w_0(\pi)$ and π do not belong to the same cuspidal datum. Hence, by Lemma 1.2,

the Eisenstein series attached to π for any $G_n(\mathbb{A})$ has no poles inside $Re(s) > 0$. However, by Theorem 1.1 and taking into account property (1) i.e. [18], the poles of the Eisenstein series inside $Re(s) > 0$ coincide with the poles of the numerator of the normalizing factors $r(s, \pi, w_0)$. Taking $G_n = SO_{2n+1}$ and $G_n = SO_{2n}$ shows that the symmetric and exterior square L-functions, respectively, are holomorphic for $Re(s) > 0$. Then, by the functional equation, they are entire.

□

The following definition is based on Theorem 1.3. Namely, part (2) of that Theorem assures that every π such that $\widetilde{\pi} \cong \pi$ is either symplectic or orthogonal.

Definition 1.4 Let π be a selfdual cuspidal automorphic representation of $GL_n(\mathbb{A})$, i.e. its contragredient $\widetilde{\pi}$ is isomorphic to π. Then, we say that π is symplectic (resp. orthogonal) if the exterior (resp. symmetric) square L-function has a pole at $s = 1$ and $s = 0$.

When decomposing the residual spectrum, it is convenient to assume that a cuspidal automorphic representation π of $M(\mathbb{A})$ used to define the Eisenstein series is normalized in such a way that all the poles inside $Re(s) > 0$ of the Eisenstein series are real. This assumption is not restricting since it can be achieved by replacing π with an appropriate element of the same cuspidal datum. Namely, there is a real number $t \in \mathbb{R}$ such that $\pi \otimes |\det|^{it}$ satisfies that assumption, where $i = \sqrt{-1}$. Therefore, in what follows we assume that π is chosen in such a way. In the following theorem, combining properties of L-functions with the condition of Lemma 1.2, we obtain the decomposition of the part of residual spectrum supported in the Siegel maximal parabolic subgroup of G_n.

Theorem 1.5 *Let $P = MN$ be the Siegel maximal parabolic k-subgroup of G_n, with $n \geq 2$. Let $L^2_{\mathrm{res},P}(G_n)$ denote the part of the residual spectrum spanned by the residues of the Eisenstein series attached to cuspidal automorphic representations of $M(\mathbb{A})$ at the poles inside $Re(s) > 0$ (which are all real by our assumption on cuspidal automorphic representations of $M(\mathbb{A})$). Then, unless $G_n = SO_{2n}$ with n odd, the space $L^2_{\mathrm{res},P}(G_n)$ decomposes into*

$$L^2_{\mathrm{res},P}(G_n) \cong \bigoplus_{\widetilde{\pi}\cong\pi} \bigoplus_{\substack{0<s_0\leq 1/2 \\ L(2s_0,\pi,r)=\infty \\ L(s_0,\pi)\neq 0 \text{ if } G_n=Sp_{2n}}} \mathcal{A}(s_0, \pi),$$

with $\widetilde{\pi}$ the contragredient of π, and $r = \wedge^2$ if $G_n = Sp_{2n}, SO_{2n}$, and $r = Sym^2$ if $G_n = SO_{2n+1}$, where $\mathcal{A}(s_0, \pi)$ is the space of automorphic forms on $G_n(\mathbb{A})$ spanned by the residues

$$(s - s_0)E(s, g; f_s, \pi)\Big|_{s=s_0}$$

at $s = s_0$ of the Eisenstein series attached to π. The space of automorphic forms $\mathcal{A}(s_0, \pi)$ is isomorphic to the image of the normalized intertwining operator $N(s_0, \pi, w_0)$. If $G_n = SO_{2n}$ and n is odd, then $L^2_{\mathrm{res},P}(SO_{2n})$ is trivial.

Proof By Lemma 1.2, i.e. [14, Sect. IV.3.12], if P is not self-associate, then the Eisenstein series attached to any cuspidal automorphic representation of $M(\mathbb{A})$ does not have a pole inside $Re(s) > 0$. Thus, in the case $G_n = SO_{2n}$ and n odd, $L^2_{\text{res},P}(SO_{2n})$ is trivial.

Hence, let us assume that P is self-associate, i.e. for $G_n = SO_{2n}$ we have n is even. Since $w_0(\pi) \cong \widetilde{\pi}$, again by Lemma 1.2, i.e. [14, Sect. IV.3.12], the Eisenstein series attached to π has no pole inside $s > 0$ if $\widetilde{\pi} \ncong \pi$. This implies the condition $\widetilde{\pi} \cong \pi$ in the first sum of the decomposition.

Let $\widetilde{\pi} \cong \pi$. For the conditions in the second sum we refer to the normalization of intertwining operators discussed in this section. By Theorem 1.1, the poles inside $s > 0$ of the Eisenstein series are the same as the poles of the normalizing factors $r(s, \pi, w_0)$. The normalizing factors are given in Theorem 1.1, and since the denominators are holomorphic and non-zero at s inside $Re(s) > 0$ by Theorem 1.3(1), their poles at $s > 0$ are given by the poles of the numerators. This is precisely what the conditions in the second sum provide. There are no poles for $s > 1/2$ because the L-functions are holomorphic inside $Re(s) > 1$.

The Langlands square-integrability criterion (Sect. I.4.11 of [14]) is always satisfied because $w_0(s_0) = -s_0 < 0$ for every $s_0 > 0$. From Eq. (1.3), the constant terms of the residues of the Eisenstein series spanning the space of automorphic forms $\mathcal{A}(s_0, \pi)$ are equal to $N(s_0, \pi, w_0) f_s$ up to a non-zero constant. Hence, $\mathcal{A}(s_0, \pi)$ is isomorphic to the image of $N(s_0, \pi, w_0)$. □

2 Arthur's conjectures

In this section we recall Arthur's conjectural description of the discrete spectrum for the split connected classical groups. In particular, we study possible Arthur's parameters for the residual representations supported in the Siegel maximal parabolic subgroup P of G_n obtained in Sect. 1. The main references are certainly [1–3], but see also [4,13]. Here we follow Sect. 30 of [3].

Let G_n be one of the split connected classical groups defined over an algebraic number field k as in Sect. 1. Let $\widehat{G}_n$ be the Langlands dual group of G_n, and let

$$N = \begin{cases} 2n+1, & \text{if } G_n = Sp_{2n}, \\ 2n, & \text{if } G_n = SO_{2n+1} \text{ or } G_n = SO_{2n}. \end{cases} \tag{2.1}$$

Then,

$$\widehat{G}_n = \begin{cases} SO_N(\mathbb{C}), & \text{if } G_n = Sp_{2n} \text{ or } G_n = SO_{2n}, \\ Sp_N(\mathbb{C}), & \text{if } G_n = SO_{2n+1}. \end{cases} \tag{2.2}$$

In the former case we say $\widehat{G}_n$ is orthogonal or of orthogonal type, and in the latter $\widehat{G}_n$ is symplectic or of symplectic type.

In Definition 1.4 the concept of symplectic and orthogonal for a selfdual cuspidal automorphic representation π of $GL_n(\mathbb{A})$ is defined in terms of the poles at $s = 1$ and $s = 0$ of the exterior and symmetric square L-functions $L(s, \pi, \wedge^2)$

and $L(s, \pi, Sym^2)$, respectively. The following Theorem, which is a part of Arthur's conjectural description of the discrete spectrum for G_n (Sect. 30 of [3]), relates those concepts with the symplectic or orthogonal type of the dual group of the twisted endoscopic group associated to π. The global induction hypothesis (page 240 of [3]), which is also a part of Arthur's conjectures, implies that the dual of the twisted endoscopic group associated to π is indeed either symplectic or orthogonal group. Moreover, it is an orthogonal group if ω_π is nontrivial or n is odd.

Theorem 2.1 (Arthur, Thm 30.3. (a) of [3]) *A selfdual cuspidal automorphic representation π of $GL_n(\mathbb{A})$ is symplectic* (*resp. orthogonal*) *if and only if the dual of the twisted endoscopic group associated to π is a symplectic* (*resp. orthogonal*) *group. In particular, if the central character ω_π is nontrivial or n is odd, then π is orthogonal.*

For Arthur's description of the discrete spectrum, we first introduce Arthur's parameters for G_n.

Definition 2.2 (Arthur's parameters for G_n) Let $\mathcal{S}(G_n)$ be the set of formal sums of formal tensor products

$$\psi = (\sigma_1 \boxtimes \nu(m_1)) \boxplus \cdots \boxplus (\sigma_\ell \boxtimes \nu(m_\ell))$$

such that

(i) σ_i is a selfdual irreducible cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$ for a positive integer n_i,
(ii) m_i is a positive integer, and $\nu(m_i)$ is the unique m_i-dimensional irreducible algebraic representation of $SL_2(\mathbb{C})$,
(iii) $N = n_1m_1 + \ldots n_\ell m_\ell$, where N is given by (2.1),
(iv) for $i \neq j$ we have $\sigma_i \ncong \sigma_j$ or $m_i \neq m_j$,
(v) the product $\prod_{i=1}^{\ell} \omega_{\sigma_i}^{m_i}$ is trivial, where $\omega_{\sigma_i} : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is the central character of σ_i,
(vi)
- if $\widehat{G}_n$ is orthogonal, then, for every i, either σ_i is orthogonal and m_i odd, or σ_i is symplectic and m_i even,
- if $\widehat{G}_n$ is symplectic, then, for every i, either σ_i is orthogonal and m_i even, or σ_i is symplectic and m_i odd.

Two such formal sums $\psi = \boxplus_{i=1}^{\ell}(\sigma_i \boxtimes \nu(m_i))$, $\psi' = \boxplus_{j=1}^{\ell'}(\sigma'_j \boxtimes \nu(m'_j)) \in \mathcal{S}(G_n)$ are equivalent if and only if $\ell = \ell'$ and there is a permutation p of $\{1, \ldots, \ell\}$ such that $\sigma'_j \cong \sigma_{p(i)}$ and $m'_j = m_{p(i)}$. Arthur's parameter for G_n is an equivalence class in $\mathcal{S}(G_n)$, and abusing the notation we denote by ψ the equivalence class of an element $\psi \in \mathcal{S}(G_n)$. The set of all Arthur's parameters for G_n we denote by $\Psi_2(G_n)$.

Arthur attaches in Theorem 30.2.(a) of [3] to every A-parameter $\psi \in \Psi_2(G_n)$ a global A-packet $\widetilde{\Pi}_\psi$ of nearly equivalent representations of $G_n(\mathbb{A})$. Roughly speaking a global A-packet is built up of the local A-packets $\widetilde{\Pi}_{\psi_v}$ attached to ψ at every place v in Theorem 30.1 of [3]. At almost all non-Archimedean places, where the local components $\sigma_{i,v}$ of all σ_i in the A-parameter ψ are unramified, $\widetilde{\Pi}_{\psi_v}$ contains the unramified representation denoted by π_v° with the Satake parameter

$$c(\psi_v) = \big(c(\sigma_{1,v}) \otimes c(\nu(m_1))\big) \oplus \cdots \oplus \big(c(\sigma_{\ell,v}) \otimes c(\nu(m_\ell))\big) \in \widehat{G}_n,$$

where $c(\sigma_{i,v}) \in GL_{n_i}(\mathbb{C})$ is the Satake parameter of the unramified representation $\sigma_{i,v}$ of $GL_{n_i}(k_v)$, and

$$c(\nu(m_i)) = \mathrm{diag}\left(q_v^{\frac{m_i-1}{2}}, q_v^{\frac{m_i-3}{2}}, \ldots, q_v^{-\frac{m_i-1}{2}}\right),$$

with q_v the number of elements of the residue field of k_v. Then all the members of the global A-packet $\widetilde{\Pi}_\psi$, as defined in Theorem 30.2.(a) of [3], are representations of $G_n(\mathbb{A})$ of the form $\pi \cong \otimes_v \pi_v$ where $\pi_v \in \widetilde{\Pi}_{\psi_v}$ and $\pi_v \cong \pi_v^\circ$ at almost all v. The representations in the global A-packet are just possible representations appearing in the discrete spectrum $L^2_{\mathrm{disc}}(G_n)$. Theorem 30.2.(b) gives the precise condition required of $\pi \in \widetilde{\Pi}_\psi$ to belong to $L^2_{\mathrm{disc}}(G_n)$. However, we do not recall it here because we are interested in the A-parameters of the residual representations obtained in Sect. 1 which certainly belong to the discrete spectrum. These A-parameters are given in the following proposition. See Sect. 1 for the notation concerning the structure of G_n and the induced representations.

Proposition 2.3 *Let $n \geqslant 2$ be an integer, and let P be the Siegel maximal proper standard parabolic k-subgroup of G_n (corresponding to $\Delta\backslash\{\alpha_n\}$). Let $s_0 > 0$ be a real number, and π a selfdual cuspidal automorphic representation of the Levi factor $M(\mathbb{A}) \cong GL_n(\mathbb{A})$ of $P(\mathbb{A})$. If the induced representation*

$$\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}\left(\pi\,|\det|^{s_0}\right)$$

has a constituent in the discrete spectrum of $G_n(\mathbb{A})$ with the A-parameter $\psi \in \Psi_2(G_n)$, then $s_0 = 1/2$, π and $\widehat{G}_n$ are of the opposite type, and

$$\psi = \begin{cases} (\pi \boxtimes \nu(2)) \boxplus (\mathbf{1} \boxtimes \nu(1)), & \text{if } G_n = Sp_{2n}, \\ \pi \boxtimes \nu(2), & \text{if } G_n = SO_{2n+1} \text{ or } G_n = SO_{2n}, \end{cases}$$

where $\mathbf{1}$ is the trivial character of $k^\times\backslash\mathbb{A}^\times$.

Proof Let v be a place of k where π_v is unramified and let $c(\pi_v) \in GL_n(\mathbb{C})$ be its Satake parameter. The unramified constituent of the local component at v of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}\left(\pi\,|\det|^{s_0}\right)$$

is given by the Frobenius–Hecke conjugacy class in $\widehat{G}_n$

$$\begin{cases} \left(c(\pi_v) \otimes \mathrm{diag}\left(q_v^{s_0}, q_v^{-s_0}\right)\right) \oplus 1, & \text{for } G_n = Sp_{2n}, \\ c(\pi_v) \otimes \mathrm{diag}\left(q_v^{s_0}, q_v^{-s_0}\right), & \text{for } G_n = SO_{2n+1} \text{ or } G_n = SO_{2n}. \end{cases}$$

At almost all places, the A-parameter ψ of a constituent of the induced representation should give the same conjugacy class in $\widehat{G}_n$. Hence, ψ is of the form

$$\psi = \begin{cases} (\pi \boxtimes \nu(m)) \boxplus (\chi \boxtimes \nu(1)), & \text{for } G_n = Sp_{2n}, \\ \pi \boxtimes \nu(m), & \text{for } G_n = SO_{2n+1} \text{ or } G_n = SO_{2n}, \end{cases}$$

where $m = 2s_0 + 1$. Since $m = 2$ by condition (iii) in Definition 2.2, the real number $s_0 > 0$ is equal to $s_0 = 1/2$. The character χ must be trivial due to (v) in Definition 2.2. The condition on the type of π comes from (vi) in Definition 2.2. □

3 Consequences of Arthur's conjectures

All the results of this section depend on Arthur's conjectural description of the discrete spectrum of $G_n(\mathbb{A})$ which we recalled in Sect. 2 following Sect. 30 of [3].

Theorem 3.1 *Let $n \geqslant 2$ be an integer, and P the Siegel maximal parabolic k-subgroup of G_n (corresponding to $\Delta\backslash\{\alpha_n\}$). Assume Arthur's conjectural description of the discrete spectrum of $G_n(\mathbb{A})$ as stated in Sect. 2. Then, in the notation of Theorem 1.5, the part $L^2_{\mathrm{res},P}(G_n)$ of the residual spectrum of $G_n(\mathbb{A})$ decomposes into*

$$L^2_{\mathrm{res},P}(G_n) = \bigoplus_{\substack{\widetilde{\pi}\cong\pi \\ \pi \text{ and } \widehat{G}_n \text{ of opposite type} \\ L(1/2,\pi)\neq 0 \text{ if } G_n = Sp_{2n}}} \mathcal{A}(1/2,\pi).$$

In particular, if n is odd and $\widehat{G}_n$ orthogonal (i.e. $G_n = Sp_{2n}$ or $G_n = SO_{2n}$), then $L^2_{\mathrm{res},P}(G_n)$ is trivial.

Proof If the space $\mathcal{A}(s_0,\pi)$ for $0 < s_0 \leq 1/2$ appears in the decomposition of $L^2_{\mathrm{res},P}(G_n)$, then it is isomorphic to a constituent of the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\pi|\det|^{s_0})$. However, by Proposition 2.3, Arthur's conjectural description of the discrete spectrum of $G_n(\mathbb{A})$ implies that $s_0 = 1/2$, and π and $\widehat{G}_n$ are of the opposite type. Since π is of the opposite type of $\widehat{G}_n$, the L-function $L(s,\pi,r)$ appearing in the decomposition of Theorem 1.5 has a pole at $s = 1$ by Definition 1.4. The condition $L(1/2,\pi) \neq 0$ if $G_n = Sp_{2n}$ is already in the decomposition of Theorem 1.5, and assures that the pole is not cancelled by the zero of the principal L-function.

If n is odd, π is always orthogonal (see Theorem 2.1). Hence, if $\widehat{G}_n$ is also orthogonal, the condition on the type of π and G_n is never satisfied. Thus, $L^2_{\mathrm{res},P}(G_n)$ is trivial. □

As mentioned just before Theorem 1.3, the analytic behavior of the symmetric and exterior square L-functions inside the strip $0 < Re(s) < 1$ is not known. However, comparing the unconditional spectral decomposition of Theorem 1.5, with the spectral decomposition of Theorem 3.1 which depends on Arthur's conjectures, and having in mind the relation of the poles of the Eisenstein series and those of L-functions, one obtains the following corollary. Although this corollary does not provide new information on the spectral decomposition, it shows that once Arthur's conjectures are proved, the analytic behavior of the symmetric and exterior L-functions, which is of importance itself, would be an immediate consequence.

Corollary 3.2 *Let $n \geq 2$ be an integer, and let π be a cuspidal automorphic representation of $GL_n(\mathbb{A})$ such that $\widetilde{\pi} \cong \pi \otimes |\det|^{it}$ with $t \in \mathbb{R}$. Assume Arthur's conjectural description of the discrete spectrum of $G_n(\mathbb{A})$ as stated in Sect. 2. Then, the exterior and symmetric square L-functions $L(s, \pi, \wedge^2)$ and $L(s, \pi, Sym^2)$ are holomorphic inside the strip $0 < Re(s) < 1$.*

Proof Assume that the L-function $L(s, \pi, r)$, where $r = \wedge^2$ or $r = Sym^2$, has a pole at s_0 with $0 < Re(s_0) < 1$. Let $\pi_0 \cong \pi \otimes |\det|^{it/2}$. Then, π_0 is selfdual, and by equation (3.12) of [19] and Sect. 6 of [20], the L-function $L(s, \pi_0, r)$ has a pole at $s = Re(s_0)$.

Consider first the case $r = Sym^2$. Then, by Theorem 1.1 and Theorem 1.3, the Eisenstein series for $G_n = SO_{2n+1}$ attached to π_0 has a pole at $s = Re(s_0)/2$. Taking the residue at $s = Re(s_0)/2$ gives the space $\mathcal{A}(Re(s_0)/2, \pi_0)$ in the part $L^2_{\text{res}, P}(SO_{2n+1})$ of the residual spectrum of $SO_{2n+1}(\mathbb{A})$. It is non-trivial since it is isomorphic to the image of the normalized intertwining operator $N(Re(s_0)/2, \pi_0, w_0)$ which is holomorphic and non-vanishing. However, the decomposition of Theorem 3.1 shows that, assuming Arthur's conjectures, the space $\mathcal{A}(Re(s_0)/2, \pi_0)$ with $0 < Re(s_0)/2 < 1/2$ does not appear. This is a contradiction, and thus we have proved that, assuming Arthur's conjectures, the symmetric square L-function has no poles inside the strip $0 < Re(s) < 1$.

For the case $r = \wedge^2$, we have already proved in part (3) of Theorem 1.3, without assuming Arthur's conjectures, that $L(s, \pi, \wedge^2)$ is entire if n is odd. Hence, it remains to consider n is even. In that case, taking $G_n = SO_{2n}$ instead of $G_n = SO_{2n+1}$, the same argument as above applies. □

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